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SPLINE COLLOCATION METHODS FOR A SYSTEM OF NONLINEAR FREDHOLM-VOLTERRA INTEGRAL EQUATIONS

NGUYEN MINH CHUONG AND NGUYEN VAN TUAN

Abstract. In this paper we shall apply spline collocation methods to solve approximately a system of nonlinear Fredholm-Volterra integral equations. We shall study existence and uniqueness conditions for collocation solution, and estimate the convergence rate. Some applications are shown.

1. Introduction

Let us consider the following system

$$x(t) = \int_a^b \overset{(1)}{K}(t,s,x(s))ds + \int_a^t \overset{(2)}{K}(t,s,x(s))ds + f(t), \tag{1}$$

where $a \le t,\ s \le b$,

$$x(t) = [x_1(t), \dots, x_m(t)],$$
$$f(t) = [f_1(t), \dots, f_m(t)],$$
$$\overset{(j)}{K}(t,s,x(s)) = [\overset{(j)}{K}_1(t,s,x(s)), \dots, \overset{(j)}{K}_m(t,s,x(s))],$$

$x_i(t)$, $f_i(t) \in C[a,b]$, $\overset{(j)}{K}(t,s,x)$ are continuous in all variables $i = 1, \dots, m$, $j = 1, 2$.

Set $C = C[a,b] \times \dots \times C[a,b]$ (m times), where the norm of C is defined as follows:

$$\|x\|_c = \max_{1 \le i \le m} \max_{a \le t \le b} |x(t)|,$$

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or

$$\|x(t)\|_c = \max \|x_i(t)\|, \quad 1 \le i \le m$$

where

$$\|x_i(t)\| = \max |x_i(t)|, \quad a \le t \le b.$$

It is obvious that C with this norm is a Banach space. Denoting by T the operator defined by

$$T : C \longrightarrow C$$

$$x \mapsto Tx = \int_a^b \overset{(1)}{K}(t, s, x(s))ds + \int_a^t \overset{(2)}{K}(t, s, x(s))ds + f(t),$$

we can rewrite the system (1) in the form

$$Tx = x, \quad x \in C. \tag{2}$$

Let

$$\pi_n : a = t_0 < t_1 < \ldots < t_n = b,$$

be a partition of $[a, b]$. Set

$$h_i = t_{i+1} - t_i, i = 0, \ldots, n-1,$$
$$h_n = \max_{0 \le i \le n-1} h_i.$$

We always suppose that the sequence $\{\pi_n\}$ has the following property $\lim\limits_{n\to\infty} h_n = 0$. We shall use the set

$$S_n = \big\{\zeta_i | \zeta_i \in [a, b],\ i = 1, \ldots, N_n,\ N_n \in \mathbf{N}\big\},$$

for a collocation set in $[a, b]$, where N_n is a constant dependent of n. We put $X_n = Sp(\pi_n, p, q)$, where

$$Sp(\pi_n, p, q) = \big\{v(t) \in C^q[a, b] : v(t)\big|_{[t_i, t_{i+1}]} \in Q_p\big\},$$

$i = 0, \ldots, n-1$, p, q are integers satisfying $0 \le q \le p-1, p \ge 1$, Q_p is a set of polynomials of degree $\le p$.

Let $X = X_n \times \ldots \times X_n$ (m times) and

$$P_n : C \longrightarrow X \subset C$$

$$x = (x_1(t), \ldots, x_m(t)) \mapsto P_n x = (v_1(t), \ldots, v_m(t)) \in X, \tag{3}$$

be a continuous linear projections from C to X converging pointwise to the identity operator P in C and

$$P_n x(\zeta_i) = x(\zeta_i), \quad \forall x \in C, \ \forall \zeta_i \in S_n. \tag{4}$$

We shall approximate the solution of equation (1) by an element

$$v = (v_1(t), \dots, v_m(t)) \in X,$$

satisfying the system

$$v_i(\zeta_j) = \int_a^b \overset{(1)}{K}_i(\zeta_j, s, v(s))ds + \int_a^{\zeta_j} \overset{(2)}{K}_i(\zeta_j, s, v(s))ds + f_i(\zeta_j),$$

$i = 1, \dots, m$, $j = 1, \dots, N_n$. This system may be writen in the form

$$v(\zeta_j) = Tv(\zeta_j), \quad \zeta_j \in S_n. \tag{5}$$

From (4) we get

$$v(\zeta_j) = P_n Tv(\zeta_j), \quad \zeta_j \in S_n.$$

If we can find an element $v \in X$ such that

$$v = P_n Tv, \tag{6}$$

then it satisfies (5) and hence it is the desired collocation solution.

2. Main result

Theorem 1. *Let P_n be the projections defined by* (3). *Assume that the equation* (2) *has a solution*

$$x^{(0)}(t) = [x_1^{(0)}(t), \dots, x_m^{(0)}(t)],$$

$x_i^{(0)}(t) \in C^{l_i}[a, b]$, $0 \le \ell_i \le p$, $i = 1, \dots, m$, $\overset{(j)}{K}(t, s, x)$ *and* $\dfrac{\partial \overset{(j)}{K}(t, s, x(s))}{\partial x}$, $j = 1, 2$, *are continuous in the domain*

$$\begin{cases} a \le t, \quad s \le b \\ \|x - x^{(0)}\|_c \le \sigma_1, \quad \sigma_1 \in \mathbf{R}. \end{cases} \tag{7}$$

Moreover, suppose that the equation

$$u(t) - \int_a^b \frac{\partial \overset{(1)}{K}(t,s,x^{(0)}(s))}{\partial x} u(s)ds - \int_a^t \frac{\partial \overset{(2)}{K}(t,s,x^{(0)}(s))}{\partial x} u(s)ds = 0,$$

where

$$u(t) = [u_1(t), \dots, u_m(t)], \quad u_i(t) \in C[a,b],$$

$$\frac{\partial \overset{(j)}{K}(t,s,x^{(0)}(s))}{\partial x} u(s) = \Big[\sum_{i=1}^m \frac{\partial \overset{(j)}{K}_1(t,s,x^{(0)}(s))}{\partial x_i} u_i, \dots, \sum_{i=1}^m \frac{\partial \overset{(j)}{K}_m(t,s,x^{(0)}(s))}{\partial x_i} u_i\Big],$$

has only the trivial solution. Then

(i) *There exists a constant $r > 0$ such that $x^{(0)}$ is the unique solution of* (2) *in the ball*

$$\|x - x^{(0)}\|_c \le r.$$

(ii) *For sufficiently large n, the collocation equation* (6) *has a unique solution*

$$v_n(t) = [v_{n,1}(t), \dots, v_{n,m}(t)], \quad v_{n,i}(t) \in X_n, \ i = 1, \dots, m,$$

in the above ball.

(iii) *The sequence $\{v_n(t)\}_{n=1}^\infty$ converges to the solution $x^{(0)}(t)$, and the convergence rate is estimated by*

$$\|v_n - x^{(0)}\|_c \le M E^0(x^{(0)}),$$

where $E^0(x^{(0)}) = \max\limits_{1\le i\le m} E_i(x_i^{(0)})$, $E_i(x_i^{(0)}) = \inf\limits_{z_i \in X_n} \|x_i^{(0)} - z_i\| = O(h_n^{\ell_i} \omega({x_i^{(0)}}^{(\ell_i)}, h_n))$, $\omega({x_i^{(0)}}^{(\ell_i)}, h_n)$ denotes the modulus of continuity of the ℓ_i-th derivative of $x_i^{(0)}$ with respect to h_n, and M is a positive constant independent of n.

Proof. By the continuity of $\overset{(j)}{K}(t,s,x)$ in the domain (7) and $f(t) \in C[a,b]$, it is clear that T is continuous in C_1, where

$$C_1 = \{x(t) \in C : \ \|x - x^{(0)}\|_c \le \sigma_1\}.$$

We shall prove that T is completely continuous. Consider the set

$$A_i = \Big\{ y_i(t) = \int_a^b \overset{(1)}{K}_i(t,s,x(s))ds + \int_a^t \overset{(2)}{K}_i(t,s,x(s))ds + f_i(t) : x(t) \in C_1 \Big\},$$

$i = 1, \dots, m$. It is obvious that y_i are uniformly bounded. By the uniform continuity of $\overset{(j)}{K}_i(t,s,x)$ in the domain (7) and of $f_i(t)$ on $[a,b]$ it is not difficult to show that y_i are equicontinuous. Consequently, A_i is precompact in $C[a,b]$, $i = 1, \dots, m$. So $T(C_1)$ is precompact in C. The operator T is thus completely continuous in C_1. Further, by using the continuity of $\overset{(j)}{K}(t,s,x)$ and $\dfrac{\partial \overset{(j)}{K}(t,s,x)}{\partial x}$, $j = 1, 2$, we see that T is continuously differentiable at $x^{(0)}$ (in the Frechet sence) and

$$T'(x^{(0)})u = \int_a^b \frac{\partial \overset{(1)}{K}(t,s,x^{(0)}(s))}{\partial x} u(s)ds + \int_a^t \frac{\partial \overset{(2)}{K}(t,s,x^{(0)}(s))}{\partial x} u(s)ds.$$

By our assumption the equation

$$u - T'(x^{(0)})u = 0, \quad u \in C$$

has only trivial solution. Now by the use of Theorem 3.1 from [10] we can get the statements (i), (ii) in Theorem 1.

On the other hand, using again Theorem 3.1 we have

$$\|v_n - x^{(0)}\|_c \le M_1 \|(P_n - P)Tx^{(0)}\|_c, \quad n \ge N_1, \ N_1 \in \mathbf{N},$$

(M_1 is a constant independent of n). Consequently,

$$\|v_n - x^{(0)}\|_c \le M_1 \|P_n x^{(0)} - x^{(0)}\|_c.$$

Since $\{P_n\}$ is a sequence of continuous linear projections pointwise converging to P, C is a Banach space, and by Banach-Steinhaus Theorem, the sequence $\{\|P_n\|\}$ is bounded. That is, there exists a positive number M_2 such that $\|P_n\| \le M_2$, $\forall n$. For all $z \in X$, we get $P_n z = z$, hence

$$\begin{aligned} \|v_n - x^{(0)}\|_c &\le M_1 \big[\|x^{(0)} - z\|_c + \|P_n(z - x^{(0)})\|_c\big], \quad n \ge N_1 \\ &\le M_1(1 + \|P_n\|)\|x^{(0)} - z\|_c, \end{aligned}$$

or

$$\|v_n - x^{(0)}\|_c \le M\|x^{(0)} - z\|_c, \quad \forall z \in X, \tag{8}$$

where $M = M_1(1 + M_2)$. From (8) we get

$$\begin{aligned}\|v_n - x^{(0)}\|_c &\le M \inf_{z=(z_1,\dots,z_m)\in X} \|x^{(0)} - z\|_c \\ &\le M \inf_{(z_1,\dots,z_m)\in X} \max_{1\le i\le m} \|x_i^{(0)} - z_i\|.\end{aligned}$$

It is not hard to prove the following equality

$$\inf_{(z_1,\dots,z_m)\in X} \max_{1\le i\le m} \|x_i^{(0)} - z_i\| = \max_{1\le i\le m} \inf_{z_i\in X_n} \|x_i^{(0)} - z_i\|,$$

which implies

$$\|v_n - x^{(0)}\|_c \le M \max_{1\le i\le m} \inf_{z_i\in X_n} \|x_i^{(0)} - z_i\|.$$

Now by Theorem 1 from [9] we obtain

$$\|v_n - x^{(0)}\|_c \le ME^0(x^{(0)}), \tag{9}$$

where $E^0(x^{(0)}) = \max_{1\le i\le m} E_i(x_i^{(0)})$, $E_i(x_i^{(0)}) = \inf_{z_i\in X_n} \|x_i^{(0)} - z_i\| = O(h_n^{l_i}\omega({x_i^{(0)}}^{(l_i)}, h_n))$.

Theorem 1 is now proved.

3. Applications

Below we shall consider some applications.

We choose $d+1$ points $0 = \eta_0 < \eta_1 < \ldots < \eta_d = 1$ from the segment $[0, 1]$. Taking into account the partition

$$\pi_n : a = t_0 < t_1 < \ldots < t_n = b,$$

we denote

$$S_n = \{\zeta_{ij} = t_i + h_i\eta_j, i = 0, \dots, n-1,\ j = 0, \dots, d\}, \tag{10}$$

$$X_n = Sp(\pi_n, d, 0), \quad X = X_n \times \ldots \times X_n (m \text{ times}).$$

Define the mappings

$$p_n : C[a,b] \longrightarrow Sp(\pi_n, d, 0),$$

$$f \mapsto p_n f \in Sp(\pi_n, d, 0),$$

such that

$$p_n f(\zeta_{ij}) = f(\zeta_{ij}), \quad \forall \zeta_{ij} \in S_n.$$

Obviously, p_n is a linear projection from $C[a,b]$ into $Sp(\pi_n, d, 0)$. On each segment $[t_i, t_{i+1}]$, $i = 0, \dots, n-1$, $p_n f$ is a polynomial of order $\le d$ interpolating the function f at ζ_{ij}. Let

$$\ell_j(t) = \prod_{k=0, k\neq j}^{d} \Big(\frac{t - \zeta_{ik}}{\zeta_{ij} - \zeta_{ik}} \Big).$$

From (10) we get

$$\ell_j(t) = \prod_{k=0, k\neq j}^{d} \Big(\frac{\eta - \eta_k}{\eta_j - \eta_k} \Big) = \ell_j^*(\eta),$$

where

$$\eta = \frac{t - t_i}{t_{i+1} - t_i}.$$

Consequently

$$M_3 = \max_{t_i \le t \le t_{i+1}} \sum_{j=0}^{d} |\ell_j(t)| = \max_{0 \le \eta \le 1} \sum_{j=0}^{d} |\ell_j^*(\eta)|.$$

So M_3 is the constant independent of i and n. It is the Lebesgue constant (see [8], p.5). By the continuity of $\ell_j(t)$ in $[t_i, t_{i+1}]$ it is not difficult to show that p_n is continuous. Let p^* be the best approximation of f by a polynomial of order d in $[t_i, t_{i+1}]$ (see [6], p.43), we have

$$|p^*(t) - p_n f(t)| \le M_3 \|f - p^*\|.$$

By Jackson Theorem (see [6], p.43) we obtain

$$\max_{t_i \le t \le t_{i+1}} |p^*(t) - p_n f(t)| \le M_3 g_k(f, d), f \in C^k[a,b],$$

where

$$g_k(f,d) = \begin{cases} 6\omega\Big(f, \dfrac{h_n}{2d}\Big), & \text{when } k = 0 \\ \dfrac{3h_n}{d}\|f'\|, & \text{when } k = 1 \\ \dfrac{6^k(k-1)^{k-1}}{(k-1)!d^k} k h_n^k \|f^{(k)}\|, & \text{when } k > 1,\ d > k-1 \geq 1. \end{cases}$$

Hence

$$\|f - p_n f\| \leq (M_3 + 1) g_k(f,d).$$

When $k = 0$ we have

$$\|f - p_n f\| \leq 6(M_3 + 1)\omega\Big(f, \frac{h_n}{2d}\Big), \quad \forall f \in C[a,b].$$

We get $\omega\Big(f, \dfrac{h_n}{2d}\Big) \to 0$ as $h_n \to 0$, where $f \in C[a,b]$ and M_3 independs on n. Hence it is easy to see that $\{p_n\}$ is a sequence of continuous projections pointwise converging to the identity operator in $C[a,b]$.

Define the mappings

$$\hat{P}_n : C \longrightarrow X \subset C \tag{11}$$

$$(x_1(t), \dots, x_m(t)) \mapsto (p_n x_1(t), \dots, p_n x_m(t)).$$

It is clear that $\hat{P}_n$ are continuous linear projections from C to X, satisfying conditions (4) and converge pointwise to the identity operator P in C and

$$\|\hat{P}_n\| \leq M_3.$$

Theorem 2. *Let $\hat{P}_n$ be the projections defined by* (11)*. Assume that the conditions of as in Theorem 1 is satisfies. Then*

(i) *There exists a constant $r > 0$ such that $x^{(0)}$ is the unique solution of* (2) *in the ball*

$$\|x - x^{(0)}\|_c \leq r.$$

(ii) *For sufficiently large n the collocation equation* (6) *has a unique solution*

$$v_n(t) = [v_{n,1}(t), \dots, v_{n,m}(t)], \quad v_{n,i}(t) \in X_n, \quad i = 1, \dots, m,$$

in the above ball.

(iii) *The sequence* $\{v_n(t)\}_{n=1}^{\infty}$ *converges to the solution* $x^{(0)}(t)$, *and we get the estimate*

$$\|v_n - x^{(0)}\|_c \leq ME^0(x^{(0)}), \tag{12}$$

where $E^0(x^{(0)}) = \max_{1\leq i\leq m} E_i(x_i^{(0)})$, $E_i(x_i^{(0)}) = \inf_{z_i\in X_n} \|x_i^{(0)} - z_i\|$, $E_i(x_i^{(0)}) \leq 3g_{l_i}(x_i^{(0)}, d)$.

Proof. By using Theorem 1 we immediately get (i) and (ii). For the proof of (iii) we see from (9) that

$$\|v_n - x^{(0)}\|_c \leq ME^0(x^{(0)}),$$

where $E^0(x^{(0)}) = \max_{1\leq i\leq m} E_i(x_i^{(0)})$, $E_i(x_i^{(0)}) = \inf_{z_i\in X_n} \|x_i^{(0)} - z_i\|$. By using a Lemma from [8], p.6 and by Jackson Theorem ([6], p. 43) we have

$$\inf_{z_i\in X_n} \|x_i^{(0)} - z_i\| \leq 3g_{\ell_i}(x_i^{(0)}, d).$$

Corollary 1. *Let* $\hat{P}_n$ *be the projections defined by* (11), $f(t)$ *and* $x(t)$ *be elements of* C. *Let* $\overset{(1)}{K}(t,s,x)$ *be continuous in the domain*

$$\begin{cases} a \leq t, \ s \leq b \\ \ ||x||_c < \infty, \end{cases}$$

and $\overset{(1)}{K}$ *satisfy the Lipschitz condition in* x:

$$\|\overset{(1)}{K}(t,s,x_1) - \overset{(1)}{K}(t,s,x_2)\|_c \leq L\|x_1 - x_2\|_c, \ \forall x_1, x_2 \in C, \ \|x_i\|_c < \infty, \tag{13}$$

$i = 1, 2$ *with* $0 < L(b-a) < 1$, $\overset{(2)}{K}(t,s,x) = 0$. *Then*

(i) *The equation* (2) *has a unique solution* $x^{(0)}(t)$.

If we assume addtionally that $\dfrac{\partial \overset{(1)}{K}(t,s,x(s))}{\partial x}$ *is continuous in the domain* (7), *then*

(ii) *There exists a constant* $r > 0$ *such that for sufficiently large* n*, the collocation equation* (6) *has a unique approximate solution* $v_n \in X$ *in the ball* $\|v_n - x^{(0)}\|_c < r$*.*

(iii) *The sequence* v_n *converges to the solution* $x^{(0)}(t)$ *in* C *and the estimate* (12) *holds.*

Proof. Obviously, T is a contracting operator in C, so it has a unique fixed point $x^{(0)}(t)$. On the other hand, by the continuity of $\overset{(1)}{K}(t,s,x)$ and $\dfrac{\partial \overset{(1)}{K}(t,s,x)}{\partial x}$ in the domain (7), we see as in the proof of Theorem 1 that T is completely continuous in C_1 and T is continuously differentiable at $x^{(0)}$ and

$$T'(x^{(0)})u = \int_a^b \frac{\partial \overset{(1)}{K}(t,s,x^{(0)})}{\partial x} u(s)ds.$$

Now we consider the equation

$$T'(x^{(0)})u = u, \quad u \in C,$$

or

$$\int_a^b \frac{\partial \overset{(1)}{K}(t,s,x^{(0)})}{\partial x} u(s)ds = u(t). \tag{14}$$

We shall prove that equation (14) has only trivial solution.

Indeed, if it has a solution $u(t) \neq 0$, so $\|u\|_c > 0$. From (14) it follows that

$$\int_a^b [\overset{(1)}{K}(t,s,x^{(0)}+u) - \overset{(1)}{K}(t,s,x^{(0)})]ds - o(u) = u(t),$$

where

$$\frac{\|o(u)\|_c}{\|u\|_c} \to 0 \quad \text{as} \quad \|u\|_c \to 0.$$

Consequently,

$$\begin{aligned}\|o(u)\|_c &= \Big\| u - \int_a^b [\overset{(1)}{K}(t,s,x^{(0)}+u) - \overset{(1)}{K}(t,s,x^{(0)})]ds \Big\|_c \\ &\geq \|u\|_c - \int_a^b \|\overset{(1)}{K}(t,s,x^{(0)}+u) - \overset{(1)}{K}(t,s,x^{(0)})\|_c ds \\ &\geq \|u\|_c - L(b-a)\|u\|_c.\end{aligned}$$

Hence

$$\|o(u)\|_c + L(b-a)\|u\|_c \geq \|u\|_c.$$

It follows that

$$\frac{\|o(u)\|_c}{\|u\|_c} + L(b-a) \geq 1,$$

that is, we get $L(b-a) \geq 1$ as $||u||_c \to 0$, which gives a contradiction. So equation (14) has only trivial solution. Now applying Theorem 2 we obtain the conclusions.

In the sequel we apply Theorem 2 to a general class of nonlinear Volterra integral equation.

Let us study the following equation

$$y'(t) = H\Big(t, y(t), \int_a^t K_1(t,s,y(s))ds\Big), \tag{15}$$

with condition

$$y(a) = \gamma. \tag{16}$$

Assume that $y_0(t)$ is a solution of (15), (16), $a \leq t,\ s \leq b$, $y(t) \in C[a,b]$, K_1 is continuous in the domain

$$\Omega_1 = \begin{cases} a \leq t, \quad s \leq b \\ \|y - y_0\| \leq \sigma_2, \quad 0 < \sigma_2 \in \mathbf{R}. \end{cases}$$

and $H(t,y,z)$ is continuous in Ω_2, where

$$\Omega_2 = \begin{cases} a \leq t,\ s \leq b, \quad \|y - y_0\| \leq \sigma_2, \\ \|z - z_0\| \leq \sigma_3,\ z_0 = \int_a^t K_1(t,s,y_0(s))ds,\ \sigma_3 \in \mathbf{R}. \end{cases}$$

Setting $z(t) = \int\limits_a^t K_1(t,s,y(s))ds$ we can write (15), (16) in the form

$$\left\{\begin{aligned} & y(t) = \gamma + \int\limits_a^t H(s,y(s),z(s))ds \\ & z(t) = \int\limits_a^t K_1(t,s,y(s))ds. \end{aligned}\right. \tag{17}$$

Let $x(t) = [y(t), z(t)]$, $\overset{(2)}{K}(t,s,x(s)) = [H(s,y(s),z(s)),\ K_1(t,s,y(s))]$, $f(t) = [\gamma, 0]$, $C = C[a,b] \times C[a,b]$, $X = X_n \times X_n$, $X_n = Sp(\pi_n, d, 0)$. The system (17) leads to the form (2) with $\overset{(1)}{K}(t,s,x) = 0$. Hence we get $x^{(0)}(t) = [y_0(t), z_0(t)]$ as a solution of (17).

Corollary 2. *Let $\hat{P}_n$ be the projections defined by* (11). *We assume that $y_0(t)$ is a solution of the problem (15), (16), with $y_0(t) \in C^{l_1}[a,b]$, $z_0(t) \in C^{l_2}[a,b]$, $0 \le \ell_1, \ell_2 \le d$ and $\overset{(2)}{K}(t,s,x)$, $\dfrac{\partial \overset{(2)}{K}(t,s,x(s))}{\partial x}$ are continuous in the domain*

$$\left\{\begin{aligned} & a \le t,\ s \le b \\ & U = \|x - x^{(0)}\|_c \le \sigma_4,\ \sigma_4 = \min\{\sigma_2, \sigma_3\} \end{aligned}\right.$$

$$\|K_1(t,s,y_1) - K_1(t,s,y_2)\| \le L^* \|y_1 - y_2\|, \quad \forall y_1, y_2 \in U.$$

where L^* are positive constants, $L^*(b-a) < 1$. Moreover, suppose that the following equation

$$u(t) - \int\limits_a^t \frac{\partial \overset{(2)}{K}(t,s,x^{(0)})}{\partial x} u(s)ds = 0,$$

with $u(t) = [u_1(t), u_2(t)] \in C$, has only trivial solution. Then

(i) *there exists a constant $r > 0$ such that $y_0(t)$ is the unique solution of* (15), (16) *in the ball*

$$\|y - y_0\| \le r.$$

(ii) *For sufficiently large n, the collocation equation* (6) *has a unique solution*

$$v_n(t) = [v_{n,1}(t), v_{n,2}(t)], \quad v_{n,i}(t) \in X_n, \quad i = 1, 2,$$

in the ball $\|x - x^{(0)}\|_c \le r$, $x \in C$.

(iii) *The sequence* $\{v_n(t)\}_{n=1}^{\infty}$ *converges to the solution* $x^{(0)}(t)$ *and we have the estimate*

$$\|v_n - x^{(0)}\|_c \le M \max\{E_1(y_0), E_2(z_0)\},$$

where $E_1(y_0) \le 3g_{l_1}(y_0, d)$, $E_2(z_0) \le 3g_{l_2}(z_0, d)$.

Proof. It is easy to see that all assumptions of Theorem 2 are satisfied. It remains only to prove that if $x^{(0)}$ is the unique solution of (17) in the ball $\|x - x^{(0)}\|_c \le r$, then y_0 is the unique solution of (15), (16) in the ball $\|y - y_0\| \le r$.

Indeed, let y_1 be another solution of (15), (16) in the ball $\|y - y_0\| \le r$. Setting $z_1(t) = \int_a^t K_1(t, s, y_1(s))ds$, we see that $x^{(1)} = [y_1(t), z_1(t)]$ is a solution of (17). We have

$$\|z_1 - z_0\| = \Big\| \int_a^t \big[K_1(t, s, y_1(s)) - K_1(t, s, y_0(s))\big] ds \Big\|$$
$$\le L^*(b - a)\|y_1 - y_0\| \le r.$$

So $x^{(1)}$ is also a solution of (17) in the ball $\|x - x^{(0)}\|_c \le r$, which gives a contradiction.

Let π_n be a uniform partition of $[a, b]$:

$$\pi_n : a = t_0 < t_1 < \ldots < t_n = b, \quad h = \frac{b - a}{n}\cdot$$

Let $k \ge 1$ be a natural number and $n \ge 2k - 1$, and $S_n = \{t_0, \ldots, t_n\}$. Consider now the mappings

$$q_n : C[a, b] \longrightarrow X_n = Sp(\pi_n, 2k - 1, 2k - 2)$$
$$f \mapsto q_n f,$$

with

$$q_n f(t_i) = f(t_i), \quad i = 0, \ldots, n,$$
$$D^j(q_n f)(a) = D^j(L_{2k-1,0} f)(a), \quad j = 1, \ldots, k - 1,$$
$$D^j(q_n f)(b) = D^j(L_{2k-1,1} f)(b), \quad j = 1, \ldots, k - 1,$$

where $L_{2k-1,0}f$, $(L_{2k-1,1}f)$ are Lagrange interpolation polynomials of function f at points $t_0, t_1, \dots, t_{2k-1}, (t_{n-2k+1}, t_{n-2k}, \dots, t_n)$, respectively.

It is obvious that q_n are continuous linear projections from $C[a,b]$ to $Sp(\pi_n, 2k-1, 2k-2)$ and converge pointwise to the identity operator in $C[a,b]$ (see [9], p.347).

Define the mappings

$$\underset{=}{P}_n : C \longrightarrow X \subset C \tag{18}$$

$$(x_1(t), \dots, x_m(t)) \mapsto (q_n x_1(t), \dots, q_n x_m(t)).$$

It is clear that $\underset{=}{P}_n$ are continuous linear projections from C to X and converge pointwise to the identity operator P in C and $\underset{=}{P}_n x(t_i) = x(t_i)$, $\forall x \in C$, $\forall t_i \in S_n$. Then from Theorem 1 we immediately get the following consequence.

Theorem 3. *Let* $\underset{=}{P}_n$ *be defined by* (18)*. Let the assumptions of Theorem 1 with* $0 \le l_i \le 2k-1$, $i = 1, \dots, m$, *be satisfied. Then*

(i) *there exists a constant* $r > 0$ *such that* $x^{(0)}$ *is the unique solution of* (2) *in the ball* $\|x - x^{(0)}\|_c \le r$.

(ii) *For sufficiently large* n *the collocation equation* (6) *has a unique solution*

$$v_n(t) = [v_{n,1}(t), \dots, v_{n,m}(t)],$$
$$v_{n,i}(t) \in Sp(\pi_n, 2k-1, 2k-2), \quad i = 1, \dots, m,$$

in the above ball.

(iii) *The sequence* $\{v_n(t)\}_{n=1}^{\infty}$ *converges to the solution* $x^{(0)}(t)$, *and the following estimate holds*

$$\|v_n - x^{(0)}\|_c \le ME^0(x^{(0)}),$$

where

$$E^0(x^{(0)}) = \max_{1 \le i \le m} E_i(x_i^{(0)}),$$

$$E_i(x_i^{(0)}) = \inf_{z_i \in Sp(\pi_n, 2k-1, 2k-2)} \|x_i^{(0)} - z_i\| = O(h^{l_i} \omega(x_i^{(0)\,(l_i)}, h)).$$

References

1. H. Brunner, *Nonpolynomial spline collocation for Volterra equations with weakly singular kernels*, SIAM J. Numer. Anal. **20** (1983), 1106-1119.
2. ———, *The approximate solution of Volterra equations with nonsmooth solutions*, Utilitas Math. **27** (1985), 57-95.
3. ———, *High-order methods for the numerical solution of Volterra integro-differential equations*, J. Comp.& Appl. Math. **15** (1986), 301-309.
4. ———, *On the numerical solution of nonlinear Volterra-Fredholm integral equations by collocation methods*, SIAM J. Numer. Anal. **27** (1990), 987-1000.
5. M. L. Crasnov, A. W. Kisilev and G. I. Makarenko, *Integral Equations*, M., Nauka, 1968.
6. P. M. Prenter, *Splines and variational methods*, Wiley-Interscience, New York, 1975.
7. N. I. Ronto and T. V. Putiatina, *Application of collocation method to multipoints boundary value problems with integral boundary conditions*, Ukrai. Mat. J. **44** (1992), 1548-1555.
8. R. D. Russell and L. F. Shampine, *A collocation method for boundary value problems*, Numer. Math. **19** (1972), 1-28.
9. Thomas r. Lucas and George W. Reddien, *Some collocation methods for nonlinear boundary value problems*, SIAM J. Numer. Anal. **9** (1972), 341-356.
10. G. Vainiko, *On the convergence of the collocation method for nonlinear differential equations*, J. Vysch. Mat. i Mat. Phys. **6** (1966), 35-42.

Institute of Mathematics, Hanoi, Vietnam