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VOLTERRA RIGHT INVERSES FOR WEIGHTED DIFFERENCE OPERATORS IN LINEAR SPACES

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Abstract. Let X be a linear space over a field $\mathcal{F}$ of scalars and let X_ω be the set of all infinite sequences $x = (x_0, x_1, \dots)$, where $x_j \in X$. Let $A = (A_0, A_1, \dots)$ be a sequence in $L_0(X)$. Consider the weighted difference operator in X_ω: $D_A x = (x_{n+1} - A_n x_n)$. The scalar cases of weighted difference operators have been investigated, among others, by Przeworska-Rolewicz [3] and Kalfat [6]. In this paper we describe the set of all right inverses and the set of all initial operators for D_A. Properties of fundamental right inverses and fundamental initial operators are studied. In particular, we give conditions for a fundamental right inverse to be Volterra and apply this result to solve the corresponding initial value problem.

0. Introdution

Let X and Y be nontrivial linear spaces over a field $\mathcal{F}$ of scalars. Denote by $L(X, Y)$ the set of all linear operators with domains in X and ranges in Y. Set

$$L_0(X, Y) = \{A \in L(X, Y) : \operatorname{dom} A = X\}.$$

In the case $Y = X$, we shall write $L(X)$ instead of $L(X, X)$ and similarly $L_0(X)$ instead of $L_0(X, X)$. The set of all right invertible operators in $L(X)$ will be denoted by $R(X)$. For $D \in R(X)$ we denote by $\mathcal{R}_D$ and $\mathcal{F}_D$ the set of all right inverses and the set of all initial operators for D, respectively, i.e.

$$\mathcal{R}_D = \{R \in L_0(X) : DR = I\},$$

$$\mathcal{F}_D = \{F \in L_0(X) : F^2 = F, FX = \ker D\}.$$

For the general theory of right invertible operators and its applications we refer the reader to [3].

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Denote by X_ω the set of all infinite sequences $x = (x_0, x_1, \dots)$ of elements of X with the natural operations $\alpha x + \beta y = (\alpha x_n + \beta y_n)$ for all $x, y \in X_\omega$ and for all $\alpha, \beta \in \mathcal{F}$.

We say that $A \in L_0(X)$ is algebraic if there exists a non-zero normed polynomial $P(t) = t^N + p_1 t^{N-1} + \cdots + p_N$ with coefficients in $\mathcal{F}$ such that $P(A) = 0$ on X. An algebraic operator A is called of order N if there does not exist a normed polynomial $Q(t)$ of degree $m < N$ such that $Q(A) = 0$ on X. Such a minimal polynomial $P(t)$ is called the characteristic polynomial of A and is denoted by $P_A(t)$. The set of all algebraic operators in $L_0(X)$ will be denoted by $A(X)$.

Let $B \in L_0(X)$. If the operator $I - \lambda B$ is invertible for all $\lambda \in \mathcal{F}$, then B is said to be a Volterra operator. The set of all Volterra operators acting in X will be denoted by $V(X)$.

1. General weighted difference operators

Let $A = (A_0, A_1, \dots)$ be a sequence of linear operators, where $A_n \in L_0(X)$. Consider the following weighted difference operator in X_ω:

$$D_A x = (x_{n+1} - A_n x_n). \tag{1}$$

It is easy to see that $D_A \in L_0(X_\omega)$ and

$$\ker D_A = \{x \in X_\omega : x_0 = u, x_n = A_{n-1} \dots A_0 u, u \in X, n \geq 1\}. \tag{2}$$

Hence $\dim \ker D_A = \dim X > 0$ and D_A is not invertible.

Lemma 1. $D_A \in R(X_\omega)$.

Proof. Define an operator $R_0 \in L_0(X_\omega)$ as follows

$$R_0 x = y, \tag{3}$$

where

$$\begin{aligned} y_0 &= 0, \\ y_1 &= x_0, \\ y_n &= x_{n-1} + A_{n-1} x_{n-2} + \cdots + A_{n-1} \cdots A_1 x_0 \\ &\qquad \text{for } n \geq 2. \end{aligned} \tag{4}$$

We have

$$D_A R_0 x = D_A y = v,$$

where

$$v_0 = x_0, \quad v_1 = y_2 - A_1y_1 = (x_1 + A_1x_0) - A_1x_0 = x_1,$$
$$v_n = y_{n+1} - A_ny_n = (x_n + A_nx_{n-1} + \cdots + A_n \cdots A_1x_0)$$
$$- A_n(x_{n-1} + A_{n-1}x_{n-2} + \cdots + A_{n-1} \cdots A_1x_0) = x_n, \quad n \geq 2.$$

Thus $D_AR_0x = x$ for all $x \in X_\omega$, i.e. $D_A \in R(X_\omega)$ and $R_0 \in \mathcal{R}_{D_A}$.

Lemma 2. *$R \in \mathcal{R}_{D_A}$ if and only if there exists $V \in L_0(X_\omega, X)$ such that*

$$Rx = y, \tag{5}$$

where

$$\begin{aligned} y_0 &= Vx, \\ y_1 &= x_0 + A_0Vx, \\ y_n &= x_{n-1} + A_{n-1}x_{n-2} + \cdots + A_{n-1} \cdots A_1x_0 \\ &\qquad + A_{n-1} \cdots A_0Vx, \quad \text{for } n \geq 2. \end{aligned} \tag{6}$$

In the sequel, R defined by (5)-(6) *will be denoted by R_V.*

Proof. Let R_V be defined by (5)-(6). We find

$$D_AR_Vx = D_Ay = v,$$

where

$$v_0 = y_1 - A_0y_0 = x_0,$$
$$v_1 = y_2 - A_1y_1 = (x_1 + A_1x_0 + A_1A_0Vx) - A_1(x_0 + A_0Vx) = x_1,$$

and for $n \geq 2$ we get by an easy induction the equalities $v_n = x_n$. Hence $D_AR_Vx = x$ for all $x \in X_\omega$, i.e. $R_V \in \mathcal{R}_{D_A}$.

Conversely, if $R \in L_0(X), D_ARx = x$ and $Rx = y$, then $y_{n+1} - A_ny_n = x_n$ for $n \geq 0$. These equalities imply

$$y_0 = u \in X,$$
$$y_1 = x_0 + A_0u$$
$$y_n = x_{n-1} + A_{n-1}x_{n-2} + \cdots + A_{n-1} \cdots A_1x_0 + A_{n-1} \cdots A_0u$$
$$\text{for } n \geq 2.$$

Since $Rx = y$, every term y_n depends on x and there exists an operator V from X_ω to X such that $y_0 = Vx$. From the assumption $R \in L_0(X_\omega)$ we conclude that $V \in L_0(X_\omega, X)$, which gives us the formula (5)-(6).

Lemma 3. *Let D_A be of the form* (1) *and let $R_V \in \mathcal{R}_{D_A}$ be defined by* (5)-(6)*. Then the initial operator F_V for D_A corresponding to R_V is of the form*

$$F_V x = (x_0 - VD_A x, A_0(x_0 - VD_A x), A_1 A_0(x_0 - VD_A x), \dots). \tag{7}$$

Proof. Note that the equation $x_0 - VD_A x = u$ for every given $u \in X$ has a solution $x = (u, A_0 u, A_1 A_0 u, \dots) \in X_\omega$. Hence, if F_V is defined by (7), then $F_V X_\omega = \ker D_A$ and $D_A F_V = 0$ on X_ω. Moreover, since $D_A F_V = 0$, for any $x \in X$, we find $F_V^2 x = F_V(F_V x) = F_V x$. Thus, $F_V \in \mathcal{F}_{D_A}$.

Formulae (5)-(6) and (7) together imply $F_V R_V x = u$, where $u_0 = Vx - VD_A R_V x = Vx - Vx = 0$ and $u_n = A_{n-1} \cdots A_0 u_0 = 0$ for $n \geq 1$. Thus $F_V R_V = 0$ on X_ω, i.e. F_V is an initial operator for D_A corresponding to R_V.

Lemmas 2 and 3 show that the properties of every initial operator $F_V \in \mathcal{F}_{D_A}$ depend on the properties of $V \in L_0(X_\omega, X)$.

2. Fundamental initial operators and corresponding right inverses

Definition 2. Let $m \in IN$ be fixed and $V_m \in L_0(X)$. Then F_{V_m} defined by the formula

$$F_{V_m} x = (u, A_0 u, A_1 A_0 u, \dots), u = x_0 - V_m(x_{m+1} - A_m x_m) \tag{8}$$

is called a fundamental initial operator for D_A. Every $R \in \mathcal{R}_{D_A}$ such that $F_{V_m} R = 0$ (i.e. F_{V_m} corresponding to R) is called a fundamental right inverse of D_A and will be denoted by R_{V_m}.

By Definition 2, if $F_{V_m} R_{V_m} = 0$ for $R_{V_m} \in \mathcal{R}_{D_A}$, then R_{V_m} is defined by the formula

$$R_{V_m} x = y, \tag{9}$$

where

$$y_0 = V_m x_m,$$
$$y_1 = x_0 + A_0 V_m x_m,$$
$$y_n = x_{n-1} + \cdots + A_{n-1} \cdots A_1 x_0 + A_{n-1} \cdots A_0 V_m x_m \quad \text{for } n \geq 2.$$

Lemma 4. *Let $N \in IN$ and let $V_m \in L_0(X)$, $V_m \not\equiv 0$ on X $(m = 0, 1, \dots, N)$. Then the corresponding system of the fundamental right inverses $\{R_{V_0}, \dots, R_{V_N}\}$ is linearly independent on X_ω.*

Proof. Let $\alpha_0, \alpha_1, \dots, \alpha_N \in \mathcal{F}$ such that $\sum_{m=1}^{N} \alpha_m R_{V_m} x = 0$ for all $x \in X_\omega$. By (9),

$$\sum_{m=0}^{N} \alpha_m V_m x_m = 0 \quad \text{for all } x_m \in X. \tag{10}$$

Putting $x = (0, \dots, 0, x_k, 0, \dots)$ in (10) $x_k \in X$ we get $\alpha_k V_m x_k = 0$ which gives $\alpha_k = 0$ for every $k \in \{0, 1, \dots, N\}$.

Similarly, we can prove a similar result for any fundamental initial operator.

Lemma 5. *Let $N \in IN$ and let $V_m \in L_0(X)$, where $V_m \not\equiv 0$ on X for $m = 0, 1, \dots, N$. Then the system of fundamental initial operators $\{F_{V_0}, ..., F_{V_N}\}$ is linearly independent on X_ω.*

Proof. Suppose that there are $\alpha_0, \dots, \alpha_N \in \mathcal{F}$ such that

$$\sum_{m=0}^{N} \alpha_m F_{V_m} x = 0 \quad \text{for all } x \in X_\omega. \tag{11}$$

Let $k \in \{1, 2, \dots, N\}$ be fixed. Setting $x = R_{V_k} t$ in (11), where $t = (t_n), t_n = 0$ for all $n \neq k+1$ and $t_{k+1} = x_k$, we have

$$\alpha_k V_k x_k = 0 \quad \text{for all } x_k \in X. \tag{12}$$

Since $V_k \not\equiv 0$ on X, from (12) we get $\alpha_k = 0$ for $k = 1, 2, \dots, N$. Thus, (11) is now of the form

$$\alpha_0 F_{V_0} x = 0 \quad \text{for all } x \in X_\omega. \tag{13}$$

Again, putting $x = (x_0, A_0 x_0, 0, 0, \dots)$ in (13) we have

$$\alpha_0 (x_0 - V_0(A_0 x_0 - A_0 x_0)) = 0 \quad \text{for all } x_0 \in X,$$

which is equivalent to $\alpha_0 x_0 = 0$ for all $x_0 \in X$, i.e. $\alpha_0 = 0$. Thus, the system $\{F_{V_0}, ..., F_{V_N}\}$ is linearly independent on X_ω.

3. Volterra right inverses

We now formulate conditions for a fundamental right inverse $R_{V_m} \in \mathcal{R}_{V_A}$ to be Volterra.

Theorem 1. *Let D_A be of the form* (1) *and let F_{V_m} be a fundamental initial operator of the form* (8). *Then the corresponding right inverse R_{V_m} is Volterra if and only if the operator*

$$K_m = I - (\lambda I + A_{m-1}) \cdots (\lambda I + A_0)(\lambda + A_{-1}), \tag{14}$$

where we admit $A_{-1} = 0$, is invertible for all $\lambda \in \mathcal{F}$.

Proof. By Lemmas 2 and 3, R_{V_m} is defined by the formula

$$R_{V_m} x = (V_m x_m, x_0 + A_0 V_m x_m, x_1 + A_1 x_0 + A_1 A_0 V_m x_m, \dots) \tag{15}$$

Consider the equation

$$(I - \lambda R_{V_m})x = v, v \in X_\omega. \tag{16}$$

For $m = 0$, by (15) we can rewrite (16) in the form of the system

$$\begin{aligned} &x_0 - \lambda V_0 x_0 = v_0, \\ &x_1 - \lambda(x_0 + A_0 V_0 x_0) = v_1, \\ &x_2 - \lambda(x_1 + A_1 x_0 + A_1 A_0 V_0 x_0) = v_2, \\ &x_n - \lambda(x_{n-1} + \cdots + A_{n-1} \cdots A_0 V_0 x_0) = v_n, \quad n \geq 3. \end{aligned} \tag{17}$$

It is easy to see that the system (17) has a unique solution for any $v = (v_n) \in X_\omega$ if and only if the first equation of (17): $x_0 - \lambda V_0 x_0 = v_0$ has a unique solution for all $\lambda \in \mathcal{F}$ and for all $v_0 \in X$, i.e. the operator $I - \lambda V_0$ is invertible for all $\lambda \in \mathcal{F}$.

If $m \geq 1$, then (17) is of the form

$$\begin{aligned} &x_0 - \lambda V_m x_m = v_0, \\ &x_1 - \lambda(x_0 + A_0 V_m x_m) = v_1, \\ &x_n - \lambda(x_{n-1} + \cdots A_{n-1} \cdots A_0 V_m x_m) = v_n, \quad n \geq 2. \end{aligned} \tag{18}$$

The system (18) has a unique solution for any $v \in X_\omega$ if and only if the system of $m+1$ equations

$$\begin{aligned} &x_0 - \lambda V_m x_m = v_0, \\ &x_1 - \lambda(x_0 + A_0 V_m x_m) = v_1, \\ &x_n - \lambda(x_{n-1} + \cdots + A_{n-1} \cdots A_0 V_m x_m) = v_n, \\ &\qquad n = 2, \dots, m \end{aligned} \tag{19}$$

has a unique solution for all $v_0, \dots, v_m \in X$. From (19) we have, for $n = 0, 1, \cdots, m-1$,

$$\begin{aligned} x_0 &= v_0 + \lambda V_m x_m, \\ x_1 &= v_1 + \lambda v_0 + (\lambda I + A_0)\lambda V_m x_m, \\ &\dots \\ x_{m-1} &= v_{m-1} + \lambda(v_{m-2} + A_{m-2}v_{m-3} + \cdots + A_{m-2}\cdots A_0 v_0) \\ &\quad + \cdots + \lambda^{m-1}v_0 + (\lambda I + A_{m-2})\cdots(\lambda I + A_0)\lambda V_m x_m. \end{aligned} \tag{20}$$

Also from (19) we have, for $n = m$,

$$\begin{aligned} x_m &= (\lambda I + A_{m-1})\cdots(\lambda I + A_0)lV_m x_m + v_m \\ &\quad + \lambda(v_{m-1} + \cdots + A_{m-1}\cdots A_0 v_0) + \cdots + \lambda^m v_0. \end{aligned} \tag{21}$$

Hence, (19) has a unique solution if and only if the equation (21) has a unique solution for all $\lambda \in \mathcal{F}$ and for all $v_0, \dots, v_m \in X$, i.e. the operator (14) is invertible for all $\lambda \in \mathcal{F}$.

Corollary 1. *If $R \in \mathcal{R}_{D_A}$, $Rx = y$, where $y_0 = Vx_0$, $V \in V(X)$, then $R \in V(X_\omega)$.*

Corollary 2. *If $R \in \mathcal{R}_{D_A}$ and $Rx = y$, where $y_0 = Qx_0$, $Q \in \mathrm{A}(X)$, then $R \in V(X_\omega)$ if and only if Q is a nilpotent operator, i.e. there is $s \in I\!N^+$ such that $Q^s = 0$.*

Proof. If $Q \in \mathrm{A}(X)$ and $Q^s = 0$, then $I - \lambda Q$ is invertible for all $\lambda \in \mathcal{F}$ and $(I - \lambda Q)^{-1} = I + \lambda Q + \cdots + \lambda^{s-1}Q^{s-1}$. Hence, by Theorem 1, the corresponding right inverse R is Volterra. Conversely, if Q is not a nilpotent operator then there is a non-zero root t_0 of $P_Q(t)$. Then $1 - \lambda t_0$ is a root of $P_{I-\lambda Q}(t)$ ([4]). Let $\lambda = t_0^{-1}$. Then the operator $I - \lambda Q$ is not invertible and $Q \notin V(X_\omega)$. By Theorem 1, $R \notin V(X_\omega)$.

4. Initial value problem

Let D_A be of the form (1) and let $V_0 \in V(X)$. Consider the following initial value problem

$$\sum_{j=0}^{N} a_j D_A^j x = y, \tag{22}$$

$$F_{V_0}D^k x = u_k, u_k \in \ker D_A, k = 0, \dots, N-1, \tag{23}$$

where $a_0, a_1, \dots, a_N \in \mathcal{F}$. From now on let $\mathcal{F}$ be the field of complex numbers.

Let $R_{V_0} \in \mathcal{R}_{D_A}$ be a fundamental right inverse of D_A and F_{V_0} be an initial operator corresponding to R_{V_0}. By Theorem 1, $R_{V_0} \in V(X_\omega)$. Hence, applying the algebraic analysis method of [3], the problem (22)-(23) is equivalent to the equation

$$\sum_{j=0}^{N} a_{N-j} R_{V_0}^j x = R_{V_0}^N y + \sum_{j=0}^{N-1} R_{V_0}^j u_j. \tag{24}$$

Let $t_1, \dots, t_N \in \mathcal{F}$ be n roots of the polynomial

$$P(t) = t^N + a_{N-1}t^{N-1} + \cdots + a_1 t + a_0.$$

We can rewrite (24) in the form

$$\prod_{j=1}^{N} (I - t_j R_{V_0}) x = R_{V_0}^N y + \sum_{j=0}^{N-1} R_{V_0}^j u_j.$$

Hence the problem (22)-(23) has a unique solution of the form

$$x = \prod_{j=1}^{N} (I - t_j R_{V_0})^{-1} (R_{V_0}^N y + \sum_{j=0}^{N-1} R_{V_0}^j u_j).$$

Remark. By the same method one can solve the first mixed boundary value problem

$$\sum_{j=0}^{N} a_j D_A^j x = y,$$
$$F_{V_m} D^k x = u_k; \quad u_k \in \ker D_A, \quad k = 0, \dots, N-1$$

under the assumptions that every operator of the form

$$K_m = I - (\lambda I + A_{m-1}) \cdots (\lambda I + A_0)(\lambda I + A_{-1}) V_m$$

$(m = 0, \dots, N-1; A_{-1} := 0)$ is invertible for all $\lambda \in \mathcal{F}$.

5. Examples of applications

Example 1. Let $X = \mathrm{C}(I\!R, \mathcal{F})$ be the set of all continuous functions from $I\!R$ to $\mathcal{F}$. Given $(A_n x)(t) = x(n-t)$, $(V_0 x)(t) = x(t)$ and $a \in \mathcal{F}$. Find all sequences $\{x_n(t)\}$ $(x_n \in X)$ such that

$$x_{n+1}(t) - x_n(n-t) + ax_n(t) = y_n(t), \tag{25}$$

$$x_0(t) + x_0(-t) - x_1(t) = 0. \tag{26}$$

where $y_n \in X(n = 0, 1, \dots)$ are given.

Write (25)-(26) in the form

$$(D_A + aI)x = y, \quad F_{V_0} x = 0.$$

Then the problem (25)-(26) is equivalent to the following equation

$$(I + aR_{V_0})x = R_{V_0} y, \tag{27}$$

where R_{V_0} is defined by the formula (9).

Theorem 1 implies that the equation (27) has a unique solution if and only if the equation $x_0(t) + ax_0(t) = y_0(t)$ has a unique solution, i.e. $a \neq -1$. If that is the case, the unique solution of the problem (25)-(26) is of the form

$$x = (I + aR_{V_0})^{-1} R_{V_0} y.$$

Example 2. Let X be a linear space over $\mathbf{C}$ and let $A \in \mathrm{A}(X)$ be an algebraic operator with the characteristic polynomial

$$P_A(t) = \prod_{j=1}^{r} (t - t_j); \; t_i \neq t_j \quad \text{fo} \quad i \neq j.$$

Write:

$$P_k = \omega(A, I); \quad \omega(t, \tau) = \prod_{j=1, j\neq k}^{r} (t_k - t_j)^{-1}(t - t_j\tau).$$

Then the following equalities hold [1]-[2]:

$$P_i P_j = \delta_{ij} P_j; \quad A^k = \sum_{j=1}^{r} t_j^k P_j; \quad k \in I\!N.$$

Consider the following algebraic weighted difference operator

$$D_A x = (x_{n+1} - Ax_n) \quad \text{for } x = (x_0, x_1, \dots) \in X_\omega. \tag{28}$$

Then

$$\ker A = \Big\{ x \in X_\omega : x_n = \sum_{j=1}^{r} t_j^n P_j u,\ u \in X \Big\}.$$

Every right inverse of D_A is of the form

$$R = R_A + F_A B, \tag{29}$$

where $F_A = I - R_A D_A \in \mathcal{F}_{D_A}$, $B \in L_0(X_\omega)$ and R_A is defined by the formula

$$R_A x = y; \quad y_0 = 0; \quad y_n = \sum_{j=1}^{r} \sum_{k=0}^{n-1} t_j^k P_j x_{n-1-k}.$$

By Theorem 1, we can formulate a sufficient condition for a right inverse R of D_A to be Volterra by means of B in (29).

Corollary 3. *Let* $X_{\omega,0} = \{x \in X_\omega : x_0 = 0\}$ *and* $B \in L_0(X_\omega)$ *with* $BX_\omega \subset X_{\omega;0}$. *Then* R *defined by* (29) *is a Volterra operator.*

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