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ON LOCAL REDUCTION NUMBERS AND a-INVARIANTS OF REES ALGEBRAS OF GOOD FILTRATIONS

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1. Introduction

Let $(A, \mathbf{m}, k)$ be a Noetherian local ring of $\dim A = d > 0$ with an infinite residue field k. Let I be an ideal of A with $\mathrm{ht}(I) > 0$. The ideal J is called a reduction of I if $J \subseteq I$ and there exists an integer n such that $I^{n+1} = JI^n$. The least non-negative integer n with this property is called the reduction number of I with respect to J and we denote it by $r_J(I)$. The reduction number of I is defined by

$$r(I) := \min\{r_J(I) \mid J \text{ is a minimal reduction of } I\},$$

where J is said to be a minimal reduction of I if it is not properly contained in any other reduction of I. Let

$$\ell(I) := \dim G(I)/\mathbf{m}G(I).$$

We call this number the analytic spread of I. The analytic spread $\ell(I)$ is equal to the minimum number of generators of every minimal reduction of I (cf. [15]). It is well-known that

$$\mathrm{ht}(I) \leq \ell(I) \leq \dim A$$

and the difference

$$\mathrm{ad}(I) := \ell(I) - \mathrm{ht}(I)$$

is called the analytic deviation of I. In the case $\mathrm{ad}(I) = 0$, the ideal I is called equimultiple. The study on ideals with positive analytic deviation as a separate class is initiated by Huckaba and Huneke in [8], [9], [10]. To determine when the Rees algebra

$$R(I) := \bigoplus_{n \geq 0} I^n t^n$$

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is a Cohen-Macaulay ring in terms of the associated graded ring

$$G(I) := \bigoplus_{n\geq 0}(I^n/I^{n+1})$$

and the reduction number of I is an interesting problem. The case of equimultiple ideals was investigated by Goto-Shimoda [7], Grothe-Herrmann-Orbanz [6], Trung-Ikeda [21], Viet [22], Hoa-Zarzuela [12]. But the most general result was obtained by Hoa (see [11], Theorem 5.4). Next, one is interested in the case of ideals having small analytic deviation in Cohen-Macaulay rings. For example, Goto-Huckaba [4] and, independently, Viet [23] have proved that if A is a Cohen-Macaulay ring, I is an ideal of A with $\mathrm{ad}(I) = 1$ and I is generically a complete intersection ideal then $R(I)$ is a Cohen-Macaulay ring if and only if $G(I)$ is a Cohen-Macaulay ring and $r(I) \leq \mathrm{ht}(I)$. Trung has extended this result to ideals having analytic deviation one or two (see [20], Theorem 1.3). In [2] Aberbach-Huneke-Trung have given the criterion for the Cohen-Macaulayness of Rees Algebras of ideals having arbitrary analytic deviation in Cohen-Macaulay rings (see also [13], [16], [17]). The theorem of Aberbach-Huneke-Trung states:

Theorem [A-H-T]. *Let $(A, \mathbf{m})$ be a Cohen-Macaulay ring of* $\dim A > 0$ *and I an ideal of A with $ht(I) > 0$. Then $R(I)$ is Cohen-Macaulay if and only if the following conditions are satisfied:*
(i) *$G(I)$ is Cohen-Macaulay.*
(ii) $r(I_{\mathbf{p}}) \leq ht(\mathbf{p}) - 1$ *for every prime* $\mathbf{p} \supseteq I$ *with* $\ell(I_{\mathbf{p}}) = ht(\mathbf{p})$*.*

In this paper, we will generalize the theorems of Goto-Shimoda [7], Hoa [11], Aberbach-Huneke-Trung [2] to a similar criterion for the Cohen-Macaulay property of Rees algebras of good filtrations in Noetherian local rings, see Theorem 4.1.

A family $F := \{I_n\}_{n\geq 0}$ of ideals of A is called a filtration if the following conditions are satisfied:

(i) $I_0 = A, I_1 \neq A$,

(ii) $I_n \supseteq I_{n+1}$ for all $n \geq 0$,

(iii) $I_n I_m \subseteq I_{n+m}$ for all $n, m \geq 0$.

Let $F = \{I_n\}_{n\geq 0}$ be a filtration of A. We call the graded rings

$$R(F) := \bigoplus_{n\geq 0} I_n t^n$$

and

$$G(F) := \bigoplus_{n\geq 0}(I_n/I_{n+1})$$

the Rees algebra and the associated ring of F, respectively.

Let I be an ideal of A. F is called an I-good filtration if $II_n \subseteq I_{n+1}$ for all $n \geq 0$ and $I_{n+1} = II_n$ for all $n \gg 0$. F is called a good filtration if it is an I-good filtration for some ideal I of A [12]. Note that F is a good filtration if and only if it is an I_1-good filtration. An ideal $J \subseteq I_1$ is a minimal reduction of a good filtration F if F is a J-good filtration and it is not properly contained in any ideal $I \subseteq I_1$ such that F is an I-good filtration. A good filtration $F = \{I_n\}_{n \geq 0}$ is called equimultiple if I_1 is an equimultiple ideal.

Let $J \subseteq I_1$ be a minimal reduction of a good filtration F. The reduction number of F with respect to J is the number

$$r_J(F) := \min\{r \mid I_{n+1} = JI_n \text{ for all } n \geq r\}.$$

The reduction number of F is the number

$$r(F) := \min \{r_J(F) | \ J \text{ is a minimal reduction of } F\} \ \ [12].$$

For every $\mathbf{p} \in$ Spec A, we set $F_{\mathbf{p}} := \{I_n A_{\mathbf{p}}\}_{n \geq 0}$ and call it the local filtration of F with respect to $\mathbf{p}$ and $r(F_{\mathbf{p}})$ the local reduction number of F with respect to $\mathbf{p}$.

Throughout this paper we will assume that F is a good fitration of A such that the Rees algebra $R(F)$ is Noetherian with dim $R(F) = d + 1$. It is well-known that if $I_1 \not\subseteq \sqrt{0_A}$ then $R(F)$ is a Noetherian ring with dim $R(F) = \dim A + 1$.

We denote by R the Rees algebra $R(F)$, by M the maximal graded ideal of $R(F)$, and by R^+ the ideal generated by all homogeneous elements of positive degree of $R(F)$.

Recall that the a^*-invariant of R is defined by

$$a^*(R) := \min\{a \in Z| \ [H^i_M(R)]_n = 0 \text{ for all } \ n > a \text{ and } i \leq d+1\}.$$

Set

$$L_i(F) := \{\mathbf{p} \in \text{Spec } A| \ \ell(I_{1\mathbf{p}}) = \text{ht}(\mathbf{p}) \leq i\}; \ i \leq \ell(I_1).$$

The number

$$r_i(F) := \max\{-1, r(F_{\mathbf{p}}) - \text{ht}(\mathbf{p})| \ \mathbf{p} \in L_i(F)\} + i, \ i \leq \ell(I_1),$$

is called the *i-th* local reduction number of F.

A ring A is said to have Serre condition (S_k) if

$$\operatorname{depth} A_{\mathbf{p}} \geq \min\{k, \operatorname{ht}(\mathbf{p})\}$$

for every $\mathbf{p} \in \operatorname{Spec} A$ (see [14]). Inspired by this definition, we say that the Rees algebra $R(F)$ satisfies Serre condition (S_k^*) if

$$\operatorname{depth} R(F)_{\mathbf{P}} \geq \min\{\operatorname{ht}(\mathbf{P}), k\}$$

for all $\mathbf{P} = \mathbf{p} + R^+(F)$, $\mathbf{p} \in \operatorname{Spec} A$.

The relations between the local reduction numbers of a good fitration F and the a^*-invariant of the Rees algebra $R(F)$ satisfying Serre condition (S_ℓ^*) can be described as follows.

Theorem 3.3. *Assume that $F = \{I_n\}_{n\geq 0}$ is a good filtration of A with $\ell = \ell(I_1)$, $R := R(F)$ satisfies Serre condition (S_ℓ^*) and depth $R^+(F) > 0$. Let J be a minimal reduction of I_1. Then*

(i) $\max\{r_{\ell-1}(F) + 1, r_J(F)\} = r_\ell(F)$.

(ii) $\max\{r_{\ell-1}(F) + 1, a^*(R) + \ell\} = r_\ell(F)$.

Using Theorem 3.3 we can prove the following result which generalizes the case of Cohen-Macaulay rings in [2] and the case of equimultiple filtrations in [11].

Theorem 4.1. *Suppose that $F = \{I_n\}_{n\geq 0}$ is a good filtration with $\dim R(F) = d + 1$. Then $R(F)$ is a Cohen-Macaulay ring if and only if the following conditions are satisfied:*

(i) $[H_M^i(G(F))]_n = 0$ *for all* $n \neq -1, i = 0, ..., d-1$.

(ii) $r(F_{\mathbf{p}}) \leq ht(\mathbf{p}) - 1$ *for every prime* $\mathbf{p} \supseteq I_1$ *with* $\ell((I_1)_{\mathbf{p}}) = ht(\mathbf{p})$.

The paper is divided into three parts. In Section 2 we collect several facts about minimal reductions of ideals generated by homogeneous elements of positive degree and about generalized Cohen-Macaulay rings with respect to an ideal. In Section 3 we introduce the notion of local reduction numbers of graded factor algebras of Rees algebras and we relate the local reduction numbers to the a^*-invariant of a Rees algebra which satifies Serre condition (S_ℓ^*) (Theorem 3.3). Section 4 gives the proof of the main theorem and some applications.

2. Preliminaries

In this section we give some results and notions which will be needed in this paper.

Let $S := \bigoplus_{n\geq 0} S_n$ be a Noetherian graded algebra over a Noetherian local ring S_0 with an infinite residue field. We denote by S^+ the ideal generated by all homogeneous elements of positive degree of S. A sequence $x_1, \dots, x_r$ of homogeneous of elements of S is called $[t_1, ..., t_r]$-regular if

$$[(x_1, ..., x_{i-1}) : x_i]_n = (x_1, ..., x_{i-1})_n$$

for all $n \geq t_i, i = 1, ..., r$ [2], [18]. If all $t_1, ..., t_r$ are finite then $x_1, ..., x_r$ is called a filter-regular sequence [19]. A minimal reduction of S^+ is an ideal J generated by $\ell(S^+)$ homogeneous elements of S of degree 1 such that $J_n = S_n$ for some positive integer n. The reduction number $r_J(S^+)$ of S^+ with respect to J is the minimum number n for which $J_{n+1} = S_{n+1}$ [20]. According to [19] every minimal reduction of S^+ can be minimally generated by a filter-regular sequence of S.

Let J be a minimal reduction of S^+. We denote by $S(J)$ the least number t such that there exists a homogeneous minimal generating set $x_1, ..., x_\ell$ of J which is $[t+1, ..., t+\ell]$-regular of S [2].

Set

$$a^*(S) := \min\{a \in Z|\ [H^i_M(S)]_n = 0 \ \text{ for all } \ n > a \text{ and } i \leq \dim S\}.$$

In the same way as in the proof of [20, Corollary 2.3] and [2, Corollary 2.9] we get some relations between $S(J), r_J(S^+), \ell(S^+)$ and $a^*(S)$ as follows.

Proposition 2.1. *Let S be a Noetherian graded algebra and $J \subseteq S_1$ be a minimal reduction of S^+. Then*

(i) $a^*(S) \leq S(J)$ *if and only if* $r_J(S^+) \leq \ell(S^+) + S(J)$.

(ii) *For any integer* $b > S(J), a^*(S) = b$ *if and only if* $r_J(S^+) = \ell(S^+) + b$.

(iii) $max\{S(J), a^*(S)\} = max\{S(J), r_J(S^+) - \ell(S^+)\}$.

(iv) *If Y is a minimal reduction of S^+ such that $r(S^+) = r_Y(S^+)$ then*

$$r_J(S^+) \leq \max\{S(J) + \ell(S^+), r(S^+), S(Y) + \ell(S^+)\}.$$

Proof. Since $J \subseteq S_1$ is a minimal reduction of S^+, there exists a homogeneous minimal system of generators $x_1, ..., x_\ell$ such that $x_1, ..., x_\ell$ is a filter-regular sequence [19]. We get

$$[(x_1, ..., x_{i-1}) : x_i]_n = (x_1, ..., x_{i-1})_n$$

for all $n \geq i + S(J), i = 1, ..., \ell$. From [20, Theorem 2.2] we obtain (i) and (ii). Next, we prove (iii). If $a^*(S) \leq S(J)$ then $r_J(S^+) \leq \ell(S^+) + S(J)$ by (i) so both sides are equal to $S(J)$. If $a^*(S) > S(J)$ then by (ii), $r_J(S^+) - \ell(S^+) = a^*(S)$ and the equality holds. By (iii) we get

$$a^*(S) \leq \max\{S(Y), r_Y(S^+) - \ell(S^+)\} = \max\{\, S(Y), r(S^+) - \ell(S^+)\}$$

and

$$r_J(S^+) \leq \max\{S(J) + \ell(S^+), a^*(S) + \ell(S^+)\}.$$

From this it follows that

$$r_J(S^+) \leq \max\{S(J) + \ell(S^+), S(Y) + \ell(S^+), r(S^+)\}. \quad \square$$

Remark 2.2.

(i) $r_J(S^+) \leq \max\{\, S(J), a^*(S)\} + \ell(S^+)$.

(ii) $a^*(S) \leq \max\{\, S(J), r_J(S^+) - \ell(S^+)\}$.

An important invariant which is closely related to the reduction number is the so-called a-invariant. The notion of a-invariant is introduced by Goto and Watanabe [5]. For a Noetherian graded ring S over a local ring, the a-invariant of S is defined by

$$a(S) := \max\{n; [H^d_M(S)]_n \neq 0\},$$

where $d = \dim S$ and M is a maximal graded ideal of S and

$$a_i(S) := \max\{n, [H^i_M(S)]_n \neq 0\},$$

$i = 0, ..., \dim S$.

Remark 2.3.

(i) $a^*(S) = \max\{a_i(S) |\ i \leq \dim S\}$.

(ii) $a_d(S) = a(S)$.

$(A, \mathbf{m})$ is called a generalized Cohen-Macaulay ring with respect to an ideal I if $H^i_{\mathbf{m}}(A)$ is annilated by some powers of I for $i = 0, ..., \dim A - 1$ [21].

Proposition 2.4 ([21], Lemma 2.1). *Suppose that A is a homomorphic image of a regular local ring. Then A is a generalized Cohen-Macaulay ring respect to I iff for every prime ideal $\mathbf{p} \not\subseteq I$, $A_{\mathbf{p}}$ is a Cohen-Macaulay ring with* $\dim A_{\mathbf{p}} = d - \dim (A/\mathbf{p})$.

From the proof of [22, Lemma 1.2] we easily get the following proposition.

Proposition 2.5. *$R(F)$ is a generalized Cohen-Macaulay ring with respect to $R^+(F)$ iff $G(F)$ is a generalized Cohen-Macaulay ring with respect to $G^+(F)$. In this case, A is a generalized Cohen-Macaulay ring with respect to I_1.*

3. On the properties of Rees algebras satisfying Serre condition (S^*_ℓ)

Throughout this section, let $(A, \mathbf{m})$ be a Noetherian local ring of dim $A = d > 0$ with an infinite residue field $A/\mathbf{m}$ and $F = \{I_n\}_{n \geq 0}$ a good filtration such that dim $R(F) = d + 1$. Let S be a graded factor algebra of $R(F)$ by a homogeneous ideal.

For every prime ideal $\mathbf{p}$ of A we denote by $S_{\mathbf{p}}$ the localization of S at the multiplicative closed set $A \backslash \mathbf{p}$. It can be verified that if $\mathbf{P} = \mathbf{p} + R^+(F)$ then dim $S_{\mathbf{P}} =$ dim $S_{\mathbf{p}}$ and $R(F)_{\mathbf{p}} = R(F_{\mathbf{p}}), G(F_{\mathbf{p}}) = G(F)_{\mathbf{p}}$.

Set

$$h = \text{ht } (I_1), R = R(F), G = G(F), \ell = \ell(S^+),$$

$$L_i(S) := \{\mathbf{p} \in \text{Spec } A |\ \ell(S^+_{\mathbf{p}}) = \text{ht}(\mathbf{p}) \leq i\}; i \leq \ell(S^+),$$

$$L_i(F) := \{\mathbf{p} \in \text{Spec } A |\ \ell((I_1)_{\mathbf{p}}) = \text{ht}(\mathbf{p}) \leq i\}; i \leq \ell(I_1).$$

Definition 3.1. Let S be as above. The number

$$r_i(S) = \max \{-1; r(S^+_{\mathbf{p}}) - \text{ht}(\mathbf{p}) | \mathbf{p} \in L_i(S)\} + i,\ i \leq \ell(S^+)$$

is called the *i-th* local reduction number algebra of S with respect to $R(F)$. We call the invariant

$$r_i(F) = \max \{-1, r(F_{\mathbf{p}}) - \text{ht}(\mathbf{p}) | \mathbf{p} \in L_i(F)\} + i,\ i \leq \ell(I_1)$$

the *i-th* locall reduction number of F.

Remark 3.2. (i) Since $r(R^+_{\mathbf{p}}) = r(F_{\mathbf{p}}) = r(G^+_{\mathbf{p}})$ for every $\mathbf{p} \in \operatorname{Spec} A$, we have

$$r_i(R) = r_i(G) = r_i(F) \text{ for all } i \leq \ell(I_1).$$

(ii)

$$r_{i+1}(S) = \max\{r_i(S) + 1, r(S^+_{\mathbf{p}}) | \mathbf{p} \in L_{i+1}(S) \backslash L_i(S)\}, i \leq \ell(S^+) - 1.$$

(iii)

$$r_i(S^+_{\mathbf{p}}) \leq r_i(S), i \leq \ell(S^+_{\mathbf{p}}).$$

(iv)

$$r_i(F) = \begin{cases} i-1, & 0 \leq i \leq h \\ \max\{i-1, r(F_{\mathbf{p}}) - \operatorname{ht}(\mathbf{p}) + i | \ell(I_{1\mathbf{p}}) = & \operatorname{ht}(\mathbf{p}) \leq i, \\ \qquad I_1 \subseteq \mathbf{p} \in \operatorname{Spec}(A)\}, & h \leq i \leq \ell(I_1). \end{cases}$$

Our approach is based on an idea of Aberbach, Huneke, Trung [2] which links the local reduction numbers of an ideal with the a-invariant of the associated graded ring under the assumption that the rings A and $G(I)$ are Cohen-Macaulay. We see that the relations between the local reduction numbers of a good filtration and the a^*-invariant of its Rees algebra can be described as follows.

Theorem 3.3. *Assume that $F = \{I_n\}_{n \geq 0}$ is a good filtration of A with $\ell(I_1) = \ell$, $R := R(F)$ satisfies Serre condition (S^*_ℓ) and depth $R^+(F) > 0$. Let J be a minimal reduction of I_1. Then*

(i) *max* $\{r_{\ell-1}(F) + 1, r_J(F)\} = r_\ell(F)$.

(ii) *max* $\{r_{\ell-1}(F) + 1, a^*(R) + \ell\} = r_\ell(F)$.

The proof of Theorem 3.3 is based on the following proposition.

Proposition 3.4. *Suppose that $R(F)$ satisfies (S^*_ℓ) and $J \subseteq [R(F)]_1$ is a minimal reduction of $R^+(F)$. Further, assume that depth $R^+(F) > 0$. Then*

(i) *For any filter-regular sequence $x_1, ..., x_\ell$ of R which genarates J, $x_1, ..., x_\ell$ is $[r_0(R) + 1, ..., r_{\ell-1}(R) + 1]$ -regular.*

(ii) $r_J(R^+) \leq r_\ell(R)$.

Let us consider the follwing conditions

$(C_i) : [(x_1, ..., x_i) : x_{i+1}]_n = (x_1, ..., x_i)_n$ for all $n \geq r_i(R) + 1, 0 \leq i < \ell$.

$(C_\ell) : r_J(R^+) \leq r_\ell(R)$.

In the same way as in the proof of [2, Theorem 3.2], we need to prove the following lemma.

Lemma 3.5. *Let R, J, ℓ be as in Proposition 3.4 and $J = (x_1, ..., x_\ell)$. Fix i such that $0 \leq i < \ell$. Assume that the sequence $x_1, ..., x_\ell$ satisfies (C_j) for all $0 \leq j < i$. Let $\mathbf{P} = \mathbf{p} + R^+$ for $\mathbf{p} \in$ Spec A with ht$(\mathbf{P}) > i$. Then*

$$[H^k_{\mathbf{P}}(R_{\mathbf{p}}/(x_1, ..., x_i)_{\mathbf{p}})]_n = 0$$

for all $n \geq r_{i-1}(R) + 2, k <$ min $\{ht(\mathbf{P}), \ell\} - i$.

Proof. Suppose that $\mathbf{P} = \mathbf{p} + R^+$ for $\mathbf{p} \in \operatorname{Spec} A$. Since $R(F)$ satisfies the condition (S^*_ℓ), it follows that

$$\operatorname{depth} R_{\mathbf{p}} \geq \min\{\operatorname{ht}(\mathbf{P}), \ell\}.$$

This immediately induces

$$[H^k_{\mathbf{P}}(R_{\mathbf{p}})] = 0 \text{ for } k < \min\{\operatorname{ht}(\mathbf{P}), \ell\}.$$

Hence the conclusion holds for $i = 0$. We do by induction on i. Now let $i > 0$ and let $\operatorname{ht}(\mathbf{P}) > i$. Set

$$J_i = (x_1, ..., x_i), i = 1, ..., \ell \text{ and } J_0 = 0.$$

The exact sequence

$$0 \to (J_{i-1} : x_i)/J_{i-1} \to R/J_{i-1} \to R/(J_{i-1} : x_i) \to 0$$

yields the following exact sequence

$$[H^k_{\mathbf{P}}(R_{\mathbf{p}}/(J_{i-1})_{\mathbf{p}})]_n \to [H^k_{\mathbf{P}}(R_{\mathbf{p}}/(J_{i-1} : x_i)_{\mathbf{p}})]_n \to [H^{k+1}_{\mathbf{P}}(J_{i-1} : x_i/(J_{i-1})_{\mathbf{p}})]_n.$$

By the inductive hypothesis we have

$$[H^k_{\mathbf{P}}(R_{\mathbf{p}}/(J_{i-1})\mathbf{p})]_n = 0$$

for all $n \geq r_{i-2}(R) + 2, k < \min\{\operatorname{ht}(\mathbf{P}), \ell\} - (i-1)$. Further, since (C_{i-1}) is satisfied,

$$[H^t_{\mathbf{P}}((J_{i-1} : x_i/(J_{i-1})_{\mathbf{p}})]_n = 0$$

for all t and $n \geq r_{i-1}(R)+1$. Since $r_{i-2}(R)+2 \leq r_{i-1}(R)+1$ and using the above exact sequence we get

$$[H^k_{\mathbf{P}}(R_{\mathbf{p}}/(J_{i-1}:x_i)_{\mathbf{p}})]_n = 0$$

for all $n \geq r_{i-1}(R)+1, k < \min\{\mathrm{ht}(\mathbf{P}), \ell\} - (i-1)$. Now, we consider the exact sequence

$$0 \to [R/J_{i-1}:x_i](-1) \xrightarrow{x_i} R/J_{i-1} \to R/J_i \to 0.$$

By localizing at $\mathbf{p}$ it is easy to derive the following exact sequences.

$$[H^k_{\mathbf{P}}(R_{\mathbf{p}}/(J_{i-1})_{\mathbf{p}})]_n \to [H^k_{\mathbf{P}}(R_{\mathbf{p}}/(J_i)_{\mathbf{p}})]_n \to [H^{k+1}_{\mathbf{P}}(R_{\mathbf{p}}:x_i/(J_{i-1})_{\mathbf{p}})]_{n-1}.$$

Hence we conclude that

$$[H^k_{\mathbf{P}}(R_{\mathbf{p}}/(J_i)_{\mathbf{p}})]_n = 0$$

for all $n \geq r_{i-1}(R)+2$ and $k < \min\{\mathrm{ht}(\mathbf{P}), \ell\} - i$. Thus, this result holds for every $i < \ell$. □

Lemma 3.6. *Let R, J, ℓ be as in Proposition 3.4. Assume that (C_j) holds for all $0 \leq j < i < \ell$. Then*

$$[U_i \cap V_i]_n = (x_1, ..., x_i)_n$$

for all $n \geq r_{i-1}(R)+2$, where U_i denotes the intersection of primary components of $(x_1, ..., x_i)$ whose associated primes contain $R^+(F)$ and have the height at most i and $V_i = \cup_{n\geq 0}[(x_1, ..., x_i) : {R^+}^n]$.

Proof (cf. [2], Lemma 3.5). For every prime ideal $\mathbf{P} = \mathbf{p} + R^+$ of R, let $U(\mathbf{P})$, resp. $U_0(\mathbf{P})$, the intersection of primary components of $(x_1, ..., x_i)$ whose associated primes are contained, res. properly contained, in $\mathbf{P}$. Then

$$U_0(\mathbf{P})_{\mathbf{p}}/U(\mathbf{P})_{\mathbf{p}} = H^0_{\mathbf{P}}(R_{\mathbf{p}}/(x_1, ..., x_i)_{\mathbf{p}}).$$

When $\mathrm{ht}(\mathbf{P}) > i$, we get $\min\{\mathrm{ht}\mathbf{P}, \ell\} > i$. Thus, by Lemma 3.5,

$$[U(\mathbf{P})_{\mathbf{p}}]_n = [U_0(\mathbf{P})_{\mathbf{p}}]_n$$

for all $n \geq r_{i-1}(R)+2$. Consequently, $[U(\mathbf{P})_{\mathbf{q}}]_n = [U_0(\mathbf{P})_{\mathbf{q}}]_n$ for any prime ideal $\mathbf{q} \subseteq \mathbf{p}$ and $n \geq r_{i-1}(R)+2$. By [2, Lemma 3.3] we deduce that

$$[U(\mathbf{P})]_n = [U_0(\mathbf{P})]_n$$

for all $n \geq r_{i-1}(R)+2$. For every integer $j \geq i$, let W_j be the intersection of primary components of $(x_1, ..., x_i)$ whose associated primes contain R^+ and have the height $\leq j$. It is a plain fact that

$$W_j = \bigcap_{\mathbf{P} \supseteq R^+, \mathrm{ht}(\mathbf{P})=j} U(\mathbf{P}) \cap V_i,$$

$$W_{j-1} = \bigcap_{\mathbf{P} \supseteq R^+, \mathrm{ht}(\mathbf{P})=j} U_0(\mathbf{P}) \cap V_i.$$

Since $U(\mathbf{P})_n = U_0(\mathbf{P})_n$ for all $n \geq r_{i-1}(R)+2$, we get

$$[W_j \cap V_i]_n = [W_{j-1} \cap V_i]_n$$

for all $n \geq r_i(R)+2$ and $j > i$. By the above results we have

$$[W_{d+1} \cap V_i]_n = [W_d \cap V_i]_n = ... = [W_i \cap V_i]_n$$

for all $n \geq r_{i-1}(R)+2$. Observe further, that

$$W_{d+1} \cap V_i = (x_1, ..., x_i) \text{ and } W_i = U_i.$$

Hence

$$[U_i \cap V_i]_n = (x_1, ..., x_i)_n \text{ for all } n \geq r_{i-1}(R)+2. \quad \square$$

Proof of Proposition 3.4. Set $J_i = (x_1, ..., x_i)$. Using Lemma 3.5 and Lemma 3.6 we do by induction on the dimension of A. Let $d = \dim A = 1$. In this case we have $\ell = 1$ and $J = (x_1)$. Since $R(F)$ satisfies (S_1^*) and depth $R^+ > 0, (C_0)$ holds. In this case F is a $\mathbf{m}$-primary filtration, so $r_1(R) = r(R^+) = r_J(R^+)$ [12, Proposition 3.2]. Thus, (C_1) holds. Now let $d > 1$. We will prove by induction on i that (C_i) holds whenever $i < \ell$. The case $i = 0$, then by depth $R^+ > 0$ it follows that (C_0) holds. The case $i \geq 1$, by the inductive hypothesis then (C_j) holds for all $0 \leq j < i$. Hence, using Lemma 3.6 we obtain

$$(x_1, ..., x_i)_n = [U_i \cap V_i]_n \text{ for all } n \geq r_i(R)+1.$$

Let $\mathbf{P} \in \mathrm{Ass}(R/U_i)$ and $\mathbf{P} \cap A = \mathbf{p}$. Since $\mathbf{P} \supseteq R^+$ and $\mathrm{ht}(\mathbf{P}) \leq i < \ell$, it follows that $R_{\mathbf{p}}$ is a Cohen-Macaulay ring with $\dim A_{\mathbf{p}} < d$. Hence, by the inductive hypothesis we get

$$r_{J_{\mathbf{p}}}(R^+_{\mathbf{p}}) \leq r_k(R_{\mathbf{p}}) \leq r_k(R) < r_i(R)$$

for $k := \ell(R^+_{\mathbf{p}}) \leq \dim A_{\mathbf{p}} < i$. Note that $J_{\mathbf{P}} = (J_i)_{\mathbf{p}}$. Hence $r_{(J_i)_{\mathbf{p}}} < r_i(R)$. Thus,

$$(J_i)_{\mathbf{p}}[R_{\mathbf{p}}]_n = (J_i)_{\mathbf{p}}[R_{\mathbf{p}}]_{n+1}$$

for all $n \geq r_i(R)$. From this it follows that

$$[(U_i)_{\mathbf{p}}]_n = [R_{\mathbf{p}}]_n \text{ for all } n \geq r_i(R)+1 \text{ and } \mathbf{P} \in \operatorname{Ass}(R/U_i).$$

By [2, Lemma 3.3], this yields

$$[U_i]_n = [R)]_n \text{ for all } n \geq r_i(R)+1.$$

Using this formula, it is easy to see that for every $n \geq r_i(R)+1$,

$$\begin{aligned}[(x_1, ..., x_i) : x_{i+1}]_n &= [(U_i \cap V_i) : x_{i+1}]_n \\ &= I_n \cap [V_i : x_{i+1}]_n = I_n \cap [V_i]_n \\ &= [U_i \cap V_i]_n = (x_1, ...x_i)_n.\end{aligned}$$

Hence (C_i) holds. This fact implies

$$S(J) \leq r_{\ell-1}(R) - \ell + 1.$$

Let Y be a minimal reduction of R^+ such that $r_Y(R^+) = r(R^+)$. Thus, by Proposition 2.1, (iv),

$$r_J(R^+) \leq \max\{S(J) + \ell(R^+), S(Y) + \ell(R^+), r(R^+)\}.$$

Since $S(J) \leq r_{\ell-1}(R) - \ell + 1$ and $S(Y) \leq r_{\ell-1}(R) - \ell + 1$, it follows that

$$\max\{S(J) + \ell(R^+), S(Y) + \ell(R^+), r_J(R^+)\} \leq \max\{r_{\ell-1}(R) + 1, r(R^+)\}.$$

Using this fact and $r_J(R^+) \geq r(R^+)$, it follows that

$$r_J(R^+) \leq r_{\ell-1}(R) + 1.$$

Since $r_{\ell-1}(R) + 1 \leq r_\ell(R)$ we get $r_J(R^+) \leq r_\ell(R)$ and (C_ℓ) holds. $\square$

Proof of Theorem 3.3. From Proposition 3.4 we get

$$S(J) \leq r_{\ell-1}(R) - (\ell - 1).$$

Applying Remark 3.2 to R, it follows that

$$\begin{aligned} r_\ell(R) &= \max\{r_{\ell-1}(R)+1, r(R^+_{\mathbf{p}})|\mathbf{p}\in L_\ell(R)\backslash L_{\ell-1}(R)\} \\ &\le \max\{r_{\ell-1}(R)+1, r(R^+)\}. \end{aligned}$$

Thus,

$$r_\ell(R) = \max\{r_{\ell-1}(R)+1, r(R^+)\}.$$

By (i) and (ii) of Remark 3.2 we have

$$r_\ell(R) = \max\{r_{\ell-1}(F)+1, r(F)\}.$$

Hence (i) of Theorem 3.3 holds. Thus, by Proposition 2.1, (iii),

$$\max\{S(J)+\ell, a^*(R)+\ell\} = \max\{S(J)+\ell, r_J(R^+)\}.$$

From this equality we get

$$\max\{S(J)+\ell, r_{\ell-1}(R)+1, a^*(R)+\ell\} = \max\{S(J)+\ell, r_{\ell-1}(R)+1, r_J(R^+)\}.$$

Since $S(J)+\ell \le r_{\ell-1}(R)+1$, it follows that

$$\max\{r_{\ell-1}(R)+1, a^*(R)+\ell\} = \max\{r_{\ell-1}(R)+1, r_J(R^+)\},$$

for every J. The above facts show that

$$\max\{r_{\ell-1}(R)+1, a^*(R)+\ell\} = \max\{r_{\ell-1}(R)+1, r(R^+)\} = r_\ell(R).$$

Replacing $r_i(R)$ and $r(R^+)$ by $r_i(F)$ and $r(F)$, respectively we get (ii) of Theorem 3.3. The proof of Theorem 3.3 is now completed.

4. Cohen-Macaulay property of Rees algebras

First, in this section we will derive criteria for the Cohen-Macaulay property of Rees algebra $R(F)$ of a good filtration $F = \{I_n\}_{n\ge 0}$ in terms of local cohomology modules of the associated graded ring $G(F)$ and the local reduction numbers of F.

Let M denote the maximal graded ideal of $R(F)$. We have the following theorem.

Theorem 4.1. *Let $(A, \mathbf{m})$ be a Noetherian local ring with dim $A = d > 0$ and $F = \{I_n\}_{n \geq 0}$ a good filtration of A with ht $(I_1) > 0$. Set $\ell := \ell(I_1)$. Then $R(F)$ is a Cohen-Macaulay ring if and only if the following conditions are satisfied:*

(i) $[H^i_M(G(F))]_n = 0$ *for all* $n \neq -1, i < d$.

(ii) $r(F_\mathbf{p}) \leq ht(\mathbf{p}) - 1$ *for every prime* $\mathbf{p} \supseteq I_1$ *with* $\ell((I_1)_\mathbf{p}) = ht(\mathbf{p})$.

Proof. ($\Rightarrow$) By [22, Theorem 1.1], $[H^i_M(G)]_n = 0$ for $n \neq -1, i < d$. If $I_1 \subseteq \mathbf{p} \in \mathrm{Spec}A$ and $\mathrm{ht}(\mathbf{p}) = \mathrm{ht}(I_1)$ then $F_\mathbf{p}$ is $\mathbf{p}A\mathbf{p}$-primary and $R(F_\mathbf{p})$ is Cohen-Macaulay. By [22, Corollary 2.2] we get $r(F_\mathbf{p}) \leq \mathrm{ht}(\mathbf{P}) - 1$. Now let $\ell(R^+_\mathbf{p}) = \mathrm{ht}(\mathbf{p}) > \mathrm{ht}\ (I_1)$. We assume by induction that (ii) holds for $I_1 \subseteq \mathbf{q} \in \mathrm{Spec}A$ and $\ell((I_1)_\mathbf{q}) = \mathrm{ht}(\mathbf{q}) < \mathrm{ht}(\mathbf{p})$. Set $\ell(F_\mathbf{p}) = \ell_\mathbf{p}$. Applying Theorem 3.3 we get

$$r_{\ell_\mathbf{p}}(F_\mathbf{p}) \leq \max\{r_{\ell_\mathbf{p}-1}(F_\mathbf{p}) + 1, a^*(R_\mathbf{p}) + \ell_\mathbf{p}\}$$

and

$$\begin{aligned} r_{\ell_\mathbf{p}-1}(F_\mathbf{p}) \leq \max\{&r(F_\mathbf{q}) - \mathrm{ht}(\mathbf{q}) + \ell_\mathbf{p} - 1 | \ell(I_1\mathbf{q}) = \mathrm{ht}(\mathbf{q}) \\ &\leq \ell - 1, I_1 \subseteq \mathbf{q} \subseteq \mathbf{p}; \ell_\mathbf{p} - 2\}. \end{aligned}$$

Thus, by the inductive hypothesis, $r(F_\mathbf{q}) \leq \mathrm{ht}(\mathbf{q}) - 1$ for every $I_1 \subseteq \mathbf{q} \in \mathrm{Spec}\, A$ and $\mathrm{ht}(\mathbf{q}) < \mathrm{ht}(\mathbf{p})$. Replacing the filtration F by the filtration $F_\mathbf{p}$, we get

$$r_{\ell_\mathbf{p}-1}(F_\mathbf{p}) \leq \ell_\mathbf{p} - 2.$$

Combining this inequality, Theorem 3.3 (ii) and the inequality $a^*(R_\mathbf{p}) < 0$ yields

$$r_{\ell_\mathbf{p}}(F_\mathbf{p}) \leq \ell_\mathbf{p} - 1 = \mathrm{ht}(\mathbf{p}) - 1.$$

Thus, (ii) holds.

($\Leftarrow$) Consider the exact sequences

(1) $$0 \to R^+ \to R \to A \to 0,$$

and

(2) $$0 \to R^+(-1) \to R \to G \to 0.$$

We have by (1), $[H^i_M(R^+)]_n \simeq [H^i_M(R)]_n$ for all $n \neq 0$ and for all i. By (2) and the hypothesis on G we have an exact sequence

(3) $$0 \to [H^i_M(R^+)]_{n+1} \to [H^i_M(R)]_n$$

for all $n < -1$ and $i \le d$. Thus, for all $n < -1$ and $i \le d$, $[H^i_M(R)]_{n+1}$ can be considered as a submodule of $[H^i_M(R)]_n$. Since $[H^i_M(G)]_n = 0$ for all $n \ne -1$ and $i < d$, using [22, Lemma 1.2] we can conclude that $R(F)$ is a generalized Cohen-Macaulay ring with respect to R^+. Thus, by [21, Lemma 2.2],

$$[H^i_M(R)]_n = 0 \text{ for all } n << 0, i = 0, ..., d.$$

One can use the same argument as in the proof for [21], Lemma 3.1 to get

$$[H^i_M(R)]_n = 0 \text{ for all } n \ge 0 \text{ and } i < d,\ a(R) < 0.$$

From the above results we obtain

$$H^i_M(R) = 0, i = 0, ..., d-1$$

and

$$[H^d_M(R)]_n = 0 \ \text{ for all } \ n < 0.$$

Since depth $R \ge d$ and (3), it follows that depth $R^+ > 0$. Next, we prove by induction on dimension d of A. If $d = 1$ then F is an $\mathbf{m}$-primary. From (ii) it follows that $r(F) \le 0$. Thus, by [22, Corollary 2.2], $a(G) < 0$. Combining this inequality with the condition (i) we deduce that $R(F)$ is Cohen-Macaulay. Let $\mathbf{P} = \mathbf{p} + R^+, \mathbf{p} \in \text{Spec}A$. If $\mathbf{p} \not\supseteq I_1$ then $R_{\mathbf{p}} = A_{\mathbf{p}}[t]$. Since $A_{\mathbf{p}}$ is Cohen-Macaulay, by Proposition 2.5, $R_{\mathbf{p}}$ is also Cohen-Macaulay. If $\mathbf{p} \supseteq I_1$ and $\text{ht}(\mathbf{P}) = \ell$ then dim $R_{\mathbf{p}} = \ell \le d$. By the inductive hypothesis, $R_{\mathbf{p}}$ is a Cohen-Macaulay ring. Thus, $R_{\mathbf{p}}$ is Cohen-Macaulay for every $\mathbf{P} = \mathbf{p} + R^+, \mathbf{p} \in \text{Spec}A$ and $\text{ht}(\mathbf{P}) \le \ell$. From this fact and depth $R \ge d \ge \ell$, it follows that $R(F)$ satisfies (S^*_ℓ). By (ii), $r_\ell(F) \le \ell - 1$. Using this inequality and Lemma 3.3 we get $a^*(R) < 0$ and then $a_d(R) < 0$. Since $[H^d_M(R)]_n = 0$ for all $n < 0$, it follows that $[H^d_M(R)] = 0$. Thus, $[H^i_M(R)] = 0$, $i \le d$, it follows that $R(F)$ is a Cohen-Macaulay ring. □

The following immediate consequence of Theorem 4.1 is a generalization of [2, Theorem 5.1] for Rees algebras of good filtrations.

Theorem 4.2. *Let $(A, \mathbf{m})$ be a Cohen-Macaulay ring and $F = \{I_n\}_{n\ge 0}$ a good filtration of A with $ht(I_1) > 0$. Then $R(F)$ is Cohen-Macaulay if and only if the following conditions are satisfied:*

(i) *$G(F)$ is a Cohen-Macaulay ring.*

(ii) *$r(F_{\mathbf{p}}) \le ht(\mathbf{p}) - 1$ for every prime $\mathbf{p} \supseteq I_1$ with $\ell((I_1)_{\mathbf{p}}) = ht(\mathbf{p})$.*

Proof. ($\Rightarrow$) By [22, Corollary 2.1] we get $G(F)$ a Cohen-Macaulay ring. The condition (ii) follows by Theorem 4.1 (ii).

($\Leftarrow$) Since $G(F)$ is a Cohen-Macaulay ring, it follows that the conditions (i) and (ii) of Theorem 4.1 are satisfied. Thus, $R(F)$ is a Cohen-Macaulay ring by Theorem 4.1. □

Applying Theorem 4.1 and using the same argument as in the proof of [2, Theorem 5.6], we get the following result.

Proposition 4.3. *Let $(A, \mathbf{m})$ be a Noetherian local ring with dim $A = d > 0$ and $F = \{I_n\}_{n\geq 0}$ a good filtration of A with ht $(I_1) > 0$. Set $\ell := \ell(I_1)$. Then $R(F)$ is a Cohen-Macaulay ring if and only if the following conditions are satisfied:*

(i) $[H^i_M(G)]_n = 0$ *for all* $n \neq -1, i < d$.

(ii) $r(F_{\mathbf{p}}) \leq ht(\mathbf{p}) - 1$ *for every prime* $\mathbf{p} \supseteq I_1$ *with* $\ell((I_1)_{\mathbf{p}}) = ht(\mathbf{p}) < \ell(I_1)$.

(iii) $r_J(F) \leq \ell(I_1) - 1$ *for some (or every) minimal reduction* J *of* I_1.

The following theorem is a generalization of [11, Theorem 5.4] for Rees algebras of good filtrations.

Theorem 4.4. *Let $(A, \mathbf{m})$ be a Noetherian local ring with dim $A = d > 0$ and $F = \{I_n\}_{n\geq 0}$ an equimultiple filtration of A with dim $R(F) = d + 1$. Then $R(F)$ is a Cohen-Macaulay ring if and only if the following conditions are satisfied:*

(i) $[H^i_M(G(F))]_n = 0$ *for all* $n \neq -1, i < d$,

(ii) $r(F) \leq ht(I_1) - 1$.

Proof. ($\Rightarrow$) The conditions (i) and (ii) follow from Proposition 4.3.

($\Leftarrow$) Since $\ell(I_1) = \mathrm{ht}(I_1)$,

$$\mathrm{ht}(\mathbf{p}) \geq \mathrm{ht}(I_1) = \ell(I_1)$$

for every prime $\mathbf{p} \supseteq I_1$. Thus,

$$\{\mathbf{p} \in \operatorname{Spec} A | I_1 \supseteq \mathbf{p} \text{ and } \mathrm{ht}(\mathbf{p}) < \ell(I_1)\} = \emptyset.$$

From this it follows that (ii) of Proposition 4.3 holds. Since $r(F) \leq$ ht $(I_1) - 1$, there exists a minimal reduction J of I_1 such that $r_J(F) \leq \ell(I_1) - 1$. This implies that the condition (iii) of Proposition 4.3 holds. Hence $R(F)$ a Cohen-Macaulay ring.

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