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ON THE POINTWISE BEHAVIOR OF SOME LACUNARY WAVELET SERIES

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Abstract. We show that the pointwise behavior of certain lacunary wavelet series at a neighborhood of a singularity, under certain conditions on the coefficients, cannot be worse than that of the logarithm's at zero.

1. The main result

Wavelets have been found to be efficient tools in studying local regularity properties of functions, an aspect where the methods in the traditional Fourier analysis have its shortcomings. For example in [2], Jaffard has shown how local Hölder continuity of a function is reflected in the size of its coefficients corresponding to the wavelets localized at the point in question. In [3,p.116], we find a necessary condition for differentiability at a given point in terms of the decay of the periodic wavelet coefficients.

In this paper, we pursue the idea that one may detect singularities of a function by locating those points at which the corresponding wavelet coefficients are exceptionally large. The focus of our investigation will be wavelet series of the form

$$F(x) = \sum_{j=1}^{\infty} c_j \psi(2^{n_j} x - k_j) \tag{1}$$

where $\{k_j\}_{j>0}$ is a sequence of integers and $\{n_j\}_{j>0}$ is an increasing sequence of positive integers relatively dense in $\mathbf{N}$ in the sense that for some positive integer M,

$$\{\ell+1, \dots, \ell+M\} \cap \{n_1, n_2, \dots\} \neq \phi, \tag{2}$$

for any integer $\ell \geq 0$. Assuming that for some real number x_0, the sequence

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$$\theta_j = 2^{n_j} x_0 - k_j, \qquad j = 1, 2, 3, \dots \tag{3}$$

converges, it may likely happen that the series in (1) will be divergent at $x = x_0$ if the coefficients c_j's do not tend to zero fast enough. Our main result describes the behavior of $F(x_0 + \delta)$ as δ tends to zero.

In what follows, we shall assume that ψ is a real-valued function, differentiable everywhere on the real line, and with a bounded derivative. We also want ψ to have sufficiently fast decay at infinity:

$$|\psi(x)| \leq C\big(1 + |x|^N\big)^{-1} \tag{4}$$

for all real x, where C and N are positive constants.

Before proceeding to the statement of our main result, we observe that there can only be at most one real number x_0 for which the corresponding sequence $\{\theta_j\}$ will be bounded. If it happens that there is such a real number x_0, then the only point of discontinuity that F may possibly have is at $x = x_0$ itself, provided the coefficients $\{c_j\}$ form a bounded sequence.

Indeed, given $\epsilon > 0$, choose a positive integer j_0 such that

$$|\theta_j| \leq 2^{n_j - 1}\varepsilon, \quad j \geq j_0.$$

Starting from the identity

$$2^{n_j}x - k_j = \theta_j + 2^{n_j}(x - x_0),$$

one finds that

$$|\phi(2^{n_j}x - k_j)| \leq C\ \left(2\varepsilon^{-1}\right)^N \left(2^{-N}\right)^{n_j}$$

for $|x - x_0| > \varepsilon$ and $j \geq j_0$. These inequalities and the boundedness of the sequence $\{c_j\}$ guarantee the continuity of F on $(-\infty, x_0 + \epsilon] \cup [x_0 + \varepsilon, \infty)$.

Theorem 1. *Suppose that for some real number x_0, the sequence $\{\theta_j\}$ defined in* (3) *is convergent. Let $\{c_j\}_{j>0}$ be a convergent sequence of complex numbers. Given $0 < \delta_0 < 1$, there exists a constant K_1 with $0 \leq K_1 < \infty$, such that the function F defined in* (1) *satisfies*

$$|F(x_0 + \delta)| \leq K_1 \log\left(|\delta|^{-1}\right) \tag{5}$$

whenever $0 < |\delta| < \delta_0$. Here, K_1 is independent of δ.

In particular, if the sequence $\{j\, n_j^{-1}\}$ is convergent, then

$$\lim_{\delta\to 0} \frac{F(x_0 + \delta)}{\log\left(|\delta|^{-1}\right)} = \lambda_0\ \psi(\theta^*) \tag{6}$$

where θ^* is the limit of the convergent sequence defined in (3) *and*

$$\lambda_0 = (\log 2)^{-1}\, c^* \lim_{j\to\infty} j\, n_j^{-1},$$

c^* being the limit of the sequence $\{c_j\}$.

2. Proof of theorem 1

Given a positive integer r and a real number α, we may write

$$F(x_0 + \alpha 2^{-n_r}) = S_r(\alpha) + R_r(\alpha) + T_r(\alpha) + r\, c^*\, \psi(\theta^*) \tag{7}$$

where

$$S_r(\alpha) = \sum_{j=1}^{r} c_j(\psi(\theta_j + \alpha 2^{n_j - n_r}) - \psi(\theta^*)),$$

$$R_r(\alpha) = \sum_{j=r+1}^{\infty} c_j \phi(\theta_j + \alpha 2^{n_j - nr})$$

$$T_r(\alpha) = \psi(\theta^*) \sum_{j=1}^{r} (c_j - c_*).$$

First of all , we shall show that $R_r(\alpha)$ is uniformly bounded for $|\alpha| > 1$ and $r \in \mathbf{N}$. To this end, choose $j_0 \in \mathbf{N}$ such that $|\theta_j| \le 2^{j_0 - 1}$, whenever $j \ge 1$. It follows that

$$|\theta_j| \le 2^{j-r-1} \le 2^{n_j - n_r - 1}$$

whenever $j - j_0 \ge r \ge 1$, so that for these values of j and r,

$$|\theta_j + \alpha 2^{n_j - n_r}| \ge 2^{n_j - n_r - 1},$$

where we have used the condition $|\alpha| > 1$. Consequently, whenever $r \in \mathbf{N}$ and $|\alpha| > 1$,

$$\begin{aligned} \sum_{j > r + j_0} \left(1 + |\theta_j + \alpha 2^{n_j - n_r}|^N\right)^{-1} &\le 2^N \sum_{j > r + j_0} \left(2^{n_r - n_j}\right)^N \\ &\le \left(1 - 2^{-N}\right)^{-1}. \end{aligned} \tag{8}$$

Now, we split the summation defining $R_r(\alpha)$ into two: one sum ranging over all j with $r+1 \le j \le r+j_0$ and the other ranging over all j with $j > r+j_0$. From (8) and (4), it follows that

$$|R_r(\alpha)| \le Bj_0\|\psi\| + BC\big(1-2^{-N}\big)^{-1} := C_1$$

where $B = \sup\{|c_j| : j > 0\}$ and $\|\psi\|$ denotes the maximum of $|\psi(x)|$ as x ranges over $\mathbf{R}$.

So given any positive integer r and any real number α satisfying $|\alpha| > 1$,

$$(9) \qquad |R_r(\alpha)| \ \le \ C_1,$$

where the constant C_1 is independent of r and α.

Now, let us obtain an estimate for $S_r(\alpha)$ under the condition that $|\alpha| \le 2^M$, where M was defined in (2). An application of the mean value theorem yields

$$|S_r(\alpha)| \le B\colon \|\psi'\|\colon \sum_{j=1}^{r} |\theta_j - \theta^*| + B\colon \|\psi'\| 2^{M+1},$$

whenever $r \in \mathbf{N}$ and $|\alpha| \le 2^M$.

Combining this last inequality with (9) and (7) leads to the following estimate, valid for any positive integer r and any real number α satisfying $1 < |\alpha| \le 2^M$:

$$(10) \qquad |F(x_0 + \alpha 2^{-n_r}) - r\, c^*\, \psi(\theta^*)| \ \le \ C_2 \cdot \sigma_r$$

where the constant C_2 is independent of α and r, and

$$\sigma_r \ = \ 1 \ + \sum_{j=1}^{r} |c_j - c^*| + \sum_{j=1}^{r} |\theta_j - \theta^*|.$$

Finally, given $0 < |\delta| < 1$, the relative denseness of $\{n_j\}$ in $\mathbf{N}$ guarantees the existence of a positive integer $r = r(\delta)$ such that

$$1 < 2^{n_r}|\delta| \le 2^M.$$

Taking $\alpha = 2^{n_r}\delta$, inequality (10) may be written

$$(11) \qquad |F(x_0 + \delta) \ - \ A(\delta)\log\left(|\delta|^{-1}\right)| \le B(\delta)\log\left(|\delta|^{-1}\right)$$

where

$$A(\delta) = \frac{c^* r(\delta)\psi(\theta^*)}{\log(|\delta|^{-1})}, \quad B(\delta) = \frac{C_2 \sigma_{r(\delta)}}{\log(|\delta|^{-1})}.$$

Observe that $r(\delta)$ tends to infinity as δ tends to zero, and moreover,

$$\lim_{\delta\to 0} \frac{n_r}{\log(|\delta|^{-1})} = (\log 2)^{-1}, \quad \lim_{r\to\infty} r^{-1}\sigma_r = 0.$$

In order to establish (5), we first observe that $\lim_{\delta\to 0} B(\delta) = 0$ while given $0 < \delta_0 < 1$, $\sup\{|A(\delta)| : 0 < |\delta| < \delta_0\} < \infty$. It then follows from (11) that (5) holds with $K_1 = \sup\{|A(\delta)| + |B(\delta)| : 0 < |\delta| < \delta_0\}$. Furthermore, if the sequence $\{jn_j^{-1}\}$ converges, then

$$\lim_{\delta\to 0} A(\delta) = (\log 2)^{-1} c^* \, \psi(\theta^*) \lim_{j\to\infty} jn_j^{-1},$$

from which follows equation (6). □

3. An example

In the following corollary, we construct some lacunary wavelet series with a prescribed singularity at a rational point x_0. In this situation, there are integers j_0, j_1, and p with $j_1 \geq j_0$ and $p > 0$, such that the binary expansion

$$x_0 = \sum_{k=j_0}^{\infty} a_k 2^{-k}$$

satisfies $a_j = a_{j+p}$ whenever $j \geq j_1$. Here $a_j = 0$ or 1 for all j.

Corollary. *Let* $\{\ell_j\}_{j\geq j_1}$ *be the sequence defined by*

$$\ell_j = 2^j \sum_{k=j_0}^{j} a_k 2^{-k}.$$

Given a convergent sequence $\{c_j\}_{j\geq j_1}$ *of complex numbers, define*

$$F(x) = \sum_{j=j_1}^{\infty} c_j \psi(2^j x - l_j).$$

Then

$$\lim_{\delta\to 0}\frac{F(x_0+\delta)}{\log(|\delta|^{-1})} \;=\; \lambda_0\sum_{s=0}^{p-1}\psi(\theta_s^*) \tag{12}$$

where

$$\lambda_0=(p\log 2)^{-1}\lim_{j\to\infty}c_j,\quad \theta_s^* \;=\; \sum_{k\geq 1}2^{-k}a_{k+j_1+s}. \tag{13}$$

Proof of Corollary. First of all, we decompose F as follows:

$$F(x)=\sum_{s=0}^{p-1}F_s(x) \tag{14}$$

where

$$F_s(x)=\sum_{j\in N_s}c_j\psi(2^jx-\ell_j)$$

and $N_s=\{j\geq j_1 : j\equiv j_1+s \pmod p\}$. With a fixed $s\in\{0,1,\cdots,p-1\}$, we shall apply Theorem 1 to $F_s(x)$. By the "periodicity" of the sequence $\{a_j\}_{j\geq j_1}$, it follows that

$$2^jx_0-\ell_j=\theta_s^*,\quad j\in N_s.$$

With the fact that $\lim_{q\to\infty}q(j_1+s+qp)^{-1}=p^{-1}$ an application of Theorem 1 yields

$$\lim_{\delta\to 0}\frac{F_s(x_0+\delta)}{\log(|\delta|^{-1})} \;=\; \lambda_0\psi(\theta_s^*),$$

where λ_0 is defined in (13). From this and the decomposition of $F(x)$ given in (14), we obtain (12). $\square$

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