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THE INHERITANCE OF THE LINEAR TOPOLOGICAL INVARIANT (DN)

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ABSTRACT. It is shown that if X is a Stein space and S is a closed set in X with $H_{2\dim X-1}(S) = 0$, then $H(X) \in (DN)$ if and only if $H(X \setminus S) \in (DN)$. Moreover it is also shown that the property (DN) is invariant under holomorphic surjections between Stein spaces with connected fibres having the property (DN).

Let F be a Frechet space with a fundamental system of semi-norms $\Big\{\|\cdot\|_k\Big\}_{k\geq 1}$. We say that E has the property (DN) (shortly write $E \in (DN)$) if

$$\exists p\ \forall q,\ d>0\ \exists k,\ c>0 : \big\|\cdot\big\|_q^{1+d} \leq c\big\|\cdot\big\|_k\ \big\|\cdot\big\|_p^d.$$

The property (DN) and other properties were introduced and investigated by Vogt [8, 9]. The aim of the present paper is to study the inheritance of the property (DN). The main results are the following

Theorem A. *Let X be a locally irreducible Stein space and S a closed subset of X such that $H_{2\dim X-1}(S) = 0$. Then $H(X) \in (DN)$ if and only if $H(X \setminus S) \in (DN)$.*

Here $H_{2\dim X-1}(S)$ denotes the $(2\dim X - 1)$-Hausdorff measure of $S \cap R(X)$ where $R(X)$ is the regular locus of X.

Theorem B. *Let $\theta : X \to Y$ be a holomorphic surjection between locally irreducible Stein spaces with connected fibres. Assume that*

$$H(\theta^{-1}(y)) \in (DN) \quad \textit{for all} \quad y \in Y.$$

Then $H(X) \in (DN)$ if and only if $H(Y) \in (DN)$.

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For other linear topological invariants, these theorems are not true. The proofs of Theorems A and B are given in Section 3 and Section 4 respectively. In Section 1 we show that Theorem B is true when θ is a proper surjection. Recently [3] Le Mau Hai and Dinh Huy Hoang have proved this result for the linear topological invariants $\overline{\Omega}$, $\widetilde{\Omega}$ and finite proper surjections. For the definition of $\overline{\Omega}$, $\widetilde{\Omega}$ we refer to the papers of Vogt [8, 9]. In Section 2 we extend Zaharjuta's result [10] to the Stein space case.

1. The property (DN) and finite proper holomorphic surjections

In this section we shall prove the following.

Theorem 1.1. *Let $\theta : X \to Y$ be a finite proper holomorphic surjection between Stein spaces. Then $H(X) \in (DN)$ if and only if $H(Y) \in (DN)$.*

Proof. Assume that $H(Y) \in (DN)$.

(i) First consider the case that Y is a normal space. Then by the integrity lemma [2] θ is a branched covering map. Moreover there exists a natural number p such that for each $f \in H(X)$ we can find a polynomial $P_f(\lambda)$ of degree p with coefficients in $H(Y)$:

$$P_f(\lambda) = \lambda^p + a_{p-1}(f)\lambda^{p-1} + \cdots + a_0(f)$$

such that

$$P_f(f) = 0,$$

where $a_{p-1}, \dots, a_0$ are continuous symmetric polynomials on $H(X)$ with values in $H(Y)$.

To prove $H(X) \in (DN)$, by Vogt [10] it suffices to check that every continuous linear map from the space $\Lambda_1(\alpha)$ to $H(X)$ is bounded on a neighbourhood of $0 \in \Lambda_1(\alpha)$ for every exponent sequence $\alpha = (\alpha_n)$, where

$$\Lambda_1(\alpha) = \Big\{ (\xi_j) \subset \mathbf{C} : \sum_{j \geq 1} |\xi_j| r^{\alpha_j} < \infty \quad \text{for} \quad 0 < r < 1 \Big\}.$$

Given such a map T. Since a_j $(o \leq j \leq p-1)$ are continuous polynomials on $H(X)$ with values in $H(Y)$ and by the hypothesis $H(Y) \in (DN)$, again by Vogt [10] we can find a neighbourhood U of $0 \in \Lambda_1(\alpha)$ such that $a_j(T)$ are bounded on U. From the relation

$$(T\xi)^p + a_{p-1}(T\xi)(T\xi)^{p-1} + \cdots + a_0(T\xi) = 0 \quad \text{for} \quad \xi \in U,$$

it follows that T is bounded on U. Hence $H(X) \in (DN)$.

(ii) In the case where Y is not normal, consider the normalization $\gamma : \tilde{Y} \to Y$ of Y. Let J denote the coherent sheaf on Y given by

$$J_y = \big\{ f \in \mathcal{H}_{Y,y} : f(\gamma_* \mathcal{H}_{\tilde{Y}})_y \subseteq \mathcal{H}_{Y,y} \big\},$$

where $\mathcal{H}_{\tilde{Y}}$ and $\mathcal{H}_Y$ are structure sheaves of $\tilde{Y}$ and Y respectively and $\gamma_* \mathcal{H}_{\tilde{Y}}$ is the direct image of $\mathcal{H}_{\tilde{Y}}$ under γ. Then $J_y \neq 0$ for $y \in Y$ [2]. By Cartan Theorem A [2] we have $H^0(Y, J) \neq 0$. Moreover, there exists $f \in H^0(Y, J)$ such that $f \neq 0$ on every irreducible branch of Y. Indeed, write $Y = \bigcup\limits_{i \geq 1} Y_i$, where Y_i are irreducible branches of Y. For each $i \geq 1$ put

$$\begin{aligned} G_i &= \big\{ f \in H^0(Y, J) : f\big|_{Y_i} \neq 0 \big\} \\ &= H^0(Y, J) \setminus \big\{ f \in H^0(Y, J) : f\big|_{Y_i} = 0 \big\}. \end{aligned}$$

Thus G_i is open. We prove that G_i is dense in $H^0(Y, J)$ for $i \geq 1$. For $i \geq 1$ take $y_i \in R(Y_i)$, the regular locus of Y_i. Since $1_{y_i} \in J_{y_i}$, by Cartan Theorem A [2] there exist $g_1, \ldots, g_m \in H^0(Y, J)$ and $\delta_1, \ldots, \delta_m \in \mathcal{H}_{Y, y_i}$ such that

$$\sum_{1 \leq j \leq m} g_{j, y_i} \delta_{j, y_i} = 1_{y_i}.$$

This yields the existence of j_0 such that $g_{j_0} \in G_i$. Thus $G_i \neq \emptyset$ for $i \geq 1$, and hence G_i is dense in $H^0(Y, J)$ for $i \geq 1$. By Baire Theorem there exists $f \in \bigcap\limits_{i \geq 1} G_i$.

Thus $H(\tilde{Y}) \cong f H(\tilde{Y})$. Since $f H(\tilde{Y}) \subset H(Y) \in (DN)$, it follows that $f H(\tilde{Y})$ and hence $H(\tilde{Y}) \in (DN)$.

(iii) Finally consider the following commutative diagram of finite proper surjections of Stein spaces

$$\begin{array}{ccc} Z & \xrightarrow{\quad \tilde{\theta} \quad} & \tilde{Y} \\ {\scriptstyle \tilde{\gamma}} \big\downarrow & & \big\downarrow {\scriptstyle \gamma} \\ X & \xrightarrow{\quad \theta \quad} & Y \end{array}$$

where $Z = X \times_Y \tilde{Y}$ is the fibre product of X and $\tilde{Y}$ and $\tilde{\theta}$, $\tilde{\gamma}$ are canonical projections.

By (ii) $H(\tilde{Y}) \in (DN)$ and by (i) $H(Z) \in (DN)$. Since $H(X)$ is a subspace of $H(Z)$, we have $H(X) \in (DN)$. This completes the proof for the sufficiency.

Since $H(Y)$ is a subspace of $H(X)$, the necessity is trivial.

Corollary 1.2. *Let X be a Stein space. Then $H(X) \in (DN)$ if and only if $H(Z) \in (DN)$ for every irreducible branch Z of X and X has a finite number of irreducible branches.*

Proof. Let $H(X) \in (DN)$. By Theorem 1.1, $H(\tilde{X}) \in (DN)$, where $\gamma : \tilde{X} \to X$ is the normalization of X. Since every irreducible branch $\tilde{Z}$ of $\tilde{X}$ is open - closed in $\tilde{X}$ [2], it follows that $H(\tilde{Z}) \in (DN)$. Given Z an irreducible branch of X. Then there exists an irreducible branch $\tilde{Z}$ of $\tilde{X}$ such that $\gamma(\tilde{Z}) = Z$. Applying Theorem 1.1 to $\gamma\big|_{\tilde{Z}} : \tilde{Z} \to Z$ we get $H(Z) \in (DN)$. On the other hand, by the definition of the property (DN), $H(X)$ has a continuous norm. This means that there exists a compact subset K of X such that $\big\|f\big\|_K = 0$ for $f \in H(X)$ implies that $f = 0$. Hence by Cartan Theorem B, X has a finite number of irreducible branches.

Conversely, assume that X has a finite number of irreducible branches $Z_1, \ldots, Z_m$ and let $H(Z_i) \in (DN)$ for $i = 1, \ldots, m$. For each i take an irreducible branch $\tilde{Z}_i$ of $\tilde{X}$ such that $\gamma(\tilde{Z}_i) = Z_i$. Since $\tilde{Z}_i$ is open-closed in $\tilde{X}$, it follows from the relations $H(Z_i) \in (DN)$ and from Theorem 1.1 that

$$H(\tilde{X}) = \prod_{1 \le i \le m} H(\tilde{Z}_i) \in (DN)$$

Again by Theorem 1.1, $H(X) \in (DN)$.

2. The property (DN) and extremal plurisubharmonic functions

Let X be a complex space and E a subset of X. Define the function $\omega(X, E, .)$ on X by

$$\omega(X, E, x) = \lim_{y \to x} \sup \big\{ \sup \varphi(y) : \varphi \in PSH(X),\ \varphi\big|_E \le 0,\ \varphi \le 1 \big\}.$$

In [13] Zaharjuta proved that if X is a Stein manifold, then $H(X) \in (DN)$ if and only if there exists a compact set K in X such that

$$\omega(X, K, x) = 0 \quad \text{for} \quad x \in X.$$

The following theorem is an extension of this result to the case of Stein spaces.

Theorem 2.1. *Let X be a locally irreducible Stein space. Then $H(X) \in (DN)$ if and only if there exists a compact set K in X such that*

$$\omega(X, K, x) = 0 \quad \text{for} \quad x \in X.$$

Proof. Let $H(X) \in (DN)$. Then there exists p such that

$$\forall q,\ 0 < \mu < 1,\ \exists r(q,\mu),\ c(q,\mu) > 0:$$

$$\|f\|_q \leq c\|f\|_r^{\mu}\ \|f\|_p^{1-\mu} \quad \text{for} \quad f \in H(X).$$

As in [3, Proposition 1.4] we have

$$\frac{\log\big(|f(x)|/\|f\|_p\big)}{\log\big(\|f\|_r/\|f\|_p\big)} \leq \mu \tag{1}$$

for $x \in K_q$ and $f \in H(X)$.

Now assume that $\varphi \in PSH(X)$ with $\varphi\big|_{K_p} \leq 0$ and $\varphi \leq 1$. Since X is Stein, by Fornaess and Narasimhan [3] there exists a decreasing sequence of continuous plurisubharmonic functions φ_j on X such that

$$\varphi_j(x) \downarrow \varphi(x) \quad \text{for} \quad x \in X.$$

Using Hartogs Lemma we may assume that

$$(\varphi_j - \varepsilon_j)\big|_{K_p} \leq 0 \quad \text{and} \quad (\varphi_j - \varepsilon_j)\big|_{K_r} \leq 1,$$

where $\varepsilon_j \downarrow 0$. Again by the Steiness of X as in the regular case [7] for each $j \geq 1$, we can write

$$\varphi_j = \lim_{n\to\infty} \max_{1\leq k\leq m_n^j} c_{jk}^n \log|f_{jk}^n|,$$

where $f_{jk}^n \in H(X)$ and $0 < c_{jk}^n < 1$ for all $j, k \geq 1$ and the convergence is uniform on compact sets in X.

Without loss of generality we may assume that

$$c_{jk}^n \log|f_{jk}^n| - 2\varepsilon_j \leq 0 \quad \text{on} \quad K_p \quad \text{for} \quad j, k, n \geq 1$$

and

$$c_{jk}^n \log|f_{jk}^n| - 2\varepsilon_j \le 1 \quad \text{on} \quad K_r \quad \text{for} \quad j,k,n \ge 1.$$

Since

$$c_{jk}^n \log|f_{jk}^n| - 2\varepsilon_j = c_{jk}^n \log\Big(\frac{\|f_{jk}^n\|_r}{\exp(2\varepsilon_j/c_{jk}^n)}\Big) \times \frac{\log\Big(\dfrac{|f_{jk}^n|}{\exp(2\varepsilon_j/c_{jk}^n)}\Big)}{\log\Big(\dfrac{\|f_{jk}^n\|_r}{\exp(2\varepsilon_j/c_{jk}^n)}\Big)}$$

$$\le \frac{\log\Big(\dfrac{|f_{jk}^n|}{\exp(2\varepsilon_j/c_{jk}^n)}\Big)}{\log\Big(\dfrac{\|f_{jk}^n\|_r}{\exp(2\varepsilon_j/c_{jk}^n)}\Big)}$$

by (1) we get

(2) $$c_{jk}^n \log|f_{jk}^n(x)| - 2\varepsilon_j \le \mu \quad \text{for} \quad x \in K_q \quad \text{and} \quad j,k,n \ge 1.$$

(2) yields

(3) $$\varphi_j(x) - 2\varepsilon_j \le \mu \quad \text{for} \quad x \in K_q \quad j \ge 1.$$

From (3) we get as $j \to \infty$

(4) $$\varphi(x) \le \mu \quad \text{for} \quad x \in K_q \quad \text{and} \quad 0 < \mu < 1.$$

Hence, as $\mu \to 0$, it follows that

(5) $$\varphi(x) \le 0 \quad \text{for} \quad x \in K_q.$$

Since q is arbitrary, we have

$$\omega(X, K_p, x) = 0 \quad \text{for} \quad x \in X.$$

Conversely, assume that $\omega(X, K, x) = 0$ for some compact set K in X. Then we can choose a holomorphic function $\hat{\delta}$ on X such that $\hat{\delta}$ is not zero on every irreducible branch of X and the singular locus $S(X)$ of X is contained in the zero set $Z(\hat{\delta})$ of $\hat{\delta}$. Indeed, let $\{X_j\}$ be the system of irreducible branches of X. For each $j \ge 1$ take

$$x_j \in X_j \setminus \Big(\bigcup_{k \ne j} X_k \bigcup S(X)\Big).$$

Since $\{X_j\}$ is locally finite, the set $\{x_j\}$ is discrete and hence $Y = \{x_j\} \cup S(X)$ is an analytic set in X. Define a holomorphic function σ on Y by

$$\sigma(x) = \begin{cases} 1 & \text{for} \quad x \in \{x_j\} \\ 0 & \text{for} \quad x \in S(X). \end{cases}$$

By Cartan Theorem B there exists $\hat{\delta} \in H(X)$ such that $\hat{\delta}\big|_Y = \sigma$ and hence $\hat{\delta}$ is not zero on X_j for each $j \geq 1$ and $S(X) \subseteq Z(\hat{\delta})$.

Now, we find a compact set L in $X \setminus Z(\hat{\delta})$ such that

$$\sup_K \varphi \leq \sup_L \varphi \quad \text{for each } \varphi \in PSH(X). \tag{6}$$

By the desingularization theorem of Hironaka there exists a proper holomorphic map γ from a complex manifold $\hat{X}$ onto X such that $Z(\hat{\delta}\gamma)$ is a locally finite union of smooth hypersurfaces having normal crossings everywhere. For each $x \in K$ take $z \in \hat{X}$ such that $\gamma(z) = x$. If $z \notin Z(\hat{\delta}\gamma)$ we can take a sufficiently small neighbourhood W_z of z such that $W_z \cap Z(\hat{\delta}\gamma) = \emptyset$. Put $L_z = \gamma(\partial W_z)$ and $V_z = \gamma(W_z)$. Then $L_z \cap Z(\hat{\delta}) = \emptyset$ and

$$\sup_{W_z} \varphi\gamma = \sup_{\partial W_z} \varphi\gamma \quad \text{for} \quad \varphi \in PSH(X)$$

or

$$\sup_{V_z} \varphi = \sup_{L_z} \varphi \quad \text{for} \quad \varphi \in PSH(X).$$

If $z \in Z(\hat{\delta}\gamma)$, we choose a neighbourhood W_z of z in $\hat{X}$ which is biholomorphic to the closed unit polydisc $\overline{\Delta}^n$ of $\mathbf{C}^n$ by a biholomorphic map θ such that

$$\overline{\Delta}^n \cap \theta\big(Z(\hat{\delta}\gamma)\big) = \bigcup_{i=1}^{p} \Big\{ z = (z_1, \ldots, z_n) : z_i = 0 \Big\} \quad \text{for some } p.$$

Put $M_z = \partial\Delta \times \Delta^{n-1} \cup \Delta \times \partial\Delta \times \Delta^{n-2} \cup \cdots \cup \underbrace{\Delta \times \Delta \times \cdots \times \partial\Delta}_{p \text{ times}} \times \Delta^{n-p}$,
and $L_z = \gamma\theta^{-1}(M_z)$, $V_z = \gamma(W_z)$. Then $L_z \cap Z(\hat{\delta}) = \emptyset$ and

$$\sup_{W_z} \varphi\gamma = \sup_{\theta^{-1}(M_z)} \varphi\gamma \quad \text{for} \quad \varphi \in PSH(X)$$

or

$$\sup_{V_z} \varphi = \sup_{L_z} \varphi \quad \text{for} \quad \varphi \in PSH(X).$$

Since γ is proper, it follows that $\gamma^{-1}(K)$ is compact. Cover $\gamma^{-1}(K)$ by a finite system of neighbourhoods $W_{z_1}, W_{z_2}, \dots, W_{z_q}$. Put

$$L = \bigcup_{j=1}^{q} L_{z_j}.$$

Then

$$\sup_K \varphi \leq \sup_{\gamma\big(\bigcup\limits_{j=1}^{q} W_{z_j}\big)} \varphi = \sup_L \varphi \quad \text{for} \quad \varphi \in PSH(X)$$

and $L \cap Z(\hat{\delta}) = \emptyset$.

By the inequality (6) and hypothesis, it follows that

$$\omega(X, L, x) = 0 \quad \text{for} \quad x \in X.$$

On the other hand, by Zeriahi [14] we may assume that L is locally L-regular. Take an increasing exhaustion sequence of compact sets K_p with $K_1 = L$. For each $p \geq 1$ and $x \in \mathrm{Int} K_p$, put

$$\tilde{\omega}(K_p, L, x) = \lim_{y \to x} \sup \Big\{ \sup \varphi(y) : \varphi \in PSH(X), \varphi\big|_L \leq 0, \varphi\big|_{K_p} \leq 1 \Big\}$$

and $\tilde{\omega}(K_p, L, x) = 0$ for $x \in X \setminus \mathrm{Int}\, K_p$.

Since L is locally L-regular, we have

$$\tilde{\omega}(K_p, L, .)\big|_L \leq 0 \quad \text{for} \quad p \geq 1$$

and hence the function $\tilde{\omega}$ on X defined by

$$\tilde{\omega}(x) = \lim_{p \to \infty} \tilde{\omega}(K_p, L, x) \quad \text{for} \quad x \in X$$

satisfies the condition

$$\tilde{\omega} \in PSH(X), \quad \tilde{\omega}\big|_L \leq 0 \quad \text{and} \quad \tilde{\omega} \leq 1 \quad \text{on} \quad X.$$

This yields $\tilde{\omega} = 0$. By Hartogs Lemma for each $q > 1$ and $0 < \mu < 1$ there exists $r(q, \mu)$ such that

$$\tilde{\omega}(K_r, L, x) \leq \mu \quad \text{for} \quad x \in K_q.$$

It follows that

$$\frac{\log\big(|f(x)|/\|f\|_1\big)}{\log\big(\|f\|_r/\|f\|_1\big)} \le \mu$$

for $x \in K_q$ and $f \in H(X)$. This means that

$$\big\|f\big\|_q \le \big\|f\big\|_r^{\mu} \cdot \big\|f\big\|_1^{1-\mu} \quad \text{for} \quad f \in H(X).$$

Hence $H(X) \in (DN)$.

Theorem 2.2. *Let X be a locally irreducible Stein space. Then $H(X) \in (DN)$ if and only if every plurisubharmonic function φ on X which is bounded from above is constant on every irreducible branch of X and X has a finite number of irreducible branches.*

Proof. Let $H(X) \in (DN)$. By Corollary 1.2, X has a finite number of irreducible branches. Given $\varphi \in PSH(X)$ such that

$$\sup_X \varphi = M < \infty.$$

Let Z be an irreducible branch of X. By Corollary 1.2, $H(Z) \in (DN)$, and by Theorem 2.1, there exists a compact set $K \subset Z$ such that

$$\omega(Z, K, x) = 0$$

for $x \in Z$.

Let $m = \sup\limits_K \varphi$. Applying the two constant theorem which is proved as in the non-singular case [5] we have for $x \in Z$

$$\varphi(x) \le M\omega(Z, K, x) - m\big(\omega(Z, K, x) - 1\big) = m.$$

By the connectivity of Z and the maximum principle it follows that φ is constant on Z.

Conversely, let X have a finite number of irreducible branches. Choose a compact set K in X such that $\mathrm{Int}(K \cap Z) \ne \emptyset$ for every irreducible branch Z of X. By the hypothesis

$$\omega(X, K, x)\big|_Z = 0 \quad \text{for} \quad x \in Z$$

and, hence, $\omega(X, K, x) = 0$ for $x \in X$. By Theorem 2.1, $H(X) \in (DN)$.

3. Proof of Theorem A

(i) First we prove that $H(X) \hookrightarrow H(X \setminus S)$. It suffices to show that for every compact set K in X there exists a compact set L in $X \setminus S$ for which

$$\|f\|_K \le \|f\|_L \quad \text{for all} \quad f \in H(X).$$

By the desingularization theorem of Hironaka there exists a proper map γ from a complex manifold Z onto X. Since $H_{2\dim Z-1}\big(\gamma^{-1}(S)\big) = H_{2\dim X-1}(S) = 0$ and by [6], it follows that for every $z \in \gamma^{-1}(K)$ there exists a neighbourhood W_z (which we can assume that $W_z \overset{\theta}{\cong} \overline{\Delta}^n$, the closed unit polydisc in $\mathbf{C}^n$) such that

$$\theta^{-1}\big(\overline{\Delta}^{n-1} \times \partial\Delta\big) \cap \gamma^{-1}(S) = \emptyset.$$

From the maximum principle we have, with $W_z^1 = \theta^{-1}\big(\overline{\Delta}^{n-1} \times \partial\Delta\big)$,

$$\sup_{W_z} |g| = \sup_{W_z^1} |g| \quad \text{for} \quad g \in H(Z).$$

This yields the existence of $W_j = W_{z_j}$ and $W_j^1 = W_{z_j}^1$ $(j = 1, \dots, m)$ such that

$$\gamma^{-1}(K) \subseteq \bigcup_{j=1}^{m} W_j$$

and

$$\sup_{W_j} |g| \le \sup_{W_j^1} |g| \quad \text{for} \quad g \in H(Z) \quad \text{and} \quad j = 1, \dots, m.$$

Put

$$L = \gamma\Big(\bigcup_{j=1}^{m} W_j^1\Big).$$

This is a compact set in X for which

$$\sup_{K} |f| \le \sup_{L} |f| \quad \text{for} \quad f \in H(X)$$

and

$$L \cap S = \gamma\Big(\bigcup_{j=1}^{m} W_j^1 \cap \gamma^{-1}(S)\Big) = \emptyset.$$

(ii) Assume that $H(X) \in (DN)$. Consider the normalization $\gamma : \tilde{X} \to X$ of X. By Theorem 1.1, $H(\tilde{X}) \in (DN)$ and hence by Theorem 2.2 every plurisubharmonic function on $\tilde{X}$ which is bounded from above is constant on every irreducible branch of $\tilde{X}$. By the Steiness and the normality of $\tilde{X}$ each irreducible branch of $\tilde{X} \setminus \gamma^{-1}(S)$ is extended to an irreducible branch of $\tilde{X}$ and each $\varphi \in PSH\big(\tilde{X} \setminus \gamma^{-1}(S)\big)$ is extended to a $\tilde{\varphi} \in PSH(\tilde{X})$. Hence each $\varphi \in PSH\big(\tilde{X} \setminus \gamma^{-1}(S)\big)$ which is bounded from above is constant on every irreducible branch of $\tilde{X} \setminus \gamma^{-1}(S)$.

Let $\{K_p\}$ be an increasing exhaustion sequence of compact sets in $\tilde{X} \setminus \gamma^{-1}(S)$ with $\mathrm{Int}K_1 \neq \emptyset$. For each $p \geq 1$ and $z \in \mathrm{Int}K_p$ put

$$\omega_p(z) = \limsup_{z' \to z} \Big\{ \sup \varphi(z') : \varphi \in PSH\big(\tilde{X} \backslash \gamma^{-1}(S)\big),\ \varphi\big|_{K_1} \leq 0,\ \varphi\big|_{K_p} \leq 1 \Big\}$$

and

$$\omega_p(z) = 0 \quad \text{for} \quad z \in \big(\tilde{X} \setminus \gamma^{-1}(S)\big) \setminus \mathrm{Int}K_p.$$

Then ω_p are plurisubharmonic functions on $\tilde{X} \setminus \gamma^{-1}(S)$ and they are decreasing to $\omega \in PSH\big(\tilde{X} \setminus \gamma^{-1}(S)\big)$. Since ω_p are bounded from above on $\tilde{X} \setminus \gamma^{-1}(S)$ and $\mathrm{Int}K_1 \neq \emptyset$ we have

$$\omega_p \equiv 0 \quad \text{for every} \quad p \geq 1.$$

Hence $\omega \equiv 0$.

Given $q \geq 1$ and $0 < \mu < 1$. By Hartogs Lemma there exists $r(q, \mu)$ such that

$$\omega_r(z) = \omega(K_r, K_1, z) \leq \mu \quad \text{for} \quad z \in K_q.$$

It follows that

$$\frac{\log\Big(\dfrac{|f(z)|}{\|f\|_1}\Big)}{\log\Big(\dfrac{\|f\|_r}{\|f\|_1}\Big)} \leq \mu$$

for $z \in K_q$ and $f \in H\big(\tilde{X} \setminus \gamma^{-1}(S)\big)$. This means that

$$\|f\|_q \leq \|f\|_r^{\mu}\ \|f\|_1^{1-\mu} \quad \text{for} \quad f \in H\big(\tilde{X} \setminus \gamma^{-1}(S)\big).$$

Hence $H\big(\tilde{X} \setminus \gamma^{-1}(S)\big) \in (DN)$. On the other hand, by the inclusion $H(X \setminus S) \hookrightarrow H\big(\tilde{X} \setminus \gamma^{-1}(S)\big)$ we also have $H(X \setminus S) \in (DN)$.

Conversely, assume that $H(X \setminus S) \in (DN)$. By (i) we have

$$H(X) \hookrightarrow H(X \setminus S).$$

Hence $H(X) \in (DN)$.

4. Proof of Theorem B

Assume that $H(Y) \in (DN)$. Consider the commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{\quad\tilde{\theta}\quad} & \tilde{Y} \\ \tilde{\gamma}\big\downarrow & & \big\downarrow\gamma \\ X & \xrightarrow{\quad\theta\quad} & Y \end{array}$$

where $\gamma : \tilde{Y} \longrightarrow Y$ is the normalization of Y and $Z = X \times {}_{\theta}\tilde{Y}$ the fiber product of X and $\tilde{Y}$ for θ. By Theorem 1.1, $H(\tilde{Y}) \in (DN)$. Since $\tilde{\gamma}$ is proper, it suffices to show that $H(Z) \in (DN)$. Let A denote the critical set of $\tilde{\theta}$. Consider $\tilde{\theta} : A \to \tilde{Y}$. Then $\tilde{\theta}(A) = B$ is a union of analytic sets of dimension which is smaller than $\dim \tilde{Y}$ [4]. Thus $H_{2\dim\tilde{Y}-1}(B) = 0$. Put

$$Z_0 = Z \setminus \tilde{\theta}^{-1}(B) \quad \text{and} \quad \tilde{\theta}_0 = \tilde{\theta}\big|_{Z_0} : Z_0 \to \tilde{Y}_0 = \tilde{Y} \setminus B.$$

First check that Z_0 is open and $\tilde{\theta}_0$ is open. Given $z_0 \in Z_0$. Then we can find neighbourhoods U and V of z_0 and $\tilde{y}_0 = \tilde{\theta}(z_0)$ respectively and a biholomorphism $\alpha : U \to V \times W$ for some complex space W such that the following diagram

$$U \xrightarrow{\quad\alpha\quad} V \times W$$

$$\tilde{\theta}_0 \qquad\qquad\qquad \pi$$

$$V$$

is commutative, where $\pi : V \times W \to V$ is the canonical projection. Then $\tilde{\theta}(U) \cap B = \emptyset$ and hence $V = \tilde{\theta}(U) \subseteq \tilde{Y}_0$. This means that Z_0 and $\tilde{Y}_0$ are open in Z and $\tilde{Y}$ respectively and hence B is a closed pluri-polar set in $\tilde{Y}$. Since $H_{2\dim\tilde{Y}-1}(B) = 0$ and $H(\tilde{Y}) \in (DN)$, by Theorem A, it follows that $H(\tilde{Y}_0) = H(\tilde{Y} \setminus B) \in (DN)$.

On the other hand, since

$$\tilde{\theta}^{-1}(\tilde{y}) = \Big\{(x, \tilde{y}) \in X \times \tilde{Y} : \theta(x) = \gamma(\tilde{y})\Big\} \cong \theta^{-1}\big(\gamma(\tilde{y})\big)$$

we have $\tilde{\theta}^{-1}(\tilde{y})$ is connected and $H(\tilde{\theta}^{-1}(y)) \in (DN)$ for all $\tilde{y} \in \tilde{Y}$. Given $\varphi \in PSH(Z)$ such that φ is bounded from above. Then by Theorem 2.2,

$$\varphi\big|_{\tilde{\theta}^{-1}(\tilde{y})} = \text{const} \quad \text{for} \quad \tilde{y} \in \tilde{Y}.$$

Since $\tilde{\theta}_0$ is open, we can write $\varphi\big|_{Z_0} = \psi\tilde{\theta}_0$ for some $\psi \in PSH(\tilde{Y}_0)$ which is bounded from above.

By $H(\tilde{Y}_0) \in (DN)$ and Theorem 2.2, it follows that ψ is constant on every irreducible branch of $\tilde{Y}_0$. Put

$$\psi^*(\tilde{y}) = \limsup_{\tilde{y}' \to \tilde{y}, \tilde{y}' \in \tilde{Y}_0} \psi(\tilde{y}') \qquad \text{for} \quad \tilde{y} \in \tilde{Y}.$$

Since B is closed pluri-polar, by the normalization of $\tilde{Y}$ we have $\psi^* \in PSH(\tilde{Y})$. Moreover, the intersection of every irreducible branch of $\tilde{Y}$ with $\tilde{Y}_0$ is connected. Hence ψ^* is constant on every irreducible branch of $\tilde{Y}$. For each $z \in Z$, $z' \in Z_0$ put $\tilde{y} = \theta(z)$, $\tilde{y}' = \theta_0(z')$. By $\varphi \in PSH(Z)$ we have

$$\varphi(z) = \limsup_{z' \to z, z' \in Z_0} \varphi(z') = \limsup_{z' \to z, z' \in Z_0} \psi\tilde{\theta}_0 = \limsup_{\tilde{y}' \to \tilde{y}, \tilde{y}' \in \tilde{Y}_0} \psi(\tilde{y}')$$

and hence φ is constant on every irreducible branch of Z. On the other hand, since Z has a finite number of irreducible branches, by Theorem 2.2, it follows that $H(Z) \in (DN)$.

Conversely, assume that $H(X) \in (DN)$. Then every plurisubharmonic function on X which is bounded from above is constant on every irreducible branch of X. Since θ is surjective, this holds for Y. By Theorem 2.2 and since Y has a finite number of irreducible branches we have $H(Y) \in (DN)$.

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