ACTA MATHEMATICA VIETNAMICA
Volume 21, Number 1, 1996, pp. 135–145

REGULARIZATION BY LINEAR OPERATORS

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ABSTRACT. The purpose of this paper is to investigate convergence rates for an operator version of Tikhonov regularization for ill-posed problems involving monotone operators. The obtained results are presented in combination with finite-dimensional approximations for the space. An iterative process for regularized solution is studied for illustration.

1. INTRODUCTION

Let X be a real reflexive Banach space and X^* be the dual space of X. For the sake of simplicity, the norms of X and X^* will be denoted by one symbol $\|.\|$. We write $\langle x^*, x\rangle$ instead of $x^*(x)$ for $x^* \in X^*$ and $x \in X$. Let A be a continuous and bounded operator with domain of definition $D(A) = X$ and range $R(A) \subseteq X^*$.

We are interested in solving the ill-posed problem

$$A(x) = f, \quad f \in R(A). \tag{1.1}$$

By this we mean that the solutions of (1.1) do not depend continuously on the data (A, f). To solve it we have to use stable methods. A widely used and effective method is Tikhonov regularization that consists of minimizing some functional $F^\alpha_{h\delta}(x)$ depending on a parameter $\alpha > 0$, where

$$F^\alpha_{h\delta}(x) = \|A_h(x) - f_\delta\|^2 + 2\alpha\Omega(x), \tag{1.2}$$

(A_h, f_δ) are approximations for (A, f) such that

$$\|A_h(x) - A(x)\| \leq h\|x\|, \quad \forall\, x \in X,\ \|f_\delta - f\| \leq \delta,$$

Received May 23, 1995; in revised form January 21, 1996.
1991 *Mathematics Subject Classification.* 47H17.
Key words and phrases. Monotone operators, regularization, Fréchet differentiable, iterative method.
Supported by the National Program for Fundamental Researches in Natural Sciences.

with wellknown levels $(\delta, h) \to 0$ and $\Omega(x)$ is some functional on X. The variational method of regularization (1.2) is studied intensively, for arbitrarily linear operator A in [8], [10], [13], [15], and, in particular, recently for the nonlinear operator A (see [5]-[7], [9], [16]-[18], [20], [21]). If A and A_h are monotone, a regularized solution can be constructed by a solution of the operator equation

$$A_h(x) + \alpha U^s(x) = f_\delta, \tag{1.3}$$

where U^s is the dual mapping of X satisfying the condition

$$\langle U^s(x), x\rangle = \|x\|^s, \quad \|U^s(x)\| = \|x\|^{s-1}, \quad s \geq 2.$$

It is indicated in [2, 20] that Eq. (1.3) has a unique solution, henceforth denoted $x_\alpha^{h\delta}$, and if $h/\alpha,\ \delta/\alpha \to 0$, as $\alpha \to 0$, then the sequence $\{x_\alpha^{h\delta}\}$ converges to a solution x_0 of (1.1):

$$\|x_0\| = \min_{x \in S_0} \|x\|,$$

where S_0 denotes the set of all solutions of (1.1) ($S_0 \neq \emptyset$). Moreover, the solution $x_\alpha^{h\delta}$ can be approximated by a solution of the finite-dimensional problem

$$A_h^n(x) + \alpha U^{sn}(x) = f_\delta^n, \tag{1.4}$$

where $A_h^n = P_n^* A_h P_n$, $U^{sn}(x) = P_n^* U^s P_n(x)$, and $f_\delta^n = P_n^* f_\delta$, P_n denotes a (linear) projection from X onto its subspace X_n, P_n^* is the adjoint of P_n,

$$X_n \subset X_{n+1},\ \forall n, \quad P_n x \to x,\ \forall\, x \in X\ (\|P_n\| \leq c,\ \text{a constant}).$$

For each $\alpha > 0$, Eq. (1.4) has a unique solution $x_{\alpha n}^{h\delta}$ and the sequence $\{x_{\alpha n}^{h\delta}\}$ converges to $x_\alpha^{h\delta}$, as $n \to +\infty$ (see [19]).

The solutions $x_\alpha^{h\delta}$ of (1.3) and $x_{\alpha n}^{h\delta}$ of (1.4) can be found by iterative methods because of uniformly monotone property of U^s and U^{sn} when X is uniformly convex (see [23]), but under very complex conditions on parameter choice (see [1], [3]). In [6] the author studied another approach to solve (1.1) by considering the equation

$$A_h(x) + \alpha Bx = f_\delta, \tag{1.5}$$

where B is a linear and selfadjoint operator, i.e. $B^* = B$, such that

$$\langle Bx, x\rangle \geq m_B\|x\|^2,\ \forall x \in D(B),\ m_B > 0,\ S_0 \subset D(B), \quad \overline{D(B)} = X.$$

Eq. (1.5), for every $\alpha > 0$, has a unique solution $x^\alpha_{h\delta}$, if $h/\alpha,\ \delta/\alpha \to 0$, as $\alpha \to 0$, the sequence $\{x^\alpha_{h\delta}\}$ converges to $x_1 \in S_0$:

$$\langle Bx_1, x - x_1\rangle \geq 0, \quad \forall x \in S_0. \tag{1.6}$$

And the solution $x^\alpha_{h\delta}$ can be approximated by solution of the finite-dimensional problem

$$A^n_h(x) + \alpha B_n x \ = \ f^n_\delta \tag{1.7}$$

under the conditions

$$X_n \subset D(B), \quad B_n x = P^*_n B P_n x \to Bx, \quad \forall x \in D(B).$$

Evidently, the last requirement is equivalent to $BP_nB^{-1}y \to y,\ \ \forall y \in R(B)$ which is proposed and studied in [11]. In many cases, we can use differential operator like B (see [12]) and thus the smoothness of solution is preserved in regularization. This is one of the advantages of B over U^s in regularization. The convergence rates of $x^{h\delta}_\alpha$ and $x^{h\delta}_{\alpha n}$ were considered in [5] and [7]. In this paper we shall answer the question on convergence rates for $\{x^\alpha_{h\delta}\}$, when $x^{\alpha n}_{h\delta} \to x_1$ and its convergence rates. After that we give an iterative method for regularized solution and an example for illustration.

Later, the symbols $\rightharpoonup$ and $\to$ denote weak convergence and convergence in norm, respectively.

In the following sections we suppose that all the above conditions are satisfied.

2. Main results

First, we prove a result about convergence rates for $\{x^\alpha_{h\delta}\}$.

Theorem 2.1. *Let the following conditions hold:*

(i) *A is Fréchet differentiable in some neighbourhood $\mathcal{U}(S_0)$ of S_0,*

(ii) *There exists a constant $L > 0$ such that*

$$\|A'(x) - A'(y)\| \ \leq \ L\|x - y\|, \quad \forall\, x \in S_0,\ y \in \mathcal{U}(S_0).$$

(iii) *There exists element* $z \in D(B)$ *such that*

$$A'^*(x_1)z = Bx_1.$$

(iv) $L\|z\| \leq 2m_B$.

Then, if α *is chosen as* $\alpha \sim (h+\delta)^\mu$, $0 < \mu < 1$, *we obtain*

$$\|x_{h\delta}^\alpha - x_1\| = O((h+\delta)^\theta), \quad \theta = \min\{1-\mu,\ \mu/2\}.$$

Proof. From (1.1), (1.5) and the monotone property of A, A_h and B it follows

$$\alpha m_B\|x_{h\delta}^\alpha - x_1\|^2 \ \leq\ \big(\delta + h\|x_{h\delta}^\alpha\|\big)\|x_{h\delta}^\alpha - x_1\| + \alpha\big\langle Bx_1,\ x_1 - x_{h\delta}^\alpha\big\rangle.$$

By virtue of the last inequality and condition (iii) of the theorem we can write

$$\text{(2.1)}\quad \alpha m_B\|x_{h\delta}^\alpha - x_1\|^2 \leq \big(\delta + h\|x_{h\delta}^\alpha\|\big)\|x_{h\delta}^\alpha - x_1\| + \alpha\big\langle z,\ A'(x_1)(x_1 - x_{h\delta}^\alpha)\big\rangle.$$

Using the Taylor's expression of [22] we have

$$A'(x_1)(x_1 - x_{h\delta}^\alpha) = A(x_1) - A(x_{h\delta}^\alpha) + r_{h\delta}^\alpha$$

with

$$\|r_{h\delta}^\alpha\| \ \leq\ \frac{L}{2}\|x_{h\delta}^\alpha - x_1\|^2.$$

As

$$\begin{aligned}\big\langle z,\ A(x_1) - A(x_{h\delta}^\alpha)\big\rangle &= \big\langle z,\ f - f_\delta + A_h(x_{h\delta}^\alpha) - A(x_{h\delta}^\alpha) + f_\delta - A_h(x_{h\delta}^\alpha)\big\rangle \\ &\leq\ \|z\|\big(\delta + h\|x_{h\delta}^\alpha\|\big) + \big\langle z,\ Bx_{h\delta}^\alpha\big\rangle,\end{aligned}$$

(2.1) implies

$$\begin{aligned}&\alpha\Big(m_B - \frac{L}{2}\|z\|\Big)\|x_\alpha^{h\delta} - x_1\|^2 \\ &\quad\leq\ \big(\delta + h\|x_{h\delta}^\alpha\|\big)\|x_{h\delta}^\alpha - x_1\|\ \alpha\|z\|\big(\delta + h\|x_{h\delta}^\alpha\|\big) + \alpha\|Bz\|\,\|x_{h\delta}^\alpha\|.\end{aligned}$$

By using the following relation of [15]:

$$a,\ b,\ c\ \geq\ 0,\ p\ >\ q,\ a^p \leq ba^q + c \Longrightarrow a^p = O(b^{p/(p-q)} + c)$$

we obtain

$$\|x^{\alpha}_{h\delta} - x_1\| = O\big((h+\delta)^{\theta}\big), \quad \theta = \min\big(1-\mu,\ \mu/2\big) \qquad \square$$

In the case of Hilbert spaces $X = X^* = H$, this result was obtained in [5] with $B = I$, the identity operator of H. Obviously, the theorem is valid if condition (ii) of the theorem is satisfied only for $x = x_1,\ \forall y \in \mathcal{U}(S_0)$.

Now, we shall answer the second question.

Theorem 2.2. *Assume that the following conditions hold:*

(i) *Conditions* (i) *and* (ii) *of Theorem 2.1,*

(ii) $\alpha = \alpha(h,\ \delta,\ n) \to 0$ *such that* $h/\alpha,\ \delta/\alpha,\ \alpha \to 0$ *and*

$$\big(\gamma_n(x) + L\|(I-P_n)x\|^2\big)\alpha^{-1} \to 0, \quad \forall x \in S_0,$$

as $n \to \infty$, *where* $\gamma_n(x)$ *is defined by*

$$\gamma_n(x) \ = \ \|A'(x)(I-P_n)x\|.$$

Then the sequence $\{x^{\alpha n}_{h\delta}\}$ *converges to* x_1.

Proof. From (1.1), (1.7) and the properties of A^n_h, P_n and B it follows

$$A^n_h(x^{\alpha n}_{h\delta}) - A^n_h(x_n) + \alpha B_n\big(x^{\alpha n}_{h\delta} - x_n\big) = f^n_\delta - A^n_h(x_n) - \alpha B_n x_n - f_n + P^*_n A(x),$$

$$f_n = P^*_n f, \quad x_n = P_n x, \quad x \in S_0.$$

Multiplying both sides of this equality by $x^{\alpha n}_{h\delta} - x_n$ and using the monotone property of $A^n_h,\ B$ and $P^2_n = P_n$ we get

$$\begin{aligned}(2.2)\qquad \alpha m_B\|x^{\alpha n}_{h\delta} - x_n\|^2 &\le \alpha\big\langle B(x^{\alpha n}_{h\delta} - x_n),\ x^{\alpha n}_{h\delta} - x_n\big\rangle \\ &= \alpha\big\langle P^*_n\big(B(x^{\alpha n}_{h\delta} - x_n)\big),\ x^{\alpha n}_{h\delta} - x_n\big\rangle \\ \le \big\langle P^*_n\big(f_\delta - f + A(x) - A(x_n) &+ A(x_n) - A_h(x_n)\big),\ x^{\alpha n}_{h\delta} - x_n\big\rangle \\ + \alpha\big\langle P^*_n B x_n,\ x_n - x^{\alpha n}_{h\delta}\big\rangle &\le \big(\delta + hc\|x\|\big)\|x^{\alpha n}_{h\delta} - x_n\| \\ + \big\langle A(x) - A(x_n),\ x^{\alpha n}_{h\delta} - x_n\big\rangle &+ \alpha\big\langle P^*_n B x_n,\ x_n - x^{\alpha n}_{h\delta}\big\rangle.\end{aligned}$$

On the other hand,

$$A(x_n) - A(x) = A'(x)(P_n - I)x + r_n, \quad \|r_n\| \ \le \frac{L}{2}\|(I-P_n)x\|^2.$$

Therefore, from (2.2) we get

$$\alpha m_B\|x^{\alpha n}_{h\delta}-x_n\|^2 \le \Big(\delta + hc\|x\| + \|A'(x)(I-P_n)x\| + \frac{L}{2}\|(I-P_n)x\|^2\Big)\times$$

$$\|x^{\alpha n}_{h\delta}-x_n\| + \alpha\big\langle P_n^*Bx_n,\ x_n - x^{\alpha n}_{h\delta}\big\rangle \tag{2.3}$$

with $\big\langle P_n^*Bx_n,\ x_n - x^{\alpha n}_{h\delta}\big\rangle \le \text{const.}\ \|x^{\alpha n}_{h\delta}-x_n\|$. Together with the conditions of the theorem the two last inequalities guarantee that the sequence $\{x^{\alpha n}_{h\delta}\}$ is bounded. Without loss of generality, let $x^{\alpha n}_{h\delta} \rightharpoonup x_1$, as $h,\, \delta,\, \alpha \to 0$ and $n \to +\infty$.

Now, we write the monotone property for $A^n = P_n^*AP_n$

$$\big\langle A^n(x_n) - A^n(x^{\alpha n}_{h\delta}),\ x_n - x^{h\delta}_{\alpha n}\big\rangle \ge 0, \quad \forall\, x \in X.$$

As $P_n^*P_n^* = P_n^*$, the last inequality can be written in the form

$$\big\langle A(x_n) - A^n(x^{\alpha n}_{h\delta}),\ x_n - x^{\alpha n}_{h\delta}\big\rangle \ \ge\ 0.$$

Thus,

$$\big\langle A(x_n) - f_\delta\big\rangle + \alpha\big\langle B_n x^{\alpha n}_{h\delta},\ x_n - x^{\alpha n}_{h\delta}\big\rangle + h\|x^{\alpha n}_{h\delta}\| \ge 0$$

or

$$\big\langle A(x_n) - f_\delta\big\rangle + \alpha\big\langle B_n x_n,\ x_n - x^{\alpha n}_{h\delta}\big\rangle + h\|x^{\alpha n}_{h\delta}\| \ge 0, \quad \forall x \in D(B).$$

Letting $h,\, \delta,\, \alpha \to 0$ and $n \to +\infty$ in this inequality we obtain

$$\big\langle A(x) - f,\ x - x_1\big\rangle \ \ge\ 0, \quad \forall\, x \in D(B).$$

By Minty's lemma (see [21], p. 257), $x_1 \in S_0$.

From (2.3) we also obtain $\big\langle Bx,\ x - x_1\big\rangle \ge 0,\ \ \forall x \in S_0$. Replacing x by $tx_1 + (1-t)x$ in this inequality and using the linear property of B and convex and closed property of S_0 we have $\big\langle tBx_1 + (1-t)Bx,\ x - x_1\big\rangle \ge 0$, $\forall x \in S_0, t \in (0,\, 1)$. Letting $t \to 1$ in this inequality we get $\big\langle Bx_1,\ x - x_1\big\rangle \ge 0,\ \ \forall x \in S_0$. Since the element x_1 is defined by (1.6) uniquely, the entire sequence $\{x^{\alpha n}_{h\delta}\}$ converges weakly to x_1. By putting $x_n = x_1^n = P_n x_1$ in (2.3) we can see that the sequence $\{x^{\alpha n}_{h\delta}\}$ converges strongly to x_1, as $h,\ \delta,\ \alpha \to 0$ and $n \to +\infty$.

Put

$$\beta_n = \|P_n^*BP_nx_1 - Bx_1\|.$$

We shall prove a result about convergence rates for the sequence $\{x^{\alpha n}_{h\delta}\}$.

Theorem 2.3. *Let the following conditions hold:*

(i) *Conditions* (i)-(iii) *of Theorem 2.1,*

(ii) α *is chosen as* $\alpha \sim (h+\delta+\gamma_n)^\mu + \beta_n$, *where* $\gamma_n = \|(I-P_n)x_1\|$.

Then

$$\|x_{h\delta}^{\alpha n} - x_1\| \;=\; O\Big((h+\delta+\gamma_n)^{\tilde{\mu}}) + \beta_n^{1/2}\Big),$$

where $\tilde{\mu} = \min\{1-\mu,\ \mu\}$.

Proof. Since

$$\|A_h(x_1) - A_h(x_1^n)\| \;\le\; h\|x_1\| + \delta + \|f_\delta - A_h(x_1^n)\|,$$

$$\begin{aligned}\big\langle P_n^* B x_1^n,\, x_1^n - x_{h\delta}^{\alpha n}\big\rangle > &= \big\langle P_n^* B x_1^n - Bx_1,\, x_1^n - x_{h\delta}^{\alpha n}\big\rangle + \big\langle Bx_1,\, x_1^n - x_{h\delta}^{\alpha n}\big\rangle \\ &\le\; \beta_n\|x_{h\delta}^{\alpha n} - x_1^n\| + \big\langle Bx_1,\ x_1^n - x_{h\delta}^{\alpha n}\big\rangle,\end{aligned}$$

from (2.3) we get

$$\begin{aligned}(2.4)\qquad \alpha m_A\|x_{h\delta}^{\alpha n} - x_1^n\|^2 \;\le \\ \big(\delta + hc\|x_1\| + \gamma_n + L\gamma_n^2/2 + \alpha\beta_n\big)\|x_{h\delta}^{\alpha n} - x_1^n\| + \alpha\big\langle Bx_1,\ x_1^n - x_{h\delta}^{\alpha n}\big\rangle.\end{aligned}$$

On the other hand,

$$\begin{aligned}\big\langle Bx_1,\ x_1^n - x_{h\delta}^{\alpha n}\big\rangle &= \big\langle Bx_1,\ x_1^n - x_1\big\rangle + \big\langle Bx_1,\ x_1 - x_{h\delta}^{\alpha n}\big\rangle \\ &\le\; \|Bx_1\|\gamma_n + \big\langle Bx_1,\ x_1 - x_{h\delta}^{\alpha n}\big\rangle.\end{aligned}$$

Therefore, from (2.4) we can see that

$$\begin{aligned}\alpha m_B\|x_{h\delta}^{\alpha n} - x_1^n\|^2 \le \Big(\delta + h\|x_1\| + \gamma_n + L\gamma_n^2/2 + \alpha\beta_n\Big)\|x_{h\delta}^{\alpha n} - x_1^n\| \\ + \alpha\|Bx_1\|\gamma_n + \alpha\big\langle Bx_1,\ x_1 - x_{h\delta}^{\alpha n}\big\rangle,\end{aligned}$$

and as

$$\begin{aligned}\big\langle Bx_1, x_1 - x_{h\delta}^{\alpha n}\big\rangle &= \big\langle z, A'(x_1)(x_1 - x_{h\delta}^{\alpha n})\big\rangle = \big\langle z, A(x_1) - A(x_{h\delta}^{\alpha n}) + r_{h\delta}^{\alpha n}\big\rangle \\ &= \big\langle z, f - f_\delta + f_\delta - A_h(x_{h\delta}^{\alpha n}) + A_h(x_{h\delta}^{\alpha n}) - A(x_{h\delta}^{\alpha n})\big\rangle \\ &\quad + \big\langle z, r_{h\delta}^{\alpha n}\big\rangle \le \|z\|\big(\delta + h\|x_{h\delta}^{\alpha n}\|\big) + \big\langle B^*z, x_{h\delta}^{\alpha n}\big\rangle + \big\langle z, r_{h\delta}^{\alpha n}\big\rangle \\ \|r_{h\delta}^{\alpha n}\| &\le \frac{L}{2}\|x_{h\delta}^{\alpha n} - x_1\|^2 \le \frac{L}{2}\|x_{h\delta}^{\alpha n} - x_1^n\|^2 + O(\gamma_n),\end{aligned}$$

we obtain that

$$\begin{aligned}\alpha\Big(m_B - \frac{L}{2}\|z\|\Big)\|x_{h\delta}^{\alpha n} - x_1^n\|^2 \le \\ O\big(h + \delta + \gamma_n + \alpha\beta_n\big)\|x_{h\delta}^{\alpha n} - x_1^n\| + \alpha O\big(h + \delta + \gamma_n + \alpha\big).\end{aligned}$$

Therefore

$$\|x^{\alpha n}_{h\delta} - x^n_1\| \;=\; O\big((h+\delta+\gamma_n)^{\tilde{\mu}} + \beta_n^{1/2}\big)$$

and

$$\|x^{\alpha n}_{h\delta} - x_1\| \;=\; O\big((h+\delta+\gamma_n)^{\tilde{\mu}} + \beta_n^{1/2}\big). \qquad \square.$$

3. Iterative method for regularized solution

Consider an iterative method for finding a solution of the following equation

$$F(x) \equiv Bx + A(x) = f, \tag{3.1}$$

where the operators B and A are defined as above.

Let x^1 be an arbitrary element of $D(B)$. The sequence of iterations is constructed by the formula

$$\begin{aligned} x^{n+1} &= x^n - t_n B^{-1}(F(x^n) - f)/\tau_n, \quad n = 1, 2, \dots \\ \tau_n &= \big\langle B^{-1}(F(x^n) - f),\ F(x^n) - f\big\rangle^{1/2}, \end{aligned} \tag{3.2}$$

where $\{t_n\}$ is a sequence of real numbers.

Theorem 3.1. *If the real numbers t_n satisfy the conditions*

$$t_n > 0, \quad t_n \searrow 0, \quad \sum_{n=1}^{\infty} t_n = +\infty, \quad \sum_{n=1}^{\infty} t_n^2 < +\infty,$$

then the sequence $\{x^n\}$ converges to $\tilde{x}$, the unique solution of (3.1), as $n \to +\infty$.

Proof. Put

$$\lambda_n := \ \langle B(x^n - \tilde{x}),\ x^n - \tilde{x}\rangle.$$

It is easy to see that

$$\lambda_{n+1} \le \lambda_n + 2\langle B(x^{n+1} - x^n),\ x^n - \tilde{x}\rangle + \langle B(x^{n+1} - x^n),\ x^{n+1} - x^n\rangle.$$

From this inequality and (3.2) we get

$$\lambda_{n+1} \le \lambda_n - 2t_n\lambda_n/\tau_n + t_n^2.$$

Therefore, the sequence $\{\lambda_n\}$ is bounded. Consequently, the sequences $\{x^n\}$ and $\{A(x^n)\}$ are bounded, too.

Since

$$\tau_n^2 = \big\langle B^{-1}\big(A(x^n) + Bx^n - (A(\tilde{x}) - B\tilde{x})\big),\ A(x^n) + Bx^n - \big(A(\tilde{x}) - B\tilde{x}\big)\big\rangle$$
$$\leq \frac{1}{m_B}\|A(x^n) - A(\tilde{x})\|^2 + 2\|A(x^n) - A(\tilde{x})\|\|x^n - \tilde{x}\| + \lambda_n^2,$$

and A is bounded, the sequence τ_n also is bounded, i.e. there exists a constant $C > 0$ such that

$$\lambda_{n+1} \leq \lambda_n - 2\lambda_n t_n/C + t_n^2.$$

Basing on the current relation of [3] we can conclude that $\lambda_n \to 0$, as $n \to +\infty$. So, the sequence $\{x^n\}$ converges to $\tilde{x}$.

For each $\alpha > 0$ the regularized solutions $x_{h\delta}^{\alpha}$ and $x_{h\delta}^{\alpha n}$ can be found by this method. In many cases we can choose the operator B with prior knowledge on B^{-1}. We see that the iterative process (3.2) is much simpler than others of [1], [3]. This is the second advantage of B over U^s in regularization for ill-posed equations of the first kind involving monotone operators.

4. Example

Consider the linear operator equation of the first kind

$$Ky = f, \quad f \in L_q([0,1]), \quad 2 < q < +\infty \tag{4.1}$$

where K is defined by

$$\big(Kx\big)(t) \ = \int_0^1 k(t,s)x(s)ds,$$

such that $\big\langle Kx,\ x\big\rangle \geq 0,\ \forall x \in L_p([0,1]),\ p^{-1} + q^{-1} = 1$, (see [4]).

Note here that the choice of space $L_p([0,1])$ in solving (4.1) is very important for achieving effectiveness in computations (see [14]).

Asumme that the solution $y(t)$ is twice-differentiable and satisfies the condition $y(0) = y(1) = 0$. In this case, U^s is certainty a nonlinear operator, hence the regularized equation (1.3) is also a nonlinear equation, while (1.5) and (1.7) are linear. This is the thirth advantage of B over U^s in regularization. Operator B, in this case, can be determined by

$Bx(t) = -\dfrac{d^2x(t)}{dt^2} + p_0(t)x(t),\ \ p_0(t) \geq p_0 > 0$, where $D(B)$ is the closure in the norm W_q^2 of all functions from $C^2[0,1]$ satisfying the condition $u(0) = u(1) = 0$ (see [7]). Then $B^{-1}v(t) = \int\limits_0^1 g(t,s)v(s)ds$ with

$$g(t,s) = u_1(t)u_2(s), \quad t \leq s; \quad = u_2(t)u_1(s) \quad t \geq s,$$

where u_1, u_2 are the nontrivial solutions of $Bu = 0$ such that $u(0) = u(1) = 0$ (see [12]). The derivatives are understood in generalized sense.

The verification of other conditions depend completely on the concrete problem and P_n.

References

1. I. J. Alber and A. I. Notik, *Geometrical properties of Banach spaces and approximation methods solving nonlinear operator equations*, Dokl. Acad. Nauk SSSR **276** (5) (1984), 1033-1037 (in Russian).
2. I. J. Alber and I. P. Ryazantseva, *On solutions of nonlinear problems involving monotone discontinuous operators*, Differential Uravnenia SSSR **25** (2) (1979), 331-342 (in Russian).
3. I. J. Alber, *Recurrent relations and variational inequalities*, Dokl. Acad. Nauk SSSR **270** (1) (1983), 11-16 (in Russian).
4. S. Banach, *Théorie des opérations linéaires*, Warsaw 1932.
5. Nguyen Buong, *Convergence rates for nonlinear ill-posed problems involving monotone operators for monotone perturbations*, Preprint, Institute of Computer Science N.**5** (1991), to appear in JJIAM "On nonlinear ill-posed problems".
6. ———, *On regularization for variational inequality and general scheme of approximations in Banach space*, Ukrainian Math. J. **9** (1991), 1273-1276 (in Russian).
7. ———, *Convergence rates and finite-approximations for nonlinear ill-posed problems involving monotone operators in Banach spaces*, Preprint ICTP N. **385** (1992), Italy, to appear in Math. Nachr.
8. H. W. Engl and C. W. Groetsch, *Projection-Regularization for linear operator equations of the first kind*, in: R. S. Anderson and G. N. Neusan, eds., Special Programs on Inverse Problems, Proc. of the Center of Math. Anal. Australial Nat. Univ. **17** (1988), 17-31.
9. H. W. Engl, K. Kunisch and A. Neubauer, *Convergence rates for Tikhonov regularization of non-linear ill-posed problems*, Inverse Problems **5** (1989), 523-540.
10. C. W. Groetsch, *On a Regularization-Ritz Method for Fredholm Equations of the First Kind*, J. Integral Equations **4** (1982), 173-182.
11. C. W. Groetsch and M. Hanke, *Regularization by Projection for Unbounded Operator Arizing in Inverse Problems*, in: Proceed. Intern. Workshop on Inverse Problems and applications to Geophysics, Industry, Medicine and Technology (HoChiMinh City, Jan. 17-19, 1995), 61-70.

12. T. Kato, *Perturbation theory for linear operators*, Springer-Verlage, New-York 1966.

13. V. A. Morozov, *Solutions for ill-posed problems*, Springer-Verlage, Berlin 1984.

14. A. C. Nemirovskii and D. B. Udin, *Complexity of problems and effectiveness of methods of optimization*, Nauka, Moscow 1979 (in Russian).

15. A. Neubauer, *Tikhonov Regularization of Ill-posed Linear Operator Equations on Convex Sets*, J. of Approx. Theory **53** (1988), 304-320.

16. ——— , *An a-poerioti parameter choice for Tikhonov regularization in Hilbert scales leading to optimal convergence rates*, SIAM J. Num. Math. **25** (1988), 1313-1326.

17. ——— , *Tikhonov regularization for nonlinear ill-posed problems: Optimal convergence rates and finite-dimensional approximation*, Inverse Problems **55** (1989), 541-577.

18. ——— and O. Scherzer, *Finite-dimensional approximation of Tikhonov regularization of nonlinear ill-posed problems*, Numer. Funct. Analysis and Optim. **11** (1990), 85-89.

19. I. P. Ryazantseva, *On Galerkin's method solving equations with discontinuous monotone operators*, Izvestia Vyschix Uchebnix Zav., ser. Math. SSSR (7) (1978), 68-72 (in Russian).

20. I. P. Ryazantseva, *On select of regularizating parameters for equations under monotone perturbations*, Izvestia Vyschix Uchebnix Zav., ser. Math. SSSR (8) (1981), 39-43 (in Russian).

21. U. Tautenhahn, *Error estimate for regularized solutions of nonlinear ill-posed problems*, Inverse Problems **10** (1994), 485-500.

22. M. M. Vainberg, *Variational method and method of monotone operators*, Nauka, Moscow 1972 (in Russian).

23. V. V. Yurgelas, *On some geometrical characteristics of Banach spaces and accretive operators*, Izvestia VUZ, Math. (5) (1982), 63-69 (in Russian).

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