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ON THE TWO-SIDED PREDICTABLE APPROXIMATION FOR STOCHASTIC PROCESSES

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Abstract. The two-sided predictable stochastic processes are introduced to show that they can approximate every bounded measurable stochastic process $(X_t(\omega),\ 0 < t \le 1)$ P-almost surely uniformly in t. Consequently, it is proved that the two-sided predictable algebra also generates the product σ-field $\mathcal{B}((0,1]) \otimes \mathcal{F}$.

I. Notations and notions

Let $(\Omega, \mathcal{F}, P)$ be the standard Wiener space with the canonical Brownian motion $(W_t, 0 \le t \le 1)$, $\mathcal{F}_t$ and $\mathcal{F}^t$ the completion of $\sigma\{W_s, s \le t\}$ and $\sigma\{W_1 - W_s, 1 - t \le s \le 1\}$ respectively, and $\mathcal{F}_{(s,t]^c} = \mathcal{F}_s \vee \mathcal{F}^{1-t}$.

Put

$\Omega^* = (0,1] \times \Omega$,

$\mathcal{F} =$ the product σ-field, that is $\mathcal{B}((0,1]) \otimes \mathcal{F}$,

$\mathcal{T} =$ algebra generated by $\{(s,t] \times G,\ s < t,\ G \in \mathcal{F}_{(s,t]^c}\}$ which is called the two-sided predictable algebra,

$S =$ algebra generated by $\{(s,t] \times G,\ s < t,\ G \in \mathcal{F}\}$, which generates the product σ-field $\mathcal{F}^*$.

Let $(X_t, 0 < t \le 1)$ be an integrable stochastic process, then $(X_t, 0 < t \le 1)$ induces a signed measure on $\mathcal{T}$ defined by

$$\lambda_X((s,t] \times G) = \int_\Omega (X_t - X_s) 1_G dP.$$

Then X is said to be an S-martingale (resp. an S-quasi-martingale) iff $\lambda_X \equiv 0$ (resp., λ_X is a measure of bounded variation) (see [D-N-S]).

Definition 1.1. The two-sided predictable stochastic processes are the simple stochastic processes of the form

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$$X_t = \sum_{i\geq 1} \lambda_i 1_{A_i}(t,\omega), \tag{1}$$

where λ_i's are real numbers, $A_i \in \mathcal{T}$ for all i, and A_i's are disjointed.

In this short note we shall show that every bounded measurable stochastic process $(X_t(\omega), 0 < t \leq 1)$ can be approximated P-almost surely uniformly in t by two-sided predictable stochastic processes. This result will be used to prove that $\mathcal{T}$ also generates the product σ-field $\mathcal{F}^*$.

II. The main results

For $C \subset \Omega^*$ we denote by $\Pi(C)$ the projection of C on Ω and $A\Delta B$ denotes the symmetric difference of the two sets A and B. First of all we shall prove the following lemma.

Lemma 2.1. *For any $A \in S$ and $\varepsilon > 0$, there exists a $B \in \mathcal{T}$ such that*

$$P(\Pi(A\Delta B)) < \varepsilon.$$

Proof. Since every set in S can be written as a finite union of sets of form: $(s,t] \times G$, $s < t$, $G \in \mathcal{F}$, we need only to consider the case $A = (s,t] \times G$, with $G \in \mathcal{F}$. Since $\mathcal{F}$ is generated by $\{W_u - W_v; u \geq v \geq t\} \cup \{W_t - W_u, W_u - W_s, s \leq u \leq t\} \cup \{W_u - W_v, v \leq u \leq s\}$, we can take without lost of generality $G = \bigcap_{0\leq i\leq n} (W_{t_{i+1}} - W_{t_i} \in I_i)$, where $0 = t_0 < t_1 < \cdots < t_n = 1$ such that $s = t_\ell$, $t = t_{k+1}$ for some $\ell < k$, and I_i are some open intervals with rational boundaries. We can write

$$\begin{aligned} A &= \bigcup_{1\leq i\leq k} (t_i, t_{i+1}] \times G \\ &= \bigcup_{1\leq i\leq k} \big\{(t_i, t_{i+1}] \times (G_i \cap (W_{t_{i+1}} - W_{t_i} \in I_i))\big\}, \end{aligned}$$

where

$$G_i = \bigcap_{j\neq i} (W_{t_{j+1}} - W_{t_j} \in I_j) \in \mathcal{F}_{(t_i,t_{i+1}]^c}.$$

Thus it suffices to prove our lemma for a set A of the form

$$A = (s,t] \times G, \text{ with } G = \{W_t - W_s \in I\},$$

where $I = (c, d)$ is an open interval.

Let $s = t_0 < t_1 < \cdots < t_{m+1} = t$, and define

$$\begin{aligned} G^0 &= \{W_t - W_{t_1} \in I\}, \\ G^m &= \{W_{t_m} - W_s \in I\}, \\ G^i &= \{(W_t - W_{t_{i+1}} + W_{t_i} - W_s\} \in I\},\ 0 \le i \le m-1. \end{aligned}$$

We want to show

$$P\Big(\bigcup_{0 \le i \le m} (G^i \Delta G)\Big) \to 0, \tag{2}$$

as $\delta := \max_i (t_{i+1} - t_i) \to 0$.

Since for any $u > v$, the probability for $\{W_u - W_v = c,\ \text{or}\ d\}$ is zero and the paths of W are continuous, we have for P-a.s. W with δ small enough

$$W \in G \Longleftrightarrow W \in G^i, \quad \text{for all } 0 \le i \le m.$$

Therefore $W \notin \bigcup_{0 \le i \le m} \{G^i \Delta G\}$ P - a.s., if δ is small enough. This implies

$$1_{\bigcup_{0 \le i \le m} \{G^i \Delta G\}}(W) \to 0,\ P - a.s.$$

as $\delta \to 0$. This proves (2).

To prove our lemma, we take

$$B = \bigcup_{0 \le i \le m} ((t_i, t_{i+1}] \times G^i).$$

Obviously $B \in \mathcal{T}$ since $G^i \in \mathcal{F}_{(t_i, t_{i+1}]^c}$ and

$$P(\Pi(A \Delta B)) = P\Big(\bigcup_{0 \le i \le m} (G^i \Delta G)\Big) \to 0$$

as $\delta \to 0$. q.e.d.

Remark. The idea, which is used in the proof of Lemma 2.1 and can be seen as an advantage of the two-sided predictable case, is that if we make $(t - s)$ smaller and amaller, then $\mathcal{F}_{(s,t]^c}$ approximates $\mathcal{F}$.

The main result of this note is the following.

Theorem 2.2. *Suppose that* $(X_t, 0 < t \leq 1)$ *is a bounded measurable stochastic process. Then there exists a sequence of two-sided predictable stochastic processes of the form* (1)

$$(X_t^{(n)}(\omega), 0 < t \leq 1), \quad n = 1, 2, \dots$$

such that

$$\lim_{n \to +\infty} X_t^{(n)}(\omega) = X_t(\omega) \quad P\text{-almost surely, uniformly in } t \in (0,1]. \tag{3}$$

Proof. Since S generates the product σ-field $\mathcal{F}^*$ and the process $(X_t, 0 < t \leq 1)$ is bounded, it is enough to prove (3) in the case

$$X_t(\omega) = 1_A(t, \omega),$$

where $A = (a, b] \times G$, $0 \leq a < b \leq 1$ and $G \in \mathcal{F}$. By Lemma 2.1 for every $n = 1, 2, \dots$ there exists a subset $B_n \in \mathcal{T}$ such that $P(\Pi(B_n \Delta A)) < \dfrac{1}{2^n}$. Put

$$X_t^{(n)} = 1_{B_n}(t, \omega).$$

It is clear that $X_t^{(n)}(\omega) = X_t(\omega)$ for all $(t, \omega) \notin B_n \Delta A$. On the other hand, since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$ is finite, by the Borell-Cattelli law, it follows that for P - a.s. every ω belongs only to a finite number of sets $B_n \Delta A$. Therefore for P - a.s. there exists $n_0 \in \{1, 2, \dots\}$ for every ω such that for all $n \geq n_0$ we have

$$X_t^{(n)}(\omega) = X_t(\omega) \quad \text{for all } 0 < t \leq 1$$

From Theorem 2.2 we can derive the following interesting result.

Corollary 2.3. *The two-sided predictable algebra* $\mathcal{T}$ *also generates the product* σ*-field* $\mathcal{F}^*$.

Proof. Put $\sigma(\mathcal{T})$ = the σ-field generated by the algebra $\mathcal{T}$. Clearly, $\mathcal{T} \subset S$ implies that $\sigma(\mathcal{T}) \subset \mathcal{F}^*$. For every $A \in S$, it follows from Theorem 2.2 that the process $(1_A(t, \omega), 0 < t \leq 1)$ is $\sigma(\mathcal{T})$-measurable. That means $A \in \sigma(\mathcal{T})$.

Therefore $S \subset \sigma(\mathcal{T})$ and thus $\mathcal{F}^* = \sigma(\mathcal{T})$.

Remark. Suppose that $(X_t, 0 \leq t \leq 1)$ and $(Y_t, 0 \leq t \leq 1)$ are right continuous, integrable processes with increasing paths, and $X_0 = Y_0 = 0$.

These processes induce two nonnegative measures on the product space Ω^* defined by

$$\lambda_X((s,t] \times G) = \int_\Omega (X_t - X_s) 1_G dP,$$
$$\lambda_Y((s,t] \times G) = \int_\Omega (Y_t - Y_s) 1_G dP,$$

for every $0 \le s < t \le 1$, $G \in \mathcal{F}$.

Assume that $\lambda_X = \lambda_Y$ on $\mathcal{T}$, i.e. $(X_t - Y_t, 0 \le t \le 1)$ is an S-martingale. Since $\mathcal{T}$ generates the product σ-field $\mathcal{F}^*$, it follows from Corollary 2.3 that $\lambda_X = \lambda_Y$ on $\mathcal{F}^*$. Therefore for all $0 < t \le 1$ and for all $G \in \mathcal{F}$ we have

$$\int_G X_t dP = \int_G Y_t dP.$$

It means that $X_t = Y_t$ P- a.s,

Thus for P- a.s. we have $X_t = Y_t$ for all $0 \le t \le 1$. So, by methods of measure theory we have showed the uniqueness of the Doob-Meyer decomposition for anticipating processes which was formulated and proved in [D-N-S] Theorem 3.3 by using the smooth Wiener functionals.

References

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