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CHARACTERISTIC FUNCTION FOR RANDOM SETS AND CONVERGENCE OF SUMS OF INDEPENDENT RANDOM SETS

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ABSTRACT. The purpose of this paper is to present a notion of characteristic function for a random set. Some basic properties of this characteristic function are given. Then, we extend classical results on the convergence of independent random variables to the case of independent random sets taking closed and bounded convex values.

0. INTRODUCTION

In this paper, we extend the notion of characteristic function to the case of random sets and we study the convergence of sums of independent random sets.

Section 1 contains the basic definitions and notations.

In Section 2 a new and useful notion of characteristic function for a random set taking values in closed bounded convex subsets of a Banach space with separable topological dual is presented. Stability results, pointwise convergence and Lévy-Cramer convergence theorems for multifunctions are given. The main interest of this characteristic function is that it connects distributions of random sets with distributions of random vectors.

In Section 3, we shall discuss the convergence of sums of independent random sets, with closed bounded convex sets, and we prove for this class of multifunctions that almost sure convergence, convergence in probability and convergence in distribution with respect to the Hausdorff metric are equivalent. Theorem 3.4 is the multivalued version of a classical result in probability theory. The difficulties encountered in this extension come from the fact that the class of closed bounded convex subsets endowed with the Minkowski addition is not a group. In the real case, relations

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between almost sure convergence and convergence in probability were first studied by Lévy [9] and extended to metric group valued random variables by Tortrat [13]. Kahane [8] proved it using Lévy's inequalities. Relations between convergence in probability and convergence in distribution were first studied by Tortrat [13], connecting this result with the convolution equation of measures $\nu^*\nu' = \nu$, and then by Itô and Nisio [7], using characteristic functions.

Before closing this introduction, we also recall that convergence in probability and convergence in distribution of random sets were studied by Salinetti and West [10], [11].

1. Notations

Let (Ω, Σ, P) be a probability space, E a Banach space. Throughout this paper, the topological dual space E^* of E is assumed to be separable with respect to its strong topology.

The set of nonempty closed bounded convex subsets of E will be denoted by $cb(E)$.

For each open subset U of E, we shall set

$$U^- := \Big\{C \in cb(E) : C \cap U \neq \emptyset\Big\}$$

and we shall denote the *Effrös σ-algebra* of $cb(E)$ by $\mathcal{E}$, that is the smallest σ-algebra over $cb(E)$ containing the class $\big\{U^- : U \text{ open in } E\big\}$.

We recall that the *support functional* $\delta^*(., A)$ of $A \in cb(E)$ is the function from E^* to $\mathbb{R}$ defined by

$$\delta^*(x^*, A) = \sup\big\{\langle a, x^*\rangle, a \in A\big\}.$$

A *random set* will be a multifunction $X : \Omega \to cb(E)$ which is measurable with respect to the σ-algebras Σ and $\mathcal{E}$.

If C_1 and C_2 are two elements of $cb(E)$, the *Hausdorff metric* between C_1 and C_2 is given by

$$h(C_1, C_2) = \sup\Big(\big\{d(x, C_1) : x \in C_2\big\} \cup \big\{d(x, C_2) : x \in C_1\big\}\Big).$$

The set $cb(E)$ will be equipped with the Hausdorff metric topology τ_H.

A *strongly random set* is a multifunction $X : \Omega \to cb(E)$ which is measurable with respect to the σ-algebras Σ and $\beta(\tau_H)$, where $\beta(\tau_H)$ denotes the Borel σ-field of $(cb(E), \tau_H)$.

The probability distribution P_x of a random set X is defined by

$$P_x(T) = P[X^{-1}(T)] \quad \text{for every } T \text{ in } \mathcal{E}.$$

Two random sets X_1 and X_2 are independent if and only if, for any S and T in $\mathcal{E}$, we have

$$P(X_1^{-1}(S) \cap X_2^{-1}(T)) = P(X_1^{-1}(S)) \cdot P(X_2^{-1}(T)).$$

For more details about the notions of distribution and independence of random sets, we refer the read to Hess' paper [6].

2. Characteristic function for random sets

In this section, we present a definition of characteristic function or Fourier transform for random sets and probability measures on $(cb(E), \mathcal{E})$.

Let $\ell_0(\mathbb{R})$ denote the space of real sequences with a finite number of non zero terms.

For an integrable random real variable f on (Ω, Σ, P), we denote its *expectation* by

$$\mathbb{E}(f) = \int_\Omega f(\omega)P(d\omega).$$

In this paper $D = \left(z_j^*\right)_{j\geq 1}$ will denote a countable subset of E^* which is norm-dense in the closed unit ball B^* of E^*.

We need to introduce the notations below.

Notations 2.1. Let X be a random set taking values in $cb(E)$. We set for each $u = \left(u_j\right)_{j\geq 1}$ in $\ell_0(\mathbb{R})$

$$\varphi_X^D(U) := \left(\varphi_X^{p,D}(u)\right)_{p\geq 1}$$

with

$$\varphi_X^{p,D}(u) = \mathbb{E}\Big[\exp\Big(i\sum_{j=1}^{p} u_j \delta^*(z_j^*, X)\Big)\Big].$$

For each positive integer p, for every $C \in cb(E)$ and for each probability measure ν on $(cb(E), \mathcal{E})$, we define a mapping Δ_p from $cb(E)$ to $\mathbb{R}^p$ by

$$\Delta_p(C) = \left(\delta^*(z_j^*, C)\right)_{1\leq j\leq p},$$

and we set

$$\varphi_\nu^D(u) := \big(\varphi_\nu^{p,D}(u)\big)_{p\geq 1}$$

with

$$\varphi_\nu^{p,D}(u) = \int_{\mathbb{R}^p} \exp\Big[i\sum_{j=1}^{p} t_j \cdot u_j\Big](\nu\circ\Delta_p^{-1})(dt),$$

where $t = (t_1, \dots, t_p)$.

Here $\nu\circ\Delta_p^{-1}$ denotes the image measure of ν by Δ_p.

$\mathcal{A}_D$ will be the algebra of elements of $cb(E)$ which are the subsets

$$\Delta_n^{-1}(G) = \Big\{C\in cb(E) : \big(\delta^*(z_1^*, C), \dots, \delta^*(z_n^*, C)\big) \in G\Big\}$$

for all $n\in\mathbb{N}^*$ and all borelian subsets G in $\mathbb{R}^n$ and $\mathcal{J}(\mathcal{A}_D)$ will denote the σ-algebra generated by $\mathcal{A}_D$.

Let us recall that, for each $n\geq 1$, the Borel σ-field $\mathcal{B}(\mathbb{R}^n)$ is generated by the subsets $G = B_1\times B_2\times\cdots\times B_n$, where B_j are borelian subsets in $\mathbb{R}$, for j such that $1\leq j\leq n$. Then, for this subset G, we have

$$\Delta_n^{-1}(G) = \bigcap_{j\geq 1}\big\{C\in cb(E)/\delta^*(z_j^*, C)\in B_j\big\}.$$

We start with a result concerning the restriction of the Effrös σ-field to $cb(E)$.

Lemma 2.2. *$\mathcal{J}(\mathcal{A}_D)$ is equal to the Effrös σ-algebra $\mathcal{E}$ of $cb(E)$.*

Proof. For each r in $\mathbb{Q}$ and z^* in E^*, we set

$$W(z^*, r) := \big\{x\in E : \langle z^*, x\rangle \leq r\big\}.$$

Then, we have

$$\big\{C\in cb(E) : \delta^*(z^*, C) > r\big\} = \big[E\setminus W(z^*, r)\big]^-$$

which implies, for each z^* in E^*, the measurability of the map $C\to\delta^*(z^*, C)$ with respect to the σ-algebra $\mathcal{E}$.

Further, for each $n\geq 1$ and for all subset $G = B_1\times B_2\times\cdots\times B_n$ in $\mathcal{B}(\mathbb{R}^n)$, we have

$$\Delta_n^{-1}(G) = \bigcap_{j\geq 1}\big\{C\in cb(E)/\delta^*(z^*, C)\in B_j\big\}\in\mathcal{E}.$$

Then, for all borelian subset G in $\mathbb{R}^n$, we have

$$\Delta_n^{-1}(G) \in \mathcal{E}$$

and this entails the inclusion $\mathcal{J}(\mathcal{A}_D) \in \mathcal{E}$.

Let us show now the inclusion $\mathcal{E} \in \mathcal{J}(\mathcal{A}_D)$.

Using the separability of E, we have the existence of a countable dense subset $\{x_i, i \in \mathbb{N}\}$ of E. As E is a separable Banach space, for each U open subset of E, we can set

$$U = \bigcup_{i\in I} B(x_i, r_i)$$

where I is a countable set and $B(x_i, r_i)$ denotes the open ball of radius r_i, centered at x_i. This equality implies that

$$U^- = \bigcup_{i\in I} \big[B(x_i, r_i)\big]^-.$$

Moreover, as

$$\big[B(x_i, r_i)\big]^- = \big\{C \in cb(E)/d(x_i, C) < r_i\big\},$$

we show that $\big[B(x_i, r_i)\big]^- \in \mathcal{E}$ assuming that, for each x in E, the map $C \to d(x, C)$ is $\mathcal{E}$-measurable. But, for any x in E and C in $cb(E)$, the following equality holds true

$$d(x, C) = \sup\big[\langle z_j^*, x\rangle - \delta^*(z_j^*, C)\,\big|\, j \in \mathbb{N}\big],$$

and we obtain, for each $r > 0$,

$$\begin{aligned}\big\{C \in cb(E)\,\big|\, d(x, C) \le r\big\} &= \bigcap_{j\ge 1} \big\{C \in cb(E)\,\big|\, \langle z_j^*, x\rangle - \delta^*(z_j^*, C) \le r\big\}\\ &= \bigcap_{j\ge 1} \big\{C \in cb(E)\,\big|\, \delta^*(z_j^*, C) \ge \langle z_j^*, x\rangle - r\big\}\end{aligned}$$

Then $x \to d(x, C)$ is $\mathcal{E}$-measurable and the inclusion follows. □

Remarks 2.3. 1) If $\mathcal{A}$ is the algebra of subsets of $cb(E)$ whose elements are the subsets

$$\big\{C \in cb(E) : \big(\delta^*(x_1^*, C), \dots, \delta^*(x_n^*, C)\big) \in G\big\}$$

for all $n \in \mathbb{N}$, all borelian subsets G in $\mathbb{R}^n$ and x_i^* in B^*, and $\mathcal{J}(\mathcal{A})$ is the smallest σ-algebra containing $\mathcal{A}$, one has in the same way

$$\mathcal{J}(\mathcal{A}) = \mathcal{E} = \mathcal{J}(\mathcal{A}_D).$$

2) The above lemma shows that the σ-algebra $\mathcal{A}_D$ is independent of the choice of the countable subset D.

Corollary 2.4. *The following conditions are equivalent:*

(i) *X and Y are identically distributed random sets taking values in $cb(E)$,*

(ii) *For each finite subset J of $\mathbb{N}^*$ and each sequence $\left(z_j^*\right)_{j\in J}$ in D, $\left(\delta^*(z_j^*, X)\right)_{j\in J}$ and $\left(\delta^*(z_j^*, Y)\right)_{j\in J}$ are identically distributed random variables,*

(iii) *For each finite subset J of $\mathbb{N}^*$ and each sequence $\left(x_j^*\right)_{j\in J}$ in B^*, $\left(\delta^*(x_j^*, X)\right)_{j\in J}$ and $\left(\delta^*(x_j^*, Y)\right)_{j\in J}$ are identically distributed random variables.*

Proof. It is well known that two probability measures on a measurable space $(S, \mathcal{S})$ are equal on $\mathcal{S}$ if and only if they are equal on a generating subclass J of $\mathcal{S}$, which is stable under finite intersections. Therefore, (i) $\Leftrightarrow$ (ii) by Lemma 2.2 and (ii) $\Leftrightarrow$ (iii) by Remark 2.3. $\square$

This corollary completes Proposition 8 in [6].

Remark 2.5. The results above also hold for random sets taking weakly compact convex values, without the assumption of separability of E^*. It is well known that E^* is separable with respect to the Mackey topology whenever E is norm-separable ([2] and theorem 0.12 in [14]).

The following proposition is an immediate consequence of Corollary 2.4.

Proposition 2.6. (i) *Let ν and ν' be probability measure on $(cb(E), \mathcal{E})$. Then $\nu = \nu'$ if and only if, for each u in $\ell_0(\mathbb{R})$,*

$$\varphi_\nu^D(u) = \varphi_{\nu'}^D(u).$$

(ii) *Let X and Y be random sets taking values in $cb(E)$. Then $P_X = P_Y$ if and only if, for each u in $\ell_0(\mathbb{R})$,*

$$\varphi_X^D(u) = \varphi_Y^D(u).$$

Remark 2.7. By Proposition 2.6, we note that φ_X^D and φ_ν^D are independent of the choice of the countable dense subset D.

The definition of characteristic function can now be introduced.

Definition 2.8. If X is a random set taking values in $cb(E)$, $\varphi_X = \varphi_X^D$ will be called the characteristic function of X. If ν is a probability measure on $(cb(E), \mathcal{E})$, $\varphi_\nu = \varphi_\nu^D$ will be called the characteristic function of ν.

Remark 2.9. For each $p \in \mathbb{N}^*$, each random set X taking values in $cb(E)$ and each $u = \big(u_j\big)_{j\geq 1}$ in $\ell_0(\mathbb{R})$, we have with Notations 2.1 and $\varphi_X^p := \varphi_X^{p,D}$:

$$\varphi_X^p(u) = \phi_{(\delta^*(z_j^*, X))_{1\leq j\leq p}}(u_1, \dots, u_p)$$

where ϕ is the usual characteristic function of random vectors. This links Definition 2.8 with the classical definition of characteristic function.

Now we verify the stability by convolution of this function and several classical properties.

Proposition 2.10. *Let X and Y be independent random sets taking values in $cb(E)$ and let*

$$cl(X+Y)(\omega) = cl\big\{a + b : a \in X(\omega), b \in Y(\omega)\big\}$$

for each ω in Ω. Then, for each u in $\ell_0(\mathbb{R})$,

$$\varphi_{cl(X+Y)}(u) = \big(\varphi_X^p(u) \cdot \varphi_Y^p(u)\big)_{p\geq 1}$$

Proof. The measurability of the multifunction $\omega \to cl(X+Y)(\omega)$ is given by Lemma 3.2.1 in [5]. Moreover, because of Remark 2.3, two probability measures on $(cb(E), \mathcal{E})$ which coincide on the algebra $\mathcal{A}$ are equal. Then, it follows from Corollary 2.4 that $\big(\delta^*(z_j^*, X)\big)_{1\leq j\leq n}$ and $\big(\delta^*(z_j^*, Y)\big)_{1\leq j\leq n}$ are independent random variables for each $n \in \mathbb{N}^*$. So for all $p \in \mathbb{N}^*$ and $u \in \ell_0(\mathbb{R})$, we have

$$\begin{aligned}
\varphi_{cl(X+Y)}^p(u) &= \mathbb{E}\Big[\exp\Big(i\sum_{j=1}^p u_j\big(\delta^*(z_j^*, X) + \delta^*(z_j^*, Y)\big)\Big)\Big] \\
&= \Big(\mathbb{E}\big(\exp\big(i\sum_{j=1}^p u_j\delta^*(z_j^*, X)\big)\big) \cdot \mathbb{E}\big(\exp\big(i\sum_{j=1}^p u_j\delta^*(z_j^*, Y)\big)\big)\Big) \\
&= \varphi_X^p(u) \cdot \varphi_Y^p(u). \quad \square
\end{aligned}$$

Definition 2.11. Let $\left(X_n\right)_{n\in\mathbb{N}^*\cup\{\infty\}}$ be a sequence of random sets taking values in $cb(E)$ and let τ be a topology on $cb(E)$. We say that X_n *converges in distribution to* X_∞ *with respect to* τ if, for every bounded real-valued function f on $cb(E)$ which is continuous with respect to τ,

$$\lim_n \int_\Omega f(X_n(\omega))P(d\omega) = \int_\Omega f(X_\infty(\omega))P(d\omega).$$

The following result provides a multivalued version of Lévy-Cramer convergence theorem.

Proposition 2.12. *Let* $\left(X_n\right)_{n\in\mathbb{N}^*\cup\{\infty\}}$ *be a sequence of strongly random sets taking values in* $cb(E)$*. If* X_n *converges in distribution to* X_∞ *with respect to* τ_H *then, for each* u *in* $\ell_0(\mathbb{R})$ *and for each* p *in* $\mathbb{N}^*$*,*

$$\lim_n \varphi^p_{X_n}(u) = \varphi^p_{X_\infty}(u). \tag{1}$$

Proof. It follows from Remark 2.9 and Lévy-Cramer convergence theorem ([4], Theorem 6.3.1) that equality (1) holds if and only if, for each $p \in \mathbb{N}^*$, the sequence in n of random vectors $\left(\delta^*(z_j^*, X_n)\right)_{1\le j\le p}$ converges in distribution to $\left(\delta^*(z_j^*, X_\infty)\right)_{1\le j\le p}$. If, for each bounded continuous real-valued function g on $\mathbb{R}^p$ and for each C in $cb(E)$, we set

$$\psi_p(C) = g(\delta^*(z_1^*, C), \ldots, \delta^*(z_p^*, C)),$$

then the Hörmander's formula ([3], Theorem 2.18) implies that the real function ψ_p on $cb(E)$ is bounded and continuous with respect to τ_H. Then by the assumptions we have

$$\lim_n \int_\Omega \psi_p(X_n(\omega))P(d\omega) = \int_\Omega \psi_p(X_\infty(\omega))P(d\omega)$$

for each $p \in N$, and hence (1) is satisfied. □

3. Sums of independent random sets

This section is devoted to the convergence of sums of independent random sets and especially to an extension of Itô-Nisio theorem ([7], Theorem 3.1).

Let $(X_n)_{n\in\mathbb{N}^*\cup\{\infty\}}$ be a sequence of independent random sets taking values in $cb(E)$. For each positive integer n, we set

$$S_n = cl(X_1 + X_2 + \cdots + X_n)$$

We prove the equivalence of the following assertions:

(i) S_n converges almost surely to X_∞ with respect to τ_H,

(ii) S_n converges in probability to X_∞ with respect to τ_H,

(iii) S_n converges in distribution to X_∞ with respect to τ_H.

This will be achieved in two steps.

At first, we show the equivalence between (i) and (ii) for convex closed bounded random sets. Next, we obtain the equivalence of the three assertions for random sets taking values in a τ_H-separable subset of $cb(E)$, using a topology embedding.

We begin by establishing the following lemmas in order to prove Theorem 3.4.

Lemma 3.1. *For each positive integer n, S_n is a random set taking values in $cb(E)$.*

Proof. For every A, B in $cb(E)$, put

$$T(A, B) = cl(A + B).$$

Lemma 3.2.1 in [5] implies the measurability of T with respect to the Effrös σ-algebra $\mathcal{E}$. Then the measurability of S_n with respect to $\mathcal{E}$ follows by induction. □

Lemma 3.2. *For each i in $\mathbb{N}^*$, let u_i be an application from $cb(E)$ to $cb(E)$ which is measurable with respect to $\mathcal{E}$. Then, for each n in $\mathbb{N}^*$, the random sets $u_1(X_1), u_2(X_2), \ldots, u_n(X_n)$ are independent.*

Proof. Let $U_1, U_2, \ldots, U_{n-1}$ and U_n be open subsets of E. Then

$$\begin{aligned} P\Big[\bigcap_{i=1}^{n}(u_i \circ X_i)^{-1}(U_1^-)\Big] &= P\Big[\bigcap_{i=1}^{n} X_i^{-1}(u_i^{-1}(U_i^-))\Big] \\ &= \prod_{i=1}^{n} P(X_i^{-1}(u_i^{-1}(U_i^-))) \\ &= \prod_{i=1}^{n} P((u_i \circ X_i)^{-1}(U_i^-)) \end{aligned}$$

and the independence of $u_1(X_1), u_2(X_2), \dots, u_{n-1}(X_{n-1})$ and $u_n(X_n)$ follows. $\square$

The following result extends an inequality given by Ottaviani and Skorohod [12] in the real case.

Lemma 3.3. *For each n in $\mathbb{N}^*$ and each $t > 0$, consider the real number*

$$a_n(t) := \max_{1 \le k \le n} P\Big(\Big\{\omega : h(S_n(\omega), S_k(\omega)) > \frac{t}{2}\Big\}\Big).$$

If $a_n(t) < 1$, then

$$P\Big(\Big\{\omega : \max_{1 \le k \le n} h(0, S_k(\omega)) > t\Big\}\Big) \le \frac{1}{1 - a_n(t)} P\Big(\Big\{\omega : h(0, S_n(\omega)) > \frac{t}{2}\Big\}\Big).$$

Proof. For each $k \le n$, set

$$A_k := \Big\{\omega : \max_{j<k} h(0, S_j(\omega)) \le t, h(0, S_k(\omega)) > t\Big\}.$$

Then

$$\begin{aligned} &P\Big(\Big\{\omega : h(0, S_n(\omega)) > \frac{t}{2}\Big\}\Big) \\ &\qquad \ge P\Big(\Big\{\omega : h(0, S_n(\omega)) > \frac{t}{2}, \max_{k \le n} h(0, S_k(\omega)) > t\Big\}\Big) \\ &\qquad \ge \sum_{k=1}^{n} P\Big(A_k \cap \Big\{\omega : h(S_n(\omega), S_k(\omega)) \le \frac{t}{2}\Big\}\Big\}. \end{aligned}$$

Let

$$A := \Big\{\omega : \max_{j \le n} h(0, S_j(\omega)) > t\Big\} = \bigcup_{k=1}^{n} A_k.$$

The Hörmander formula ([13]. Theorem 2.18) implies

$$h(S_n(\omega), S_k(\omega)) = h(0, cl(X_{k+1}(\omega) + \cdots + X_n(\omega)))$$

for each ω in Ω. It follows from the independence of $(X_1, X_2, \dots, X_k)$ and $(X_{k+1}, \dots, X_n)$ and from Lemma 3.2 that

$$\begin{aligned} &P\Big(A_k \cap \Big\{\omega : h(S_n(\omega), S_k(\omega)) > \frac{t}{2}\Big\}\Big) \\ &= P(A_k) \cdot P\Big(\Big\{\omega : h(S_n(\omega), S_k(\omega)) > \frac{t}{2}\Big\}\Big). \end{aligned}$$

Therefore

$$\begin{aligned}
&\sum_{k=1}^{n} P\Big(A_k \cup \Big\{\omega : h(S_n(\omega), S_k(\omega)) \le \frac{t}{2}\Big\}\Big\} \\
&\qquad \ge P(A) - \sum_{k=1}^{n} P(A_k) \cdot P\Big(\Big\{\omega : h(S_n(\omega), S_k(\omega)) > \frac{t}{2}\Big\}\Big) \\
&\qquad \ge P(A) - P(A)\Big[\max_{k \le n} P\Big(\Big\{\omega : h(S_n(\omega), S_k(\omega)) > \frac{t}{2}\Big\}\Big)\Big] \\
&\qquad = (1 - a_n(t))P(A). \quad \square
\end{aligned}$$

We are now ready to establish our theorem about the convergence of a sum of independent random sets.

Theorem 3.4. *Let $(X_n)_{n \in \mathbb{N}^* \cup \{+\infty\}}$ be a sequence of independent random sets taking values in the metric space $(cb(E), h)$. Then the following conditions are equivalent:*

(i) *S_n converges almost surely to X_∞,*

(ii) *S_n converges in probability to X_∞,*

(iii) *S_n has a subsequence which converges in probability to X_∞.*

Proof. The implications "(i) $\Longrightarrow$ (ii)" and "(ii) $\Longrightarrow$ (iii)" are obvious. To prove "(iii) $\Longrightarrow$ (i)" let us denote by $(S_{n_k})_{k \in \mathbb{N}^*}$ a subsequence of $(S_n)_{n \in \mathbb{N}^*}$ which converges in probability to X_∞ in $(cb(E), h)$. As $(cb(E), h)$ is a complete metric space ([3], Theorem II.3), we may assume, in the same way as in the real case ([8] or [9]) that $(S_{n_k})_{k \ge 1}$ has a subsequence which converges almost surely to X_∞ in $(cb(E), h)$. We may also denote this subsequence by $(S_{n_k})_{k \ge 1}$. By the construction of this subsequence, we can choose n_k such that

$$P\Big(\Big\{\omega : h\big(S_{n_k}(\omega), S_{n_{k+1}}(\omega)\big) \ge \frac{1}{2^k}\Big\}\Big) < \frac{1}{2^k}\,.$$

Put

$$S_{n,m}(\omega) = cl\big(X_{m+1}(\omega) + \cdots + X_n(\omega)\big)$$

for each (m, n) in $\mathbb{N} \times \mathbb{N}$ with $m \le n$, and

$$T_k(\omega) = \max_{n_k \le n \le n_{k+1}} h\big(0, S_{n,n_k}(\omega)\big)$$

for each ω in Ω and each $k \geq 1$. With the notations of Lemma 3.3, if $t = \frac{1}{2^{k-1}}$ then $a_n(t) \leq \frac{1}{2}$. Therefore, applying this lemma, we obtain

$$P\Big(\Big\{\omega : T_k(\omega) > \frac{1}{2^{k-1}}\Big\}\Big) \leq 2P\Big(\Big\{\omega : h(S_{n_k}(\omega), S_{n_{k+1}}(\omega)) > \frac{1}{2^k}\Big\}\Big),$$

hence

$$\sum_{k\geq 1} P\Big(\Big\{\omega : T_k(\omega) > \frac{1}{2^{k-1}}\Big\}\Big) < +\infty.$$

It follows that $\big(T_k\big)_{k\geq 1}$ converges to 0 in probability. Hence, by the triangular inequality, (i) follows. □

We shall use the topological embedding of $(cb(E), \tau_H)$ into the space of bounded continuous real-valued functions on E^*. The following result is well known.

Proposition 3.5 ([3], Theorem II.19). *Let $\mathcal{F}$ be the vector space of positively homogeneous real-valued functions on E^* which are bounded and strongly continuous on B^* equipped with the norm*

$$\|f\| = \sup\big\{|f(x^*)| \,\big|\, \|x^*\| \leq 1\big\}.$$

For each C in $cb(E)$, we define an application ϕ of $cb(E)$ into $\mathcal{F}$ by

$$\phi(C) = \delta^*(., C).$$

Then ϕ is an isometry of $(cb(E), \tau_H)$ into $\mathcal{F}$.

The following result extends the Itô-Nisio theorem ([7], Theorem 3.1 or [13]).

Theorem 3.6. *Let $\mathcal{C}$ be a subset of the metric space $(cb(E), h)$ which is separable. Let $\big(X_n\big)_{n\in\mathbb{N}^*\cup\{\infty\}}$ be a sequence of independent random sets taking values in $(\mathcal{C}, h)$. Then the following conditions are equivalent*

(i) *S_n converges almost surely to X_∞,*

(ii) *S_n converges in probability to X_∞,*

(iii) *S_n converges in distribution to X_∞.*

Proof. It is well known that $\mathcal{F}$ equipped with the topology of uniform convergence is a Banach space. By theorem 3.4, (i) and (ii) are equivalent.

(ii) $\Longrightarrow$ (iii) can be verified as in the real case ([1], Theorem 4.3).

To prove (iii) $\Longrightarrow$ (i), let us denote by $\mathcal{F}_1$ the closed separable vector space which is generated by $\phi(\mathcal{C})$. By Proposition 3.5, ϕ is an homeomorphism. Then $\phi(S_n)$ converges in distribution to $\phi(X_\infty)$ in $\mathcal{F}_1$. It follows from Itô-Nisio theorem ([7], Theorem 3.1) that $\phi(S_n)$ converges almost surely in $\mathcal{F}_1$ to $\phi(X_\infty)$. As $S_n = \phi^{-1}[\phi(S_n)]$ and $X_\infty = \phi^{-1}[\phi(X_\infty)]$, (i) follows. □

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