ACTA MATHEMATICA VIETNAMICA
Volume 21, Number 2, 1996, pp. 201–211

ON A CLASS OF SINGULAR INTEGRAL EQUATIONS WITH ROTATIONS

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ABSTRACT. The paper deals with the singular integral equation

$$a(t)\varphi(t) + b(t)[(S+M)P_\ell\varphi](t) = f(t),$$

where $a(t) = \prod_{i=1}^{m}(t-\alpha_i)^{r_i}s(t)$.

We will reduce the equation to the well-known Riemann boundary value problem and then describe all solution of the indicate problem in a closed form.

1. INTRODUCTION

Let Γ be a Liapunov curve on the complex plane C and X be either $L_p(\Gamma)$ $(1 < p < \infty)$ or $H^\mu(\Gamma)$ $(0 < \mu < 1)$. It is well-known that the singular integral operator of Cauchy's type

$$(S\varphi)(t) = \frac{1}{\pi i}\int_\Gamma \frac{\varphi(\tau)}{\tau - t}d\tau$$

has the property $S^2 = I$, where I is the identity operator (see [7]). The Noetherian theory of singular integral equations of Cauchy's type

$$a\varphi + bS\varphi + K\varphi = f \tag{1}$$

was started with works of Noether and Carleman in 1921, and then it was developed by many others (see [5], [6] and references there). A reason that this theory attracts a lot of attention is that there is a relation between Riemann boundary value problems of analytic functions and equations of the form (1). The operator S such as the simple-layer and double-layer potentials in Neumann's and Dirichlet's problems plays an important role in theory of boundary value problems. In [2], N. V. Mau considered the operator

Received July 8, 1995; inrevised form December 9, 1995.
This work was supported in part by the National Basic Research Program in Natural Sciences, Vietnam.

$$(S_{n,k}\varphi)(t) = \frac{1}{\pi i}\int\limits_{\Gamma} \frac{\tau^{n-k-1}t^k}{\tau^n - t^n}\varphi(\tau)d\tau, \tag{2}$$

where n, k are nonegative integers, $0 \le k \le n-1$ and proved that if $n \ne 1$ then $S^3_{n,k} = S_{n,k}$ (note that $S_{1,0} = S$). Moreover, he also investigated equations of the form

$$a(t)\varphi(t) + b(t)(S_{n,k}\varphi)(t) + (N_{n,k}\varphi)(t) = f(t) \tag{3}$$

under the assumption $a(t) \ne 0$ on Γ. Like the singular integral equations of Cauchy's type, $(a\varphi + bS_{n,k}\varphi)$ is called a singular part and $N_{n,k}\varphi$ is called a regular part of equation (3). In this paper, we study equation (3) in the case when the function $a(t)$ has isolated zero-points on Γ. Our method is to reduce equation (3) to a system of singular integral equations of Cauchy's type whose matrix is diagonal. Theorem 4.2 indicates that the solvability of (3) depends on the solvability of the ℓ-th equation in the reduced system. Remark 4.2 at the end of this paper shows that complete equations induced by $S_{n,k}$ may be reduced to an equation of the form (3).

2. Two Lemmas on multiplicative operator and projections

Let $\Gamma = \{t : |t| = 1\}$ and $X = H^\mu(\Gamma)$ be the Hölder space on Γ. Consider the following operator in X (see [1])

$$(S\varphi)(t) = \frac{1}{\pi i}\int\limits_{\Gamma} \frac{\varphi(\tau)}{\tau - t}d\tau, \tag{4}$$

$$(W\varphi)(t) = \varphi(\varepsilon_1 t), \tag{5}$$

$$(M\varphi)(t) = \frac{1}{\pi i}\int\limits_{\Gamma} m(\tau,t)\varphi(\tau)d\tau, \tag{6}$$

where $\varepsilon_1 = \exp(2\pi i/n)$ (n is a positive integer) and $m(\tau,t)$ is a given function satisfying the Hölder's condition in $(\tau,t) \in \Gamma \times \Gamma$.

Denote by $P_1, P_2, \ldots, P_n$ the projections induced by the operator W. In the sequel we shall need the following identities (see [1], [2])

$$
(7) \qquad \begin{cases} P_i = \frac{1}{n}\sum\limits_{j=1}^{n} \varepsilon_i^{n-1-j} W^{j+1} \\ P_i P_j = \delta_{ij} P_j, \quad i,j = 1,2,\dots,n \\ W^k = \sum\limits_{j=1}^{n} \varepsilon_j^k P_j, \quad k = 0,1,2,\dots,n \\ SP_j = P_j S = S_{n,j}, \quad j = 1,2,\dots,n, \end{cases}
$$

where δ_{ij} is the Kronecker's symbol. For every $a \in X$ we write $(K_a\varphi)(t) = a(t)\varphi(t)$.

Lemma 2.1. *Suppose that $a \in X$ is fixed. Then for every pair (k,j), $k,j \in \{1,2,\dots,n\}$, there exists an element $b \in X$ such that $P_k K_a P_j = K_b P_j$ (such a function $b = b(t)$ will be denoted by $a_{kj}(t)$).*

Proof. From (5)-(7) we get

$$
\begin{aligned}
P_k K_a P_j &= \frac{1}{n}\sum_{\nu=1}^{n} \varepsilon_k^{n-1-\nu} W^{\nu+1} K_a P_j \\
&= \frac{1}{n}\sum_{\nu=1}^{n} \varepsilon_k^{n-1-\nu} a(\varepsilon_{\nu+1} t) W^{\nu+1} P_j \\
&= \frac{1}{n}\sum_{\nu=1}^{n} \varepsilon_k^{n-1-\nu} a(\varepsilon_{\nu+1} t) \sum_{\mu=1}^{n} \varepsilon_\mu^{\nu+1} P_\mu P_j \\
&= \Big(\frac{1}{n}\sum_{\nu=1}^{n} \varepsilon_{\nu+1}^{j-k} a(\varepsilon_{\nu+1} t)\Big) P_j = a_{kj}(t) P_j,
\end{aligned}
$$

where

$$
(8) \qquad a_{kj}(t) = \frac{1}{n}\sum_{\nu=1}^{n} \varepsilon_{\nu+1}^{j-k} a(\varepsilon_{\nu+1} t).
$$

Putting $a_{kj}(t) = b(t)$, we obtain $b \in X$ and $P_k K_a P_j = K_b P_j$.

Lemma 2.2. *Let $a \in X$ be fixed. Then for every pair (k,j), $k,j \in \{1,2,\dots,n\}$, the above identity yields*

$$
P_k K_{a_{kj}} = K_{a_{kj}} P_j,
$$

where $a_{kj}(t)$ is defined by (8).

Proof. For any $\varphi \in X$, we have

$$
\begin{aligned}
(P_k K_{a_{kj}}\varphi)(t) = P_k a_{kj}(t)\varphi(t) &= \frac{1}{n}\sum_{\nu=1}^{n} \varepsilon_k^{n-1-\nu} W^{\nu+1} a_{kj}(t)\varphi(t)\\
&= \Big(\frac{1}{n}\sum_{\nu=1}^{n} \varepsilon_k^{n-1-\nu} W^{\nu+1}\Big)\Big(\frac{1}{n}\sum_{\mu=1}^{n} \varepsilon_{\mu+1}^{j-k} a(\varepsilon_{\mu+1}t)\Big)\varphi(t)\\
&= \frac{1}{n}\sum_{\nu=1}^{n}\Big[\frac{1}{n}\sum_{\mu=1}^{n} \varepsilon_{\mu+1}^{j-k}\varepsilon_{\nu+1}^{j-k} a(\varepsilon_{\mu+1}\varepsilon_{\nu+1}t)\Big]\varepsilon_{\nu+1}^{k-j}\varepsilon_k^{n-1-\nu}W^{\nu+1}\varphi(t)\\
&= \frac{1}{n}\sum_{\nu=1}^{n} a_{kj}(t)\varepsilon_{\nu+1}^{k-j}\varepsilon_k^{n-1-\nu}W^{\nu+1}\varphi(t)\\
&= a_{kj}(t)\Big[\frac{1}{n}\sum_{\nu=1}^{n}\varepsilon_j^{n-1-\nu}W^{\nu+1}\Big]\varphi(t)\\
&= a_{kj}(t)(P_j\varphi)(t)\\
&= (K_{a_{kj}}P_j\varphi)(t).
\end{aligned}
$$

Thus $P_k K_{a_{kj}} \equiv K_{a_{kj}} P_j$. The proof is complete.

3. Reducing equation (3) to a system of singular integral equations

Now we consider the following equation in X

$$a(t)\varphi(t) + b(t)[(S+M)P_\ell\varphi](t) = f(t), \tag{9}$$

where a, b, $f \in X$ are given and S, M, P_ℓ $(1 \le \ell \le n)$ are the operators defined by (4), (5), (6), (7). Suppose that the function $a(t)$ has isolated zero-points on Γ, i.e.

$$a(t) = \prod_{i=1}^{m}(t-\alpha_i)^{r_i}s(t),$$

where $\alpha_i \in \Gamma$, r_i, $i = 1, 2, \ldots, m$, are positive integers and $s(t)$ is a non-vanishing function on Γ $^{(*)}$. Without loss of generality we may assume that $s(t) = 1$.

$^{(*)}$ The idea of this assumption was posed by Prof. Ng. V. Mau

Lemma 3.1. *Suppose that the function $m(\tau,t)$ satisfies the condition $m(\tau,t)=m(\varepsilon_1\tau,t)=m(\tau,\varepsilon_1 t)$. Then $\varphi\in X$ is a solution of (9) if and only if $\{\varphi_k=P_k\varphi,\ k=1,2,\dots,n\}$ is a solution of the following system*

$$a^*(t)\varphi_k(t)+b^*_{k\ell}(t)(S+M)\varphi_\ell(t)=f^*_k(t),\quad k=1,2,\dots,n, \tag{10}$$

where

$$\begin{aligned} a^*(t)&=\prod_{j=1}^{n}a(\varepsilon_{j+1}t)\\ b^*_{k\ell}(t)&=\frac{1}{n}\sum_{j=1}^{n}\varepsilon_{j+1}^{\ell-k}b(\varepsilon_{j+1}t)\prod_{\substack{\mu\neq j\\ \mu=1}}^{n}a(\varepsilon_{\mu+1}t)\\ f^*_k(t)&=\frac{1}{n}\sum_{j=1}^{n}\varepsilon_k^{n-1-j}f(\varepsilon_{j+1}t)\prod_{\substack{\mu\neq j\\ \mu=1}}^{n}a(\varepsilon_{\mu+1}t). \end{aligned} \tag{11}$$

Proof. Since $\varphi\in X$ is the solution of (9), it follows

$$\begin{aligned} \prod_{\mu=1}^{n}a(\varepsilon_{\mu+1}t)\varphi(t)+b(t)\prod_{\substack{\mu=1\\ \mu\neq(n-1)}}^{n}a(\varepsilon_{\mu+1}t)(S+M)P_\ell\varphi=\\ =f(t)\prod_{\substack{\mu=1\\ \mu\neq(n-1)}}^{n}a(\varepsilon_{\mu+1}t). \end{aligned} \tag{12}$$

It is clear that $K_{a^*}P_k=P_kK_{a^*}$, ($a^*(t)$ is defined by (11)). Applying the projections P_k, $k=1,2,\dots,n$ to both sides of (12) and using Lemma 2.1 we obtain

$$\begin{aligned} a^*(t)P_k\varphi+\Big[\frac{1}{n}\sum_{j=1}^{n}\varepsilon_{j+1}^{\ell-k}b(\varepsilon_{j+1}t)\prod_{\substack{\mu=1\\ \mu\neq(n-1)}}^{n}a(\varepsilon_{\mu+1}\varepsilon_{j+1}t)\Big](S+M)P_\ell\varphi\\ =\frac{1}{n}\sum_{j=1}^{n}\varepsilon_k^{n-1-j}f(\varepsilon_{j+1}t)\prod_{\substack{\mu=1\\ \mu\neq(n-1)}}^{n}a(\varepsilon_{\mu+1}\varepsilon_{j+1}t). \end{aligned} \tag{13}$$

It is easy to see that

$$\prod_{\substack{\mu=1\\ \mu\neq(n-1)}}^{n}a(\varepsilon_{\mu+1}\varepsilon_{j+1}t)\equiv\prod_{\substack{\mu=1\\ \mu\neq j}}^{n}a(\varepsilon_{\mu+1}t)\quad\text{for any } j\in\{1,2,\dots,n\}.$$

Hence, (13) is equivalent the following system

$$a^*(t)P_k\varphi + b^*_{k\ell}(t)(S+M)P_\ell\varphi = f^*_k(t), \quad k = 1, 2, \ldots, n.$$

Thus $(P_1\varphi, P_2\varphi, \ldots, P_n\varphi)$ is a solution of (10).

Conversely, suppose that there exists $\varphi \in X$ such that

$$a^*(t)P_k\varphi + b^*_{k\ell}(t)(S+M)P_\ell\varphi = f^*_k(t), \quad k = 1, 2, \ldots, n.$$

Summing by k $(k = 1, 2, \ldots, n)$ we obtain

$$a^*(t)\varphi(t) + \sum_{k=1}^{n} b^*_{k\ell}(t)(S+M)P_\ell\varphi = \sum_{k=1}^{n} f^*_k(t). \tag{14}$$

From (11) we get

$$\begin{aligned}
\sum_{k=1}^{n} b^*_{k\ell}(t) &= \sum_{k=1}^{n} \frac{1}{n} \sum_{j=1}^{n} \varepsilon_{j+1}^{\ell-k} b(\varepsilon_{j+1}t) \prod_{\substack{\mu\neq j\\ \mu=1}}^{n} a(\varepsilon_{\mu+1}t)\\
&= \sum_{j=1}^{n} \Big[\frac{1}{n}\sum_{k=1}^{n} \varepsilon_{j+1}^{\ell-k}\Big] b(\varepsilon_{j+1}t) \prod_{\substack{\mu\neq j\\ \mu=1}}^{n} a(\varepsilon_{\mu+1}t)\\
&= \sum_{j=1}^{n} \delta_{j(n-1)} b(\varepsilon_{j+1}t) \prod_{\substack{\mu\neq j\\ \mu=1}}^{n} a(\varepsilon_{\mu+1}t) \qquad (15)\\
&= b(t) \prod_{\substack{\mu=1\\ \mu\neq(n-1)}}^{n} a(\varepsilon_{\mu+1}t).
\end{aligned}$$

Similarly,

$$\sum_{k=1}^{n} f^*_k(t) = f(t) \prod_{\substack{\mu=1\\ \mu\neq(n-1)}}^{n} a(\varepsilon_{\mu+1}t). \tag{16}$$

The identities (14), (15), (16) together give

$$\begin{aligned}
\prod_{\mu=1}^{n} a(\varepsilon_{\mu+1}t)\varphi(t) + b(t) \prod_{\substack{\mu=1\\ \mu\neq(n-1)}}^{n} a(\varepsilon_{\mu+1}t)(S+M)P_\ell\varphi =\\
= f(t) \prod_{\substack{\mu=1\\ \mu\neq(n-1)}}^{n} a(\varepsilon_{\mu+1}t).
\end{aligned}$$

This implies

$$a(t)\varphi(t) + b(t)(S+M)P_\ell\varphi = f(t).$$

The proof is complete.

Lemma 3.2. *If* $(\varphi_1, \varphi_2, \dots, \varphi_n)$ *is a solution of the system* (10) *then* $(P_1\varphi_1, P_2\varphi_2, \dots, P_n\varphi_n)$ *is also its solution.*

Proof. Suppose that $(\varphi_1, \varphi_2, \dots, \varphi_n)$ is a solution of (10). Applying the projections P_k to both sides of the equations (10) we have

$$a^*(t)P_k\varphi_k + P_k b^*_{k\ell}(t)(S+M)\varphi_\ell(t) = P_k f^*_k(t). \tag{17}$$

The following identities hold

$$\begin{aligned} P_k b^*_{k\ell}(t) &= \frac{1}{n}\sum_{i=1}^n \varepsilon_k^{n-1-i} W^{i+1} \frac{1}{n}\sum_{j=1}^n \varepsilon_{j+1}^{\ell-k} b(\varepsilon_{j+1}t) \prod_{\substack{\mu\neq j\\ \mu=1}}^n a(\varepsilon_{\mu+1}t) \\ &= \frac{1}{n}\sum_{i=1}^n \Big[\frac{1}{n}\sum_{j=1}^n \varepsilon_{j+1}^{\ell-k}\varepsilon_{i+1}^{\ell-k} b(\varepsilon_{j+1}\varepsilon_{i+1}t)\times \\ &\quad\times \prod_{\substack{\mu\neq j\\ \mu=1}}^n a(\varepsilon_{\mu+1}\varepsilon_{i+1}t)\Big]\varepsilon_k^{n-1-i}\varepsilon_{i+1}^{k-\ell} W^{i+1} \\ &= \frac{1}{n}\sum_{i=1}^n b^*_{k\ell}(t)\varepsilon_\ell^{n-1-i} W^{i+1} = b^*_{k\ell}(t)P_\ell, \end{aligned} \tag{18}$$

$$\begin{aligned} P_k f^*_k(t) &= \frac{1}{n}\sum_{i=1}^n \varepsilon_k^{n-1-i} W^{i+1} \frac{1}{n}\sum_{j=1}^n \varepsilon_k^{n-1-j} f(\varepsilon_{j+1}t) \prod_{\substack{\mu\neq j\\ \mu=1}}^n a(\varepsilon_{\mu+1}t) \\ &= \frac{1}{n}\sum_{i=1}^n \varepsilon_k^n \Big[\frac{1}{n}\sum_{j=1}^n \varepsilon_k^{n-(j+i+2)} f(\varepsilon_{j+i+2}t) \prod_{\substack{\mu\neq j\\ \mu=1}}^n a(\varepsilon_{j+i+2}t) \\ &= \frac{1}{n}\sum_{i=1}^n f^*_k(t) = f^*_k(t). \end{aligned} \tag{19}$$

Substituting (18), (19) in (17) we obtain

$$a^*(t)P_k\varphi_k + b^*_{k\ell}(t)(S+M)P_\ell\varphi_\ell = f^*_k(t), \quad k = 1, 2, \dots, n.$$

Therefore $(P_1\varphi_1, P_2\varphi_2, \dots, P_n\varphi_n)$ is a solution of (10). The lemma is proved.

4. The solvability of equation (3)

Theorem 4.1. *Suppose that the function $m(\tau, t)$ satisfies the assumption of Lemma 3.1*

(i) *If $(\varphi_1, \varphi_2, \dots, \varphi_n)$ is a solution of* (10)*, then*

$$\varphi = \sum_{i=1}^{n} P_i \varphi_i$$

is a solution of equation (9)*.*

(ii) *If $\varphi \in X$ is a solution of* (9) *then $(P_1\varphi_1, P_2\varphi_2, \dots, P_n\varphi_n)$ is a solution of* (10)*.*

Proof. (i) By Lemma 3.2, $(P_1\varphi_1, P_2\varphi_2, \dots, P_n\varphi_n)$ is also a solution of (10). We put

$$\varphi = \sum_{i=1}^{n} P_i \varphi_i.$$

It is clear that $P_k\varphi = P_k\varphi_k$. It means that $(P_1\varphi, P_2\varphi, \dots, P_n\varphi)$ is a solution of (10). Hence, we have

$$a^*(t)P_k\varphi + b^*_{k\ell}(t)(S+M)P_\ell\varphi = f^*_k(t), \quad k = 1, 2, \dots, n.$$

Summing by k of both sides of the last equations and using (15), (16) we obtain

$$a^*(t)\varphi(t) + b(t) \prod_{\substack{\mu=1 \\ \mu \neq (n-1)}}^{n} a(\varepsilon_{\mu+1}t)(S+M)P_\ell\varphi = f(t) \prod_{\substack{\mu=1 \\ \mu \neq (n-1)}}^{n} a(\varepsilon_{\mu+1}t).$$

Thus

$$a(t)\varphi(t) + b(t)(S+M)P_\ell\varphi = f(t).$$

(ii) The conclusion follows immediately from Lemma 3.1. The proof is complete.

Remark 4.1. Theorem 4.1 shows that it is sufficient to solve the system (10) in the given space $X = H^\mu(\Gamma)$.

We set

$$\Omega = \{\varepsilon^{-1}_{\mu+1}\alpha_i, \quad \mu = 1, 2, \dots, n;\ i = 1, 2, \dots, m\}$$

$$\{g(t)\}_{(k,t_0)} = \frac{d^k g(t)}{dt^k}\Big|_{t=t_0}.$$

Theorem 4.2. *Suppose that the function $m(\tau, t)$ satisfies the assumption of Lemma 3.1. Then the equation* (9) *has a solution in X if and only if the equation*

$$a^*(t)\varphi(t) + b^*_{\ell\ell}(t)(S+M)\varphi = f^*_\ell(t) \tag{20}$$

has a solution $\varphi_0(t)$ satisfying the following conditions

$$\{f^*_k(t) - b^*_{k\ell}(t)(S+M)\varphi_0(t)\}_{(k,t_i)} = 0, \tag{21}$$

where $t_i \in \Omega$, $k = 0, 1, 2, \ldots, r_i$, $i = 1, 2, \ldots, m$.

Proof. Since $\varphi \in X$ is a solution of (9), it follows from Lemma 2.1 that the system (10) has a solution $(P_1\varphi, P_2\varphi, \ldots, P_n\varphi)$. It means that $P_\ell\varphi$ is the solution of the ℓ-th equation of (10)

$$a^*(t)P_\ell\varphi(t) + b^*_{\ell\ell}(t)(S+M)P_\ell\varphi = f^*_\ell(t),$$

i.e. the equation (20) has solutions. Let $\varphi_0(t)$ be a certain solution of (20). Since (10) has solutions it follows that for every $k = 1, 2, \ldots, n$, $\varphi_k(t)$ is a solution of the equation

$$a^*(t)\varphi_k(t) = f^*_k(t) - b^*_{k\ell}(t)(S+M)\varphi_0(t). \tag{22}$$

The left side of (22) is a function having zero of order r_i at $t_i = \varepsilon^{-1}_{\mu+1}\alpha_i \in \Omega$. Hence, the condition (21) is necessary. Conversely, if $\varphi_0(t)$ is a finite solution of (20) and (21) holds, then it is easy to see that (10) has a solution $(\varphi_1, \varphi_2, \ldots, \varphi_n)$, where $\varphi_i \in X$. Theorem 4.1 follows that

$$\varphi = \sum_{i=1}^{n} P_i\varphi_i$$

is the solution of (9). The proof is complete.

We set

$$D^+ = \{z \in C : |z| < 1\}$$
$$D^- = \{z \in C : |z| > 1\}.$$

Denote by $H^+(D^+)$, $H^-(D^-)$ the sets of all analytic functions in D^+, D^- respectively.

Corollary. *Suppose that the function*

$$M(\tau,t)=\frac{b^*_{\ell\ell}(t)}{a^*(t)+b^*_{\ell\ell}(t)}m(\tau,t)$$

admits an analytic continuation on D^+ with each variable (τ,t) and satisfies $M(\tau,t)=M(\varepsilon_1\tau,t)=M(\tau,\varepsilon_1 t)$. Then the equation (9) is solvable in a closed form.

Proof. Consider the ℓ-th equation of (10)

$$a^*(t)\varphi_\ell(t)+b^*_{\ell\ell}(t)(S\varphi_\ell)(t)+b^*_{\ell\ell}(t)(M\varphi_\ell)(t)=f^*_\ell(t). \tag{23}$$

Put

$$\Phi_\ell(z)=\frac{1}{\pi i}\int\limits_\Gamma \frac{\varphi_\ell(\tau)}{\tau-z}d\tau,\quad z\in C\setminus\Gamma.$$

According to Sokhotski-Plemelij formula (see [5]), equation (23) is reduced to the following boundary value problem

$$\Phi^+_\ell(t)-\frac{b^*_{\ell\ell}(t)}{a^*(t)+b^*_{\ell\ell}(t)}M\Phi^-_\ell(t)=\frac{a^*(t)-b^*_{\ell\ell}(t)}{a^*(t)+b^*_{\ell\ell}(t)}\Phi^-_\ell(t)+\frac{f^*_\ell(t)}{a^*(t)+b^*_{\ell\ell}(t)}\cdot \tag{24}$$

From Lemma 2 in [5] (p.186) and the assumption for $M(\tau,t)$, it follows that (24) is the Riemann boundary value problem for analytic functions. Denote by $(\Psi^+_\ell(z),\Psi^-_\ell(z))$ a solution of (24). We have

$$\Phi^-_\ell(t)=\Psi^-_\ell(t)$$

and

$$\Phi^+_\ell(t)-\frac{b^*_{\ell\ell}(t)}{a^*(t)+b^*_{\ell\ell}(t)}(M\Phi^-_\ell)(t)=\Psi^+_\ell(t).$$

Hence, using Sokhotski-Plemelij formula, the solution of (24) is of the form

$$\varphi_\ell(t)=\Phi^+_\ell(t)-\Phi^-_\ell(t)=\Psi^+_\ell+\frac{b^*_{\ell\ell}(t)}{a^*(t)+b^*_{\ell\ell}(t)}(M\Psi^-_\ell)(t)-\Psi^-_\ell(t). \tag{25}$$

Thus, from Theorems 4.1 and 4.2 we conclude that

(i) If neither equation (24) has solutions nor solutions $\varphi_\ell(t)$ of the form (25) do satisfy condition (21), then equation (9) has no solutions.

(ii) If there exists $\varphi_\ell(t)$ of the form (25) satisfying conditions (21), then equation (9) is solvable in a closed form. Solutions of (9) are given by the following formula

$$\varphi(t) = \sum_{k=1}^{n} (P_k \varphi_k)(t),$$

where $\varphi_\ell(t)$ is defined by (25) and $\varphi_k(t)$, $1 \leq k \neq \ell \leq n$, are defined clearly from system (10).

Remark 4.2. The complete singular integral equation induced by $S_{n,j}$ is of the form

$$a(t)\varphi(t) + b(t)(S_{n,j}\varphi)(t) + c(t)(S_{n,j}^2\varphi)(t) = f(t). \tag{26}$$

From (7) it follows $S_{n,j}^2 = P_{n,j}$. Thus (26) is of the form (9).

Acknowledgment

The author is greatly indebted to Professor Nguyen Van Mau for his suggestions and various valuable advices. The author also wishes to express his deep gratitude to Professor Nguyen Thuy Thanh for his encouragement and attention to this work.

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