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ON CERTAIN STABLE WEDGE SUMMANDS OF $B(Z/p)^n_+$

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Abstract. By using representation theory and explicit idempotents in the group ring $F_p[GL_n(Z/p)]$, we give a new splitting of $B(Z/p)^n_+$ into $p-1$ stable wedge summands in which the numbers of occurrences of the indecomposable stable wedge summands are known. As a consequence, we find an information on the Cartan matrices of $F_p[GL_n(Z/p)]$ and $F_p[M_n(Z/p)]$. Moreover we point out the occurrence of some indencomposable stable wedge summands of $B(Z/p)^n_+$ in the Campbell-Selick summands.

1. Introduction

Let p be a prime number, P an abelian p-group, and BP_+ its classifying space with a disjoint basepoint. One of the most, significant problems in homotopy theory at present is the problem of finding a stable splitting

$$BP_+ \simeq X_1 \vee X_2 \vee \cdots \vee X_N$$

into wedge summands, completed at p. With Carlsson's solution of the Segal Conjecture [1], this topological problem is reduced to the pure algebraic problem of writing the identity of the Burnside rings as a sum of orthogonal idempotents. It suffices to write the identity of the ring F_p [End(P)] as a sum of orthogonal idempotents [6]. This can be reduced to the special case of an elementary abelian p-group [8]. Harris and Kuhn have given a splitting of $B(Z/p)^n_+$ into indecomposable stable wedge summands which is equivalent to a decomposition of 1 into primitive orthogonal idempotents in $F_p[M_n(Z/p)]$ [8]. This splitting is finest, but in general most of the idempotents have not yet been known explicitly. Campbell and Selick have given a natural splitting of $H^*(B(Z/p)^n; F_p)$ into a direct sum of (p^n-1) modules over the Steenrod algebra $\mathcal{A}$ [3]. Then $B(Z/p)^n_+$ is splitted into (p^n-1) stable wedge summands that are called Campbell - Selick summands. Harris has pointed out the existence of the corresponding idempotents in $F_p[G]$, where G is a certain subgroup of

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$GL_n(Z/p)$ [7], but they also have not been described explicitly. Also in [7], Harris gives a splitting of $B(Z/p)^n_+$ into stable wedge summands by constructing explicitly the primitive orthogonal idempotents in $F_p[F^*_{p^n}]$. However, the numbers of occurrences of the indecomposable summands in these summands are very difficult to determine.

In this paper, we construct explicitly the primitive orthogonal idempotents which sum to 1 in $F_p[F^*_p]$. From that we obtain a stable splitting of $B(Z/p)^n_+$ into $p-1$ wedge summands in which the multiplicities of the indecomposable summands in these summands are known. As a consequence, we find an information on the Cartan matrices of $F_p[GL_n(Z/p)]$ and $F_p[M_n(Z/p)]$. Moreover, we describe the occurrence of some stable wedge summands of $B(Z/p)^n_+$ in the Campbell-Selick summands. In particular, when $p = 2$, every Campbell-Selick summand contains a copy of the Steinberg summand.

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2. Irreducible representations of $M_n(Z/p)$, $GL_n(Z/p)$ and $F^*_{p^n}$ over F_p

In [8], Harris and Kuhn follow the constructions of the irreducible representations of $F_p[M_n(Z/p)]$ and $F_p[GL_n(Z/p)]$ as given by James and Kerber in chapter 8, particularly Exercise 8.4 of [9].

A nonincreasing sequence $\alpha = (\alpha_1, \alpha_2, \dots , \alpha_n)$ of nonnegative integers whose sum is m is called a partition of m. This partition α can be illustrated by the corresponding Young diagram $[\alpha]$, which consists of m nodes x placed in rows. The i th row of $[\alpha]$ consists of α_i nodes, $1 \leq i$, and all the rows start in the same column. The lengths α'_i of the columns of $[\alpha]$ form another partition $\alpha' = (\alpha'_1, \alpha'_2, \dots)$ of m. This partition α' is called the partition associated with α. An α-tableau arises from $[\alpha]$ by replacing the nodes x of $[\alpha]$ by the points i of $\{1, \dots , m\}$. A generalized Young tableau of shape $[\alpha]$, and content $(\beta_1, \beta_2, \dots)$ arises from $[\alpha]$ by replacing the nodes of the diagram by positive integers in such a way that the integer i occurs exactly β_i times. A generalized Young tableau is said to be semistandard if the numbers are nondecreasing along each row and strictly increasing down each column.

Let W be the n-dimensional vector space over the arbitrary field F with basis $w_1, w_2, \dots , w_n$ on which $GL_n(F)$ acts in the natural way. Let $L^{(m)}$ denote the m-fold tensor product of W. $GL_n(F)$ acts on $L^{(m)}$ by the diagonal action and S_m acts on $L^{(m)}$ by place permutations of the subscripts.

For a given partition $\alpha = (\alpha_1, \dots, \alpha_n)$ of m, let

$$w^\alpha = w_1 \otimes \cdots \otimes w_{\alpha'_1} \otimes w_1 \otimes \cdots \otimes w_{\alpha'_2} \otimes \cdots \otimes w_1 \otimes \cdots \otimes w_{\alpha'_k},$$

$$\mathcal{V}^\alpha = \sum_{\pi \in S_{\alpha'_1} \times \cdots \times S_{\alpha'_k}} (\mathrm{sgn}\pi)\pi,$$

where $(\alpha'_1, \dots, \alpha'_k)$ is the partition associated with α. If $\mathrm{char}F = 2$ and α is 2-singular (i.e., $\alpha_{i+1} = \alpha_{i+2} > 0$ for some $i \geq 0$), let

$$W^\alpha = \left\{ w \in L^{(m)} | sw = 0 \,\text{for all}\, s \in FS_m \,\text{such that}\, s\mathcal{V}^\alpha w^\alpha = 0 \right\} \cap \mathcal{V}^\alpha L^{(m)}.$$

In all other cases, let

$$W^\alpha = \left\{ w \in L^{(m)} \mid sw = 0 \,\text{for all}\, s \in FS_m \,\text{such that}\, s\mathcal{V}^\alpha w^\alpha = 0 \right\}.$$

Since the action of $GL_n(F)$ commutes with the action of S_m on $L^{(m)}$, W^α is a $GL_n(F)$-module. It is called the Weyl module associated to α.

With each α-tableau T (in general containing repeated entries), we associate a tensor w_T in $L^{(m)}$ as follows. Let $T(1), T(2), \dots, T(m)$ be the entries in T, reading down successive columns. Then define $W_T = w_{T(1)} \otimes w_{T(2)} \otimes \cdots \otimes w_{T(m)}$.

Two tableaux T_1 and T_2 are said to be row equivalent if T_2 can be obtained from T_1 by permuting the order of appearance of the numbers in each row of T_1.

If T is an α-tableau, let

$$E_T = \mathcal{V}^\alpha \sum \left\{ w_{T_1} \mid T_1 \text{ is row-equivalent to } T \right\}.$$

Then $\{E_T \| T$ is a semistandard α-tableau $\}$ is a basis for W^α ([9], 8.1.16).

On W^α there exists a bilinear form ϕ^α such that $\phi^\alpha(mx, y) = \phi^\alpha(x, m^t y)$ for $m \in M_n(F)$, and $x, y \in W^\alpha (m^t$ is the transpose of m). Let

$$W^\alpha_\perp = \left\{ w \in W^\alpha \mid \phi^\alpha(w, v) = 0 \text{ for all } v \in W^\alpha \right\}.$$

Then $W^\alpha_\perp$ is a $M_n(F)$-module. Let $\lambda_i = \alpha_i - \alpha_{i+1}$ with $1 \leq i \leq n-1$ and $\lambda_n = \alpha_n$. Then we have the sequence $\lambda = (\lambda_1, \dots, \lambda_n)$ and we can write $S_{(\lambda_1, \dots, \lambda_n)}$ or $S_{(\lambda)}$ instead of $W^\alpha / W^\alpha_\perp$. Conversely, for each sequence

$\lambda = (\lambda_1, \dots, \lambda_n)$ of nonnegative integers, there exists a unique sequence $\alpha = (\alpha_1, \dots, \alpha_n)$ satisfying $\alpha_i - \alpha_{i+1} = \lambda_i$ $(1 \le i \le n-1)$ and $\alpha_n = \lambda_n$. Then α is a partition of m_λ, where $m_\lambda = \lambda_1 + 2\lambda_2 + \dots + n\lambda_n$. Let

$$\Lambda = \Big\{(\lambda_1, \dots, \lambda_n) \,\|\, 0 \le \lambda_k \le p-1,\ 1 \le k \le n\Big\},$$
$$\Lambda' = \Big\{(\lambda_1, \dots, \lambda_n) \,\|\, 0 \le \lambda_k \le p-1,\ 1 \le k \le n-1,\ 0 \le \lambda_n \le p-2\Big\}.$$

We obtain the following results in the case $F = F_p$.

Theorem 2.1 [8, §6]**.** *Let* $S'_{(\lambda)} = Res^{M_n}_{GL_n}(S_{(\lambda)})$. *Then*

$$\mathrm{Irr}(F_p[M_n(Z/p)]) = \big\{S_{(\lambda)} \mid \lambda \in \Lambda\big\},$$
$$\mathrm{Irr}(F_p[GL_n(Z/p)]) = \big\{S'_{(\lambda)} \mid \lambda \in \Lambda'\big\}.$$

Denote the stable summand of $B(Z/p)^n_+$ corresponding to $S_{(\lambda)}$ (resp. $S'_{(\lambda)}$) by $X_{(\lambda)}$ (resp. $X'_{(\lambda)}$). Particularly, let $M(n)$ be the summand corresponding to the Steinberg module $S'_{(p-1,\dots,p-1,0)}$ (it is also called the Steinberg summand). Then $M(n) \simeq L(n) \vee L(n-1)$, where $L(n) = \sum^{-n} SP^{p^n}(S^0)/SP^{p^{n-1}}(S^0)$ and $SP^k(S)$ denotes the kth symmetric product of the sphere spectrum S [11].

Theorem 2.2 [8, §6]**.**

(i) $B(Z/p)^n_+ \simeq \bigvee_{\lambda \in \Lambda} \dim S_{(\lambda)} X_{(\lambda)}$, $B(Z/p)^n_+ \simeq \bigvee_{\lambda \in \Lambda'} \dim S'_{(\lambda)} X'_{(\lambda)}$,

(ii) $X_{(\lambda_1,\dots,\lambda_{n-1},0)} \simeq X_{(\lambda_1,\dots,\lambda_{n-1})}$,

(iii) $X'_{(\lambda_1,\dots,\lambda_n)} \simeq X_{(\lambda_1,\dots,\lambda_n)}$ if $0 < \lambda_n < p-1$,

(iv) $X'_{(\lambda_1,\dots,\lambda_{n-1},0)} \simeq X_{(\lambda_1,\dots,\lambda_{n-1},0)} \bigvee X_{(\lambda_1,\dots,\lambda_{n-1},p-1)}$.

Corollary 2.3.

(i) $\dim S'_{(\lambda_1,\dots,\lambda_n)} = \dim S_{(\lambda_1,\dots,\lambda_n)}$ if $1 \le \lambda_n \le p-2$, and

$\dim S'_{(\lambda_1,\dots,\lambda_{n-1},0)} = \dim S_{(\lambda_1,\dots,\lambda_{n-1},0)} = \dim S_{(\lambda_1,\dots,\lambda_{n-1},p-1)}$,

(ii) $\dim S_{(\lambda_1,\dots,\lambda_k)} \le \dim S_{(\lambda_1,\dots,\lambda_k,0,\dots,0)}$.

Proof. (i) follows from Theorem 2.2 and the Krull-Schmidt Theorem. (ii) follows from 8.1.16 and 8.3.7 of [9].

In F_{p^n} choose an element ω such that ω generates the cyclic group of units in F_{p^n} and $\{\omega, \phi(\omega), \dots, \phi^{n-1}(\omega)\}$ forms a basis for F_{p^n} over F_p [4], where $\phi(a) = a^p$ is the Frobenius. Let $p(x) = a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + x^n$ be the minimal polynomial for ω. Let

$$\theta = \begin{pmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & \dots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -a_{n-1} \end{pmatrix}$$

be the $n \times n$ matrix over F_p representing multiplication by ω in the basis $\{1, \omega, \dots, \omega^{n-1}\}$. Since ω is a generator of $F^*_{p^n}$, we see that θ has order $p^n - 1$ in $GL_n(Z/p)$. Therefore we can consider

$$F^*_{p^n} \quad = \quad <\theta> \quad \subseteq GL_n(Z/p).$$

Since $F^*_{p^n}$ is abelian and p does not divide the order of $F^*_{p^n}$, there are $p^n - 1$ distinct one-dimensional representations of $F^*_{p^n}$ defined over F_{p^n}. Label them by R_j, $j \in Z/(p^n - 1)$, with $R_j(\theta) = \omega^j$. Explicit idempotents in $F_{p^n}[F^*_{p^n}]$ associated to these are $e_j = -\sum_{k=0}^{p^n-2} \omega^{-kj}\theta^k$ [2, 33.8].

We let $Z/n = <\phi>$ act on $Z/(p^n - 1)$ by $\phi(i) = ip$. Let J_i be the orbit containing i, and let I be a set consisting of one element from each orbit. The cardinality of J_i is z_i, where z_i is the smallest positive exponent k with $ip^k \equiv i \pmod{p^n - 1}$. By defining $f_i = \sum_{j \in J_i} e_j$, with $i \in I$, Harris has proved that for each $i \in I$, f_i is an idempotent in $F_p[F^*_{p^n}]$ and $\{F_p[F^*_{p^n}]f_i \mid i \in I\}$ is a full set of irreducible representations $F^*_{p^n}$ over F_p [7, 3.5].

Remark 2.4. In [3] Campbell and Selick give a very natural splitting of $H^*(B(Z/p)^n; F_p)$ into a direct sum of $(p^n - 1)$ modules over the mod-p Steenrod algebra $\mathcal{A}$, called the weight summands, $M_n(j)$ when $p = 2$ and $ME_n(j)$ when $p > 2$ for $j \in Z/(p^n - 1)$. These weight summands give a splitting of $B(Z/p)^n_+$ into $(p^n - 1)$ stable wedge summands for which Harris uses the notation $Y_n(j)$ for $j \in Z/(p^n - 1)$, where $Y_n(j) \simeq Y_n(j')$ for all $j, j' \in J_i$, and he calls them the Campell-Selick summands [7]. Let $\widehat{Y}_n(i) = \bigvee_{j \in J_i} Y_n(j)$. Then $\widehat{Y}_n(i) \simeq f_i B(Z/p)^n_+$ [7, 4.4] and

$$\widehat{Y}_n(i) \simeq \bigvee_{\lambda \in \Lambda'} z_i a'_{\lambda i} X'_{(\lambda)},$$

where $a'_{\lambda i}$ is the number of times the representation $F_p[F^*_{p^n}]f_i$ occurs in a composition series for $\mathrm{Res}^{GL_n}_{F^*_{p^n}}(S'_{(\lambda)})$ [7, 4.6].

3. Main results

Definition 3.1. For $i = 0, 1, \dots, p-2$, let $g_i = \sum\limits_{j \equiv i (\mathrm{mod} p-1)} e_j$.

Proposition 3.2.

(i) $g_i \in F_p[F^*_p]$.

(ii) $\{F_p[F^*_p]g_i \mid 0 \le i \le p-2\}$ *is a full set of irreducible representations of* F^*_p *over* F_p.

Proof. (i) We have $g_i = -\sum\limits_{k=1}^{p^n-1} \left(\sum\limits_{l=0}^{q-1} \omega^{-k(i+l(p-1))} \right) \theta^k$, where $q = \dfrac{p^n-1}{p-1}$ $(\equiv 1 (\mathrm{mod} p))$, and

$$\sum_{l=0}^{q-1} \omega^{-k(i+l(p-1))} = \begin{cases} 0 & \text{if } \ q \not| \, k, \\ q\omega^{-ki} & \text{if } \ q \mid k. \end{cases}$$

Hence $g_i = -\sum\limits_{j=1}^{p-1} \omega^{-qij}\theta^{qj}$. Since $\omega^q \in F_p$ and $\theta^q \in F^*_p$, we have $g_i \in F_p[F^*_p]$.

(ii) It is clear that the elements g_i are orthogonal idempotents which sum to the identity and $F_p[F^*_p]$ has $p-1$ distinct one dimensional representations.

Remark 3.3. Let $Z_n(i) = g_i B(Z/p)^n_+$. Then

$$B(Z/p)^n_+ \simeq \bigvee_{0 \le i \le p-2} Z_n(i) \ ,$$

$$Z_n(i) \simeq \bigvee_{\substack{j \in I \\ j \equiv i (\mathrm{mod} p-1)}} \widehat{Y_n}(j) \simeq \bigvee_{\lambda \in \Lambda'} \bigvee_{\substack{j \in I \\ j \equiv i (\bmod p-1)}} z_j a'_{\lambda j} X'_{(\lambda)}$$

$$\simeq \bigvee_{\lambda \in \Lambda'} \Big(\sum_{\substack{j \in I \\ j \equiv i (\bmod p-1)}} z_j a'_{\lambda j} \Big) X'_{(\lambda)}.$$

Lemma 3.4. *The eigenvalues of* θ *acting on the vector space* $F_p[F^*_{p^n}]f_i$ *are*

$$\big\{ \omega^i, \omega^{ip}, \dots, \omega^{ip^{s_i-1}} \big\}.$$

Proof. The action of θ on $F_p[F^*_{p^n}]f_i$ has ω^j as an eigenvalue if and only if $F_{p^n}[F^*_{p^n}]e_j$ is a composition factor of $F_{p^n}[F^*_{p^n}]f_i$. This happens if and only if $e_jf_i \neq 0$ [2, 54.12]. And since the e_j are orthogonal, this happens if and only if $j \in J_i$.

Lemma 3.5. *The eigenvalues for the action of θ on the Weyl module W^α are $\omega^{\beta(T)}$ where T is a semistandard α-tableau of content $(\beta_1, \dots, \beta_n)$ and $\beta(T) \equiv \Big(\sum_{k=1}^{n} p^{k-1}\beta_k\Big)$ $(\mathrm{mod}\, p^n - 1)$.*

Proof. Since the eigenvalues of θ are $\omega, \omega^p, \dots, \omega^{p^{n-1}}$, the eigenvalues for the actions of θ and $\mathrm{diag}(\omega, \omega^p, \dots, \omega^{p^{n-1}})$ are the same. The lemma follows from [9, 8.1.18].

Lemma 3.6. *The eigenvalues for the action of θ on $S_{(\lambda)}$ are ω^j with $j \equiv m_\lambda($ mod $p-1)$.*

Proof. Let $\alpha = (\alpha_1, \dots, \alpha_n)$ be the partition of m_λ. By Lemma 3.5 we see that the eigenvalues for the action of θ on W^α is $\omega^{\sum_{k=1}^{n} p^{k-1}\beta_k}$. Here $m_\lambda = \sum_{k=1}^{n} \beta_k \equiv \Big(\sum_{k=1}^{n} p^{k-1}\beta_k\Big)($ mod $p-1)$. The conclusion follows from the fact that S_λ is a composition factor of W^α.

Theorem 3.7.

$$Z_{n,1}(i) \simeq \bigvee_{\substack{\lambda\in\Lambda' \\ m_\lambda \equiv i (\mathrm{mod}\, p-1)}} \big(\dim S'_{(\lambda)}\big) X'_{(\lambda)} \ ,$$

$$Z_{n,1}(i) \simeq \bigvee_{\substack{\lambda\in\Lambda \\ m_\lambda \equiv i (\mathrm{mod}\, p-1)}} \big(\dim S_{(\lambda)}\big) X_{(\lambda)} \ .$$

Proof. By Remark 3.3, $X'_{(\lambda)}$ is a summand of $Z_n(i)$ if and only if there exists $j \in I$ such that $j \equiv i$ (mod $p-1$) and $a'_{\lambda j} \neq= 0$. By Lemma 3.4, Lemma 3.6 and Remark 2.4, $\{\omega^j, \dots, \omega^{jp^{z_j - 1}}\} \subseteq \{\omega^k \mid k \equiv m_\lambda($ mod $p-1)\}$ and $j \equiv i$ (mod $p-1$) for some $j \in I$. It follows that $m_\lambda \equiv i$ (mod $p-1$). Hence

$$\sum_{\substack{v\in I \\ v\equiv i (\mathrm{mod}\, p-1)}} z_v a'_{\lambda v} = \begin{cases} \dim S'_{(\lambda)} \text{ if } m_\lambda \equiv i(\text{ mod } p-1), \\ 0 \text{ otherwise .} \end{cases}$$

The proof of the first part is completed. The second part is a consequence of Theorem 2.2, Lemma 2.3 and the first part.

Corollary 3.8.

$$F_p[GL_n(Z/p)]g_i \cong \bigoplus_{\substack{\lambda\in\Lambda' \\ m_\lambda\equiv i(\bmod\ p-1)}} \big(\dim S'_{(\lambda)}\big)P'_{(\lambda)} ,$$

$$F_p[M_n(Z/p)]g_i \cong \bigoplus_{\substack{\lambda\in\Lambda \\ m_\lambda\equiv i(\bmod\ p-1)}} \big(\dim S_{(\lambda)}\big)P_{(\lambda)} ,$$

where $P'_{(\lambda)}$ (resp. $P_{(\lambda)}$) is the projective cover of $S'_{(\lambda)}$ (resp. $S_{(\lambda)}$).

Proof. The first part is immediate from Theorem 3.7 and [7, 3.8]. The second part follows Theorem 3.7 and the Krull-Schmidt Theorem.

Corollary 3.9. *For any $1 \leq k \leq n$ and $0 \leq i \leq p-2, Z_n(i)$ contains a copy of the summand $Z_k(i)$.*

Proof. This follows from Theorem 3.7, Theorem 2.2 (ii) and Lemma 2.3 (ii).

Lemma 3.10. *The eigenvalues for the actions of θ on $F_p[GL_n(Z/p)]g_i$ and $F_p[M_n(Z/p)]g_i$ are ω^j with $j \equiv i$ (mod $p-1$).*

Proof. ω^j is an eigenvalue for the action of θ on $F_p[GL_n(Z/p)]g_i$ if and only if $F_{p^n}[F^*_{p^n}]e_j$ is a composition factor of $F_{p^n}[GL_n(Z/p)]g_i$ as a $F^*_{p^n}$-module. This is equivalent to $e_j(F_{p^n}[GL_n(Z/p)]g_i) \neq 0$ [2, 54.12], which holds if and only if $e_jg_i \neq 0$ (since g_i belongs to the center of $F_{p^n}[M_n(Z/p)]$) i.e., $j \equiv i$ (mod $p-1$).

Theorem 3.11. *If $S'_{(\mu)}$ (resp. $S_{(\mu)}$) is a composition factor of $P'_{(\lambda)}$ (resp. $P_{(\lambda)}$), then $m_\mu \equiv m_\lambda$ (mod $p-1$).*

Proof. $P'_{(\lambda)}$ (resp. $P_{(\lambda)}$) is a summand of $F_p[GL_n(Z/p)]g_i$ (resp. $F_p[M_n(Z/p)]g_i$), where $i \equiv m_\lambda$(mod $p-1$) (3.8). Thus, By Lemma (3.10) the eigenvalues for the action of θ on $P'_{(\lambda)}$ (resp. $P_{(\lambda)}$) are ω^j with $j \equiv m_\lambda$ (mod $p-1$). Hence by Lemma 3.6, if $S_{(\mu)}$ is a composition factor of $P'_{(\lambda)}$ (resp. $P_{(\lambda)}$), then $m_\mu \equiv m_\lambda$ (mod $p-1$).

Theorem 3.11 means that the entries $c_{\mu\lambda}$ of the Cartan matrices for $F_p[M_n(Z/p)]$ and $F_p[GL_n(Z/p)]$ are zero when $m_\mu \not\equiv m_\lambda$(mod $p-1$) ([5]).

Example 3.12. For $p = 3$:

$$
\begin{aligned}
Z_1(0) &\simeq X'_{(0)},\\
Z_1(1) &\simeq X'_{(1)},\\
Z_2(0) &\simeq X'_{(0,0)} \vee X'_{(0,1)} \vee 3X'_{(2,0)} \vee 3X'_{(2,1)},\\
Z_2(1) &\simeq 2X'_{(1,0)} \vee 2X'_{(1,1)},\\
Z_3(0) &\simeq X'_{(0,0,0)} \vee 3X'_{(0,1,0)} \vee 3X'_{(1,0,1)} \vee 6X'_{(2,0,0)} \vee 6X'_{(0,2,0)}\\
&\qquad \vee 7X'_{(1,1,1)} \vee 15X'_{(2,1,0)} \vee 15X'_{(1,2,1)} \vee 27X'_{(2,2,0)},\\
Z_3(1) &\simeq X'_{(0,0,1)} \vee 3X'_{(0,1,1)} \vee 3X'_{(1,0,0)} \vee 6X'_{(2,0,1)} \vee 6X'_{(0,2,1)}\\
&\qquad \vee 7X'_{(1,1,0)} \vee 15X'_{(2,1,1)} \vee 15X'_{(1,2,0)} \vee 27X'_{(2,2,1)}.
\end{aligned}
$$

Example 3.13. For $p = 5$, we have:

$$
\begin{aligned}
Z_1(0) &\simeq X'_{(0)}, \quad Z_1(1) \simeq X'_{(1)},\\
Z_1(2) &\simeq X'_{(2)}, \quad Z_1(3) \simeq X'_{(3)},\\
Z_2(0) &\simeq X'_{(0,0)} \vee X'_{(0,2)} \vee 3X'_{(2,3)} \vee 3X'_{(2,1)} \vee 5X'_{(4,0)} \vee 5X'_{(4,2)},\\
Z_2(1) &\simeq 2X'_{(1,0)} \vee 2X'_{(1,2)} \vee 4X'_{(3,1)} \vee 4X'_{(3,3)},\\
Z_2(2) &\simeq X'_{(0,1)} \vee X'_{(0,3)} \vee 3X'_{(2,0)} \vee 3X'_{(2,2)} \vee 5X'_{(4,1)} \vee 5X'_{(4,3)},\\
Z_2(3) &\simeq 2X'_{(1,1)} \vee 2X'_{(1,3)} \vee 4X'_{(3,0)} \vee 4X'_{(3,2)}.
\end{aligned}
$$

For $1 \le k \le n$, let $S(k)$ denote the irreducible $F_p[M_n(Z/p)]$-representation $S_{(0,\dots,0,1,0,\dots,0)}$, where 1 is in the k-th position. Let $S(0) = S_{(0,\dots,0)}$. For $0 \le k \le n$, let $X(k)$ denote the indecomposable wedge summand of $B(Z/p)^n_+$ corresponding to $S(k)$.

Let $j = (j_{n-1}, j_{n-2}, \dots, j_0)$ be the base-p representation of j, let $\sigma(j) = j_0 + \cdots + j_{n-1}$, and let $\alpha(j)$ be the cardinality of $\{k \mid j_k \neq 0\}$. Note that $\sigma(j) = \alpha(j)$ when $p = 2$.

Theorem 4.11 in [7] is a special case ($p = 2$) of the following theorem.

Theorem 3.14. *For $0 \le j \le p^n - 2$, $Y_n(j)$ contain exactly one copy of the summand $X(k)$ if and only if $k = \alpha(j) = \sigma(j)$. Also, $Y_n(0)$ contains a copy of $X(n)$.*

Proof. $S(k)$ corresponds to the partition $\alpha = (\alpha_1, \dots, \alpha_n)$, where $\alpha_1 = \ldots = \alpha_k = 1$, and $\alpha_{k+1} = \ldots = \alpha_n = 0$. From [9, 8.3.9] we have $W_\perp^\alpha = 0$. Therefore,

$$\left\{\omega^{p^{i_1-1}+\ldots+p^{i_k-1}} \mid 1 \le i_1 < \ldots < i_k \le n\right\} = \left\{\omega^j \mid \alpha(j) = \sigma(j) = k\right\}$$

is the set of the Eigenvalues of θ acting on $S(k)$ (3.5). The theorem then follows from Remark 2.4 and Lemma 3.4.

Theorem 3.15. *For $0 \le j \le p^n - 2$, and $0 \le k \le p-1$, $Y_n(j)$ contains exactly one copy of the summand $X_{(0,\dots,0,k)}$ if and only if* $j = \dfrac{k(p^n-1)}{p-1}$.

Proof. $S_{(0,\dots,0,k)}$ is exactly $(\det)^k$, where $\det : F_p[M_n(Z/p)] \longrightarrow F_p$ is the determinant representation [8, 6.2]. Hence the Eigenvalue of θ acting on $\det^k$ is $\left(\omega^{\sum_{i=0}^{n-1} p}\right)^k = \omega^{\frac{k(p^n-1)}{p-1}}$. The conclusion then follows from Remark 2.4 and Lemma 3.4.

Theorem 3.16. *Let $p = 2$ and $n \ge 3$. For each $j \in Z/(p^n - 1)$, $Y_n(j)$ contains at least one copy of the Steinberg summand $M(n)$.*

Proof. The partition corresponding to $(\lambda) = (1, \dots, 1, 0)$ is $\alpha = (n-1, n-2, \dots, 1, 0)$. From [9, 8.1.17, 2.3.19] we have $\dim W^\alpha = \prod_{i=1}^{n-1} f(i)$, where

$$f(i) = \prod_{j=1}^{n-i} (n-i+j)/(2n-2i-2j+1) =$$
$$= \frac{n-i+1}{2(n-i)-1}\frac{n-i+2}{2(n-i)-3}\cdots\frac{2(n-i)-1}{3}\frac{2(n-i)}{1}.$$

By induction, we obtain $f(i) = 2^{n-i}$. Then $\dim W^\alpha = \prod_{i=1}^{n-1} 2^{n-i} = 2^{\frac{n(n-1)}{2}} = \dim S_{(1,\dots,1,0)}$, where the last equality is implied from 2.3 and [10, 6.5]. Hence it is enough to show that for each $j \in Z/(2^n - 1)$, there exists a semistandard α-tableau T of content $(\beta_1, \dots, \beta_n)$ such that $\sum_{k=1}^{n} 2^{k-1}\beta_k = j + (2^n - 1)$ (Theorem 2.2, Lemma 3.4). We first consider the case $j = 0$. We can construct T by induction on n. Suppose that there exists an α-tableau T. Let $\gamma_{ij} (1 \le j \le n-i, 1 \le i \le n-1)$ denote the entry of T which lies in the i th row and j th column. Adding 1 to T, we can find another α-tableau T_1. Let T' be the tableau obtained from T_1 by adding the column $(1, 2, \dots, n)^t$ to the left of T_1. Then T' is a semistandard α'- tableau of content $(\beta'_1, \dots, \beta'_{n+1})$, where $\alpha' = (n, n-1, \dots, 1, 0)$ and

$$\sum_{k=1}^{n+1} 2^{k-1}\beta'_k = \sum_{k=1}^{n} 2^{k-1}\beta_k + 1 + (2^n - 1) = 2^{n+1} - 1.$$

For example,

$$n=3: \begin{matrix} 1 & 3 \\ 2 & \end{matrix}\ , \quad n=4: \begin{matrix} 1 & 2 & 2 \\ 2 & 3 & \\ 3 & & \end{matrix}\ , \quad n=5: \begin{matrix} 1 & 1 & 1 & 3 \\ 2 & 2 & 3 & \\ 3 & 3 & & \\ 4 & & & \end{matrix}\ ,$$

$$n=6: \begin{matrix} 1 & 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 3 & \\ 3 & 3 & 3 & & \\ 4 & 4 & & & \\ 5 & & & & \end{matrix}\ , \quad n=7: \begin{matrix} 1 & 1 & 1 & 1 & 1 & 2 \\ 2 & 2 & 2 & 2 & 4 & \\ 3 & 3 & 3 & 3 & & \\ 4 & 4 & 4 & & & \\ 5 & 5 & & & & \\ 6 & & & & & \end{matrix}\ ,$$

$$n=8: \begin{matrix} 1 & 1 & 1 & 1 & 1 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 4 & \\ 3 & 3 & 3 & 3 & 3 & & \\ 4 & 4 & 4 & 4 & & & \\ 5 & 5 & 5 & & & & \\ 6 & 6 & & & & & \\ 7 & & & & & & \end{matrix}\ , \quad n=9: \begin{matrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 3 \\ 2 & 2 & 2 & 2 & 2 & 2 & 4 & \\ 3 & 3 & 3 & 3 & 3 & 3 & & \\ 4 & 4 & 4 & 4 & 4 & & & \\ 5 & 5 & 5 & 5 & & & & \\ 6 & 6 & 6 & & & & & \\ 7 & 7 & & & & & & \\ 8 & & & & & & & \end{matrix}\ ,$$

$$n=10: \begin{matrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 4 & \\ 3 & 3 & 3 & 3 & 3 & 3 & 3 & & \\ 4 & 4 & 4 & 4 & 4 & 4 & & & \\ 5 & 5 & 5 & 5 & 5 & & & & \\ 6 & 6 & 6 & 6 & & & & & \\ 7 & 7 & 7 & & & & & & \\ 8 & 8 & & & & & & & \\ 9 & & & & & & & & \end{matrix}\ .$$

For $n \geq 10$, suppose $\gamma_{1,n-2} = 1$ or 2, $\gamma_{1,n-1} = 2$ or 3, $\gamma_{2,n-2} = 4, \gamma_{i,1} = i$, and $\gamma_{k,n-k} = k$ or $k+1$ for $3 \leq k \leq n-2$. Then the entries γ'_{ij} of T' are chosen as follows.

If $\gamma_{1,n-2} = 1$, then

$$\gamma'_{ij} = \begin{cases} 2 & \text{if } i = 1 \text{ and } j = n-1, \\ i & \text{if } j = 1, \\ \gamma_{ij} & \text{otherwise;} \end{cases}$$

If $\gamma_{1,n-2} = \gamma_{1,n-1} = 2$, then

$$\gamma'_{ij} = \begin{cases} 1 & \text{if } i = 1 \text{ and } j = n-1, \\ 3 & \text{if } i = 1 \text{ and } j = n, \\ i & \text{if } j = 1, \\ \gamma_{ij} & \text{otherwise;} \end{cases}$$

If $\gamma_{1,n-2} = 2$ and $\gamma_{1,n-1} = \gamma_{3,n-3} = 3$, then

$$\gamma'_{ij} = \begin{cases} 1 & \text{if } i = 1 \text{ and } j = n-1, \\ 2 & \text{if } i = 1 \text{ and } j = n, \\ 4 & \text{if } i = 3 \text{ and } j = n-3, \\ i & \text{if } j = 1, \\ \gamma_{ij} & \text{otherwise;} \end{cases}$$

If $\gamma_{1,n-2} = 2, \gamma_{1,n-1} = 3$, and $\gamma_{3,n-3} = 4$, then

$$\gamma'_{ij} = \begin{cases} 1 & \text{if } i = 1 \text{ and } j = n-1, \\ 2 & \text{if } i = 1 \text{ and } j = n, \\ i & \text{if } i \le i_0 \text{ and } j = n-i \text{ or } i \ge i_0 + 2 \\ & \qquad\qquad \text{and } j = n-i \text{ or } j = 1, \\ i_0 + 2 & \text{if } i = i_0 + 1 \text{ and } j = n - i_0 - 1, \\ \gamma_{ij} & \text{otherwise.} \end{cases}$$

where i_0 is the smallest positive integer such that $\gamma_{i_0,n-i_0} = \gamma_{i_0+1,n-i_0-1}$.

Note that we can construct $T_i (i \ge 2)$ from T_1 by adding 1 to T_{i-1} as above. By this way we can prove the remaining cases $1 \le j \le 2^n - 2$.

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