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ON THE PROPERTY (LB^∞) OF SPACES OF GERMS OF HOLOMORPHIC FUNCTIONS AND THE PROPERTIES ($\widetilde{\Omega}, \overline{\Omega}$) OF THE HARTOGS DOMAINS IN INFINITE DIMENSION

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Abstract. The aim of this paper is to establish the property (LB^∞) on $[H(K_\varepsilon)]'$ under the assumption that E is a Frechet space with an absolute basis and K_ε is a balanced convex compact subset of E. At the same time, the properties ($\widetilde{\Omega}$,$\overline{\Omega}$) for the Hartogs domains associated to an open polydisc in a nuclear Frechet space with a basis will be also proved.

1. Introduction

Let E be a Frechet space and K a compact subset of E. By $H(K)$ we denote the space of germs of holomorphic functions on K equipped with the inductive limit topology

$$H(K) = \liminf_{U \supset K} H^\infty(U)$$

where $H^\infty(U)$ denotes the Banach space of bounded holomorphic functions on U.

In [8] N. D. Lan has proved that if E is nuclear and K a balanced convex compact subset in E then $[H(K)]' \in (LB^\infty)$ if and only if $(\widetilde{\Omega}_K)$ holds on E. The aim of this paper is to consider this result for the case where E is a non-nuclear Frechet space.

Let E be a Frechet space with an absolute basis $\{e_j\}_{j\geq 1}$. For each balanced convex compact subset K and $\varepsilon > 0$, put

$$K_\varepsilon = \overline{\text{conv}}\big(K + \varepsilon \|e_j^*\|_K^* e_j\big)$$

where $\{e_j^*\}_{j\geq 1}$ is the sequence of coefficient functionals associated to $\{e_j\}_{j\geq 1}$. Then K_ε is a balanced convex compact subset of E. The main

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result of this paper is the following

Theorem 1.1. *The following conditions are equivalent:*
(i) $E \in (\widetilde{\Omega}_{K_\varepsilon})$ *for some* $\varepsilon > 0$,
(ii) $E \in (\widetilde{\Omega}_{K_\varepsilon})$ *for every* $\varepsilon > 0$,
(iii) $\big[H(K_\varepsilon)\big]' \in (LB^\infty)$ *for some* $\varepsilon > 0$,
(iv) $\big[H(K_\varepsilon)\big]' \in (LB^\infty)$ *for every* $\varepsilon > 0$,
(v) K_ε *is not polar for some* $\varepsilon > 0$,
(vi) K_ε *is not polar for all* $\varepsilon > 0$.

Next we investigate the properties $(\widetilde{\Omega}, \overline{\Omega})$ for the Hartogs domains associated to an open polydisc in a nuclear Frechet space with a basis. In the finite dimensional case a similar result for the property $(\overline{\Omega})$ has been established by Aytuna [1].

2. Preliminaries

Let E be a Frechet space with the topology defined by an increasing system of semi-norms $\big\{\|\cdot\|_k\big\}$. For each subset $B \subset E$ define

$$\|u\|_B^* = \sup\big\{|u(x)| : x \in B\big\}.$$

Write $\|u\|_k^*$ in the case $B = U_k = \big\{x \in E : \|x\|_k \le 1\big\}$, where $u \in E'$, the topological dual space of E.

We say that E has

(a) the property $(\widetilde{\Omega})$ if $\forall p \; \exists q, \; d > 0 \quad \forall k \; \exists C > 0 \; \forall u \in E'$,

$$\|u\|_q^{*1+d} \le C\|u\|_k^* \; \|u\|_p^{*d},$$

(b) the property $(\widetilde{\Omega}_K)$ if there exists a closed absolutely convex bounded subset $K \subset E$ such that $\forall p \; \exists q, \; C > 0, \; d > 0 \; \forall u \in E'$,

$$\big\|u\big\|_q^{*1+d} \le C\big\|u\big\|_K^* \; \big\|u\big\|_p^{*d},$$

(c) the property (LB^∞) if $\forall \{\rho_n\} \uparrow +\infty \; \forall p \; \exists q \; \forall n_0 \; \exists N_0 \ge n_0 \; \exists C > 0$ $\forall u \in E' \; \exists n_0 \le k \le N_0$,

$$\|u\|_q^{*1+\rho_k} \le C\|u\|_k^* \; \|u\|_p^{*\rho_k}.$$

In the case E has the property $(\widetilde{\Omega})$ (resp. $(\widetilde{\Omega}_K)$, (LB^∞)) we write $E \in (\widetilde{\Omega})$ (resp. $E \in (\widetilde{\Omega}_K)$, $E \in (LB^\infty)$). The properties $(\widetilde{\Omega})$, $(\widetilde{\Omega}_K)$ and (LB^∞)

have been introduced and investigated by Vogt (see [10], [6]). In [10] Vogt has proved that E has the property (LB^{∞}) if and only if every continuous linear map from E to $\Lambda_{\infty}^{\infty}(\alpha)$ is bounded on some neighbourhood of $0 \in E$, where

$$\Lambda_{\infty}^{\infty}(\alpha) = \Big\{\xi = (\xi_j) \subset \omega : \|\xi\|_k = \sup_j |\xi_j| \rho_k^{\alpha_j} < +\infty,\ 0 < \rho_k < +\infty\ \forall k\Big\}.$$

Let U be an open subset of a locally convex space E and F a locally convex space. A mapping $f : U \to F$ is called holomorphic if f is continuous and Gâteaux-holomorphic. By $H(U, F)$ we denote the vector space of F-valued holomorphic functions on U, which is equipped with the open-compact topology. In the case $F = \mathbf{C}$ we write $H(U)$ instead of $H(U, \mathbf{C})$.

For details concerning holomorphic functions with values in locally convex spaces and scalar holomorphic functions as well as germs of holomorphic functions on compact subsets in a locally convex space we refer to the books of Dineen [3] and Noverraz [9].

3. Proof of the theorem

For proving the Theorem 1.1 we need the following

Lemma 3.1. *Let K be a compact set in a Frechet space E such that $\big[H(K)\big]'$ has property (LB^{∞}). Then K is an unique subset, i.e. if $f \in H(K)$ and $f\big|_K = 0$ then $f = 0$ on a neighbourhood of K in E.*

Proof. Let $\{V_n\}$ be a decreasing neighbourhood basis of K in E. Given $f \in H(K)$ with $f\big|_K = 0$. Choose $p \geq 1$ such that $f \in H^{\infty}(V_p)$. For each $n \geq 1$, put

$$\varepsilon_n = \|f\|_n = \sup\big\{|f(z)| : z \in V_n\big\}.$$

Then $\{\varepsilon_n\} \downarrow 0$. By the hypothesis $\big[H(K)\big]' \in (LB^{\infty})$ and applying (LB^{∞}) to $\{\rho_n\} = \Big\{\sqrt{\log\dfrac{1}{\varepsilon_n}}\ \Big\} \Big\uparrow \infty$ we have

$$\exists q\ \forall n_0\ \exists N_0 \geq n_0,\ C_{n_0} > 0\ \forall m > 0\ \exists k_m : n_0 \leq k_m \leq N_0 :$$

$$\big\|f^m\big\|_q^{1+\rho_{k_m}} \leq C_{n_0} \big\|f^m\big\|_{k_m} \big\|f^m\big\|_p^{\rho_{k_m}}$$

which yields

$$\big\|f\big\|_q^{1+\rho_{k_m}} \leq C_{n_0}^{\frac{1}{m}} \big\|f\big\|_{k_m} \big\|f\big\|_p^{\rho_{k_m}}.$$

Choose $n_0 \le k \le N_0$ such that Card $\{m : k_m = k\} = \infty$. Then

$$\begin{aligned}\|f\|_q &\le \|f\|_k^{\frac{1}{1+\rho_k}} \|f\|_p^{\frac{\rho_k}{1+\rho_k}} \\ &\le (\varepsilon_k)^{\frac{1}{1+\rho_k}} (\varepsilon_p)^{\frac{\rho_k}{1+\rho_k}} \longrightarrow 0\end{aligned}$$

as $k \to +\infty$. Hence $f\big|_{V_p} = 0$. Lemma 3.1 is proved. □

Proof of Theorem 1.1. We prove the theorem according to the following scheme

$$\begin{array}{ccccc} \text{(i)} & \Longrightarrow & \text{(iv)} & \Longrightarrow & \text{(iii)} \\ \Big\Uparrow & & \Big\Downarrow & & \Big\Downarrow \\ \text{(ii)} & \Longleftarrow & \text{(vi)} & \Longrightarrow & \text{(v)} \end{array}$$

(i) $\Rightarrow$ (iv). Given $a > 0$. Since $E \in (\widetilde{\Omega}_{K_\varepsilon})$ we have

$$(1) \qquad \forall p\ \exists q,\ d > 0,\ C > 0 : \|\cdot\|_q^{*1+d} \le C\|\cdot\|_{K_\varepsilon}^{*}\ \|\cdot\|_p^{*d}.$$

Since $\{e_j\}_{j\ge1}$ is an absolute basis of E, we have $\|e_j^*\|_q^* = \dfrac{1}{\|e_j\|_q}$ for all $j \ge 1$, $q \ge 1$. Using (1) to each e_j^* we get $\forall p\ \exists q,\ d > 0,\ C > 0$,

$$(2) \qquad \begin{aligned}\frac{1}{\|e_j\|_q^{1+d}} &\le C\|e_j^*\|_{K_\varepsilon}^* \frac{1}{\|e_j\|_p^d} \\ &= \frac{C(1+\varepsilon)\|e_j^*\|_K^*}{\|e_j\|_p^d} \\ &\le \frac{C(1+\varepsilon)}{a\|e_j\|_{K_a}\|e_j\|_p^d} \\ &\le \frac{C_1}{\|e_j\|_{K_a}\|e_j\|_p^d}\end{aligned}$$

where $\quad C_1 = \dfrac{C(1+\varepsilon)}{a}\cdot$

In order to prove that $[H(K_a)]' \in (LB^\infty)$, it suffices by Vogt [10] to show that every continuous linear map $T : [H(K_a)]' \to H(\mathbf{C})$ is bounded on a neighbourhood of $0 \in \big[H(K_a)\big]'$. Consider the function $f : K_a \to H(\mathbf{C})$ given by

$$f(x)(\lambda) = T(\delta_x)(\lambda) \quad \text{for} \quad x \in K_a,\ \lambda \in \mathbf{C}.$$

Then f is weakly holomorphic. Indeed, note firstly that since $E \in \big(\widetilde{\Omega}_{K_\varepsilon}\big)$ and K_ε is compact, it is easy to see that E is Schwartz and hence $\big[H(K_a)\big]'$ is also a Frechet-Schwartz space. Let $\mu \in \big[H(\mathbf{C})\big]'$ be given. Then $\mu T \in \big[H(K_a)\big]'' = H(K_a)$ which is a holomorphic extension of μf to a neighbourhood of K_a. For each $s > 0$ consider the weakly holomorphic function $h^s : K_a \to H^\infty(s\Delta)$ given by $h^s = R^s \circ f$, where $R^s : H(\mathbf{C}) \to H^\infty(s\Delta)$ is the restriction map. Since $a\big\|e_j^*\big\|_K^* e_j \subset K_a$ for every $j \geq 1$, it follows that K_a is an uniqueness subset. Then h^s is uniquely extended to a bounded holomorphic function $\widehat{h^s} : V^s \to H^\infty(s\Delta)$, where V^s is a neighbourhood of K_a in E. Moreover, we may assume that if $0 < r < s$ then $V^s \subset V^r$. The uniqueness yields that

$$\widehat{h^s}(x)\big|_{r\Delta} = \widehat{h^r}(x) \quad \text{for} \quad x \in V^s,\ \forall 0 < r < s.$$

Thus the family $\big\{\widehat{h^s}\big\}$ defines a holomorphic function g on a neighbourhood

$$W = \bigcup_{s>0} \big(V^s \times s\Delta\big) \text{ of } K_a \times \mathbf{C}.$$

Write the Taylor expansion of g at $0 \in E$ in $x \in E$

$$g(x)(\lambda) = \sum_{n \geq 0} P_n g(x)(\lambda),$$

where

$$P_n g(x)(\lambda) = \frac{1}{2\pi i} \int\limits_{|t|=\rho} \frac{g(tx, \lambda)}{t^{n+1}}\, dt.$$

Choose $p \geq 1$ such that $K_a + U_p \subset V^1$. Take q, d, $C > 0$ according to (2). For each $r > 0$, choose s, $D > 0$ such that

$$\big\|\sigma\big\|_r^{1+d} \leq D\big\|\sigma\big\|_s\ \big\|\sigma\big\|_1^d, \tag{3}$$

where $\sigma \in H(\mathbf{C})$ and

$$\|\sigma\|_r = \sup\big\{|\sigma(z)| : |z| \leq r\big\}.$$

Using (2) and (3) we get

$$\begin{aligned}\sum_{n\geq 0}\big|P_ng(x)(\lambda)\big| &\leq \sum_{n\geq 0}\sum_{j_1,\dots,j_n\geq 1}\big|\widehat{P_ng}(e_{j_1},\dots,e_{j_n})(\lambda)\big|\ \big|e^*_{j_1}(x)\big|\dots\big|e^*_{j_n}(x)\big| \\ &\leq \sum_{n\geq 0}\sum_{j_1,\dots,j_n\geq 1}\frac{D^{\frac{1}{1+d}}C_1^{\frac{n}{1+d}}\big|e^*_{j_1}(x)\big|\ \big\|e_{j_1}\big\|_q\dots\big|e^*_{j_n}(x)\big|\ \big\|e_{j_n}\big\|_q}{\big\|e_{j_1}\big\|_{K_a}^{\frac{1}{1+d}}\dots\big\|e_{j_n}\big\|_{K_a}^{\frac{1}{1+d}}\ \big\|e_{j_1}\big\|_p^{\frac{d}{1+d}}\dots\big\|e_{j_n}\big\|_p^{\frac{d}{1+d}}}\times \\ &\quad \big\|\widehat{P_ng}(e_{j_1},\dots,e_{j_n})\big\|_s^{\frac{1}{1+d}}\ \big\|\widehat{P_ng}(e_{j_1},\dots,e_{j_n})\big\|_1^{\frac{d}{1+d}} \\ &\leq D^{\frac{1}{1+d}}\sum_{n\geq 0}C_1^{\frac{n}{1+d}}\frac{n^n}{n!}\big\|P_ng\big\|_{K_a,s}^{\frac{1}{1+d}}\ \big\|P_ng\big\|_{p,1}^{\frac{d}{1+d}}\ \big\|x\big\|_q^n \\ &\leq D^{\frac{1}{1+d}}\big\|g\big\|_{K_a\times s\Delta}^{\frac{1}{1+d}}\ \big\|g\big\|_{U_p\times\Delta}^{\frac{d}{1+d}}\sum_{n\geq 0}C_1^{\frac{n}{1+d}}\frac{n^n}{n!}\delta^n<+\infty\end{aligned}$$

for $x\in\delta U_q$ with $\delta>0$ sufficiently small and $|\lambda|<r$. Hence g can be considered as a separately holomorphic function on $(\delta U_q\times\mathbf{C})\cup(V^1\times\Delta)$.

Let $\mathcal{F}$ denote the family of all finite dimensional subspaces P of E. Using a result of Nguyen and Zeriahi [7] to $g_P=g\big|_{(\delta U_q\cap P\times\mathbf{C})\cup(V^1\cap P\times\Delta)}$, we get a holomorphic extension $\overline{g}_P$ of g_P on $(V^1\cap P)\times\mathbf{C}$. By the uniqueness, the family $\{\overline{g}_P : P\in\mathcal{F}\}$ defines a Gateaux holomorphic function g_1 on $(V^1\times\mathbf{C})$ such that $g_1\big|_{K_a\times\mathbf{C}}=f$. On the other hand, g_1 is holomorphic on $V^1\times\Delta$. By Zorn's theorem, g_1 is holomorphic on $V^1\times\mathbf{C}$. Consider $\hat{g}_1 : V^1\to H(\mathbf{C})$ associated to g_1. As in the above argument, it follows that there exists a neighbourhood W of K_a in V^1 such that $\hat{g}_1$ is bounded on W.

Define the continuous linear map $S:\big[H^\infty(W)\big]'\to H(\mathbf{C})$ by

$$S(\mu)(\lambda)=\mu\big(\hat{g}_1(.,\lambda)\big).$$

Since K_a is an uniqueness subset of E, $\operatorname{span}\delta(K_a)$ is weakly dense in $\big[H(K_a)\big]'$ and hence it is dense in $\big[H(K_a)\big]'$ by the reflexivity of $\big[H(K_a)\big]'$, where $\delta:K_a\to\big[H(K_a)\big]'$ is given by

$$\delta(x)(\varphi)=\varphi(x),\quad x\in K_a,\ \varphi\in H(K_a).$$

Now we have

$$\begin{aligned} T\Big(\sum_{j=1}^m \lambda_j \delta_{z_j}\Big)(\lambda) &= \sum_{j=1}^m \lambda_j T(\delta_{z_j})(\lambda) = \sum_{j=1}^m \lambda_j f(z_j, \lambda) \\ &= \sum_{j=1}^m \lambda_j \hat{g}_1(z_j, \lambda) = \sum_{j=1}^m \lambda_j S(\delta_{z_j})(\lambda) \\ &= S\Big(\sum_{j=1}^m \lambda_j \delta_{z_j}\Big)(\lambda) \end{aligned}$$

for $\lambda \in \mathbf{C}$, $z_j \in K_a$. Hence $S\big|_{[H(K_a)]'} = T$ and $[H(K_a)]' \in (LB^\infty)$.

(iv) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (v). Let K_ε be polar. Choose a plurisubharmonic function φ on E such that $\varphi \neq -\infty$ and $\varphi\big|_{K_\varepsilon} = -\infty$. Consider the Hartogs domain Ω_φ given by

$$\Omega_\varphi = \Big\{(z, \lambda) \in E \times \mathbf{C} : |\lambda| < e^{-\varphi(z)}\Big\}.$$

Since φ is plurisubharmonic on E, Ω_φ is pseudoconvex. Hence, Ω_φ is the domain of existence of a holomorphic function f because E has an absolute basis. Write the Hartogs expansion of f as

$$f(z, \lambda) = \sum_{n=0}^\infty h_n(z) \lambda^n,$$

where

$$h_n(z) = \frac{1}{2\pi i} \int\limits_{|t| = \frac{1}{2} e^{-\varphi(z)}} \frac{f(z,t)}{t^{n+1}}\, dt.$$

By the upper-semi-continuity of φ, it implies that h_n are holomorphic on E for all $n \geq 0$.

Consider the function $g : K_\varepsilon \to H(\mathbf{C})$ given by

$$g(z)(\lambda) = f(z, \lambda).$$

We prove that g is weakly holomorphic. Indeed, let $\mu \in [H(\mathbf{C})]'$ be arbitrary. There exists $r > 0$ such that $\mu \in [H(r\overline{\Delta})]'$, where

$$\overline{\Delta} = \Big\{\lambda \in \mathbf{C} : |\lambda| \leq 1\Big\}.$$

By the openess of Ω_φ, there exists a neighbourhood V of K_ε in E such that $V \times r\Delta \subset \Omega_\varphi$. By the absolute convergence of the series $\sum_{n=0}^{\infty} h_n(z)\lambda^n$ on $V \times r\Delta$, it follows that $\mu g \in H(V)$ and, hence, $g \in H_w(K_\varepsilon, H(\mathbf{C}))$. Since $H(\mathbf{C}) \in (DN)$ and $[H(K_\varepsilon)]' \in (LB^\infty)$, we can find by [5] a neighbourhood U of K_ε in E and a bounded holomorphic function $\hat{g} \in H(U, H(\mathbf{C}))$ which is a holomorphic extension of g. We can write

$$\hat{g}(z,\lambda) = \sum_{n=0}^{\infty} \hat{g}_n(z)\lambda^n, \quad \lambda \in \mathbf{C},$$

where $\hat{g}_n(z)$ are holomorphic on U for $n \geq 0$.

Choose a neighbourhood W of K_ε such that $W \subset U$ and $W \times 2\Delta \subset \Omega_\varphi$. Define two holomorphic functions

$$\begin{aligned} H: \quad W &\longrightarrow H^\infty(\Delta) \\ z &\longmapsto (h_0(z), h_1(z), \dots, h_n(z), \dots) \end{aligned}$$

where

$$H(z)(\lambda) = \sum_{n=0}^{\infty} h_n(z)\lambda^n$$

and

$$\begin{aligned} G: \quad W &\longrightarrow H^\infty(\Delta) \\ z &\longmapsto (\hat{g}_0(z), \hat{g}_1(z), \dots) \end{aligned}$$

where

$$G(z)(\lambda) = \sum_{n=0}^{\infty} \hat{g}_n(z)\lambda^n.$$

By Lemma 3.1, from the condition $[H(K_\varepsilon)]' \in (LB^\infty)$ we infer that K_ε is an unique set. Moreover, we note that $H\big|_{K_\varepsilon} = G\big|_{K_\varepsilon}$ and $H^\infty(\Delta)$ is a Banach space. Thus there exists a neighbourhood W_1 of K_ε in W such that $\hat{g}\big|_{W_1\times\Delta} = f\big|_{W_1\times\Delta}$. Let X be a connected component of W_1. Since $X \times \mathbf{C}$ is connected and $\hat{g}\big|_{X\times\Delta} = f\big|_{X\times\Delta}$ and $X \times \Delta \subset \Omega_\varphi$ and Ω_φ is the domain of existence of f, we have $W_1 \times \mathbf{C} \subset \Omega_\varphi$. Hence $\varphi\big|_{W_1} = -\infty$. This is impossible.

(v) $\Rightarrow$ (i) follows from a result of Dineen-Meise-Vogt [4, Theorem 7].

(i) $\Rightarrow$ (iii) is similar as (i) $\Rightarrow$ (iv) by remarking that if $E \in \big(\widetilde{\Omega}_{K_\varepsilon}\big)$ then $[H(K_\varepsilon)]' \in (LB^\infty)$.

(iv) $\Rightarrow$ (vi) is similar as (iii) $\Rightarrow$ (v).

(vi) $\Rightarrow$ (ii) is similar as (v) $\Rightarrow$ (i).

(ii) $\Rightarrow$ (i) is obvious.

(vi) $\Rightarrow$ (v) is obvious.

Theorem 1.1 is now proved. □

4. The Hartogs domains and the properties $(\overline{\Omega}, \widetilde{\Omega})$

Let D be a pseudoconvex domain in $\mathbf{C}^n$ and φ a plurisubharmonic function on D. Consider the Hartogs domain Ω_φ in $\mathbf{C}^{n+1}$ given by

$$\Omega_\varphi = \Big\{(z, \lambda) \in D \times \mathbf{C} : |\lambda| < e^{-\varphi(z)}\Big\}.$$

In [1, Corollary 7] Aytuna has proved that if φ is continuous then $H(\Omega_\varphi)$ has the property $(\overline{\Omega})$ if and only if $H(D)$ has the same property. Recall that a Frechet space E has property $(\overline{\Omega})$ if

$$\exists d > 0 \ \forall p \ \exists q \ \forall k \ \exists C > 0 : \big\| \cdot \big\|_q^{*1+d} \le C \big\| \cdot \big\|_k^{*} \ \big\| \cdot \big\|_p^{*d}$$

or, equivalently [11],

$$\exists d > 0 \ \forall U \ \exists W \ \forall V \ \exists C > 0 : W \subset C r^d V + \frac{1}{r} U.$$

In this section we extend the above result to the infinite dimensional case.

Let E be a nuclear Frechet space with a basis $\{e_j\}_{j \ge 1}$. For each $a = (a_1, \dots, a_n, \dots)$, $a_j > 0$ for $j \ge 1$, define

$$\mathbf{D}_a = \Big\{x \in E : \sup_j |x_j| a_j < 1\Big\}.$$

Then $\mathbf{D}_a$ is called an open polydisc in E. Since $\mathbf{D}_a$ is finitely pseudo-convex, $\mathbf{D}_a$ is pseudo-convex in E. Let φ be a plurisubharmonic function on $\mathbf{D}_a$. Consider the Hartogs domain

$$\Omega_\varphi(\mathbf{D}_a) = \Big\{(z, \lambda) \in D_a \times \mathbf{C} : |\lambda| < e^{-\varphi(z)}\Big\}.$$

Then $\Omega_\varphi(\mathbf{D}_a)$ is also pseudoconvex.

Now we prove the following

Theorem 4.1. *Let E be a nuclear Frechet space with a basis $\{e_j\}_{j\geq 1}$ and a continuous norm. Assume that $\mathbf{D}_a$ is an open polydisc in E and φ is a continuous plurisubharmonic function on $\mathbf{D}_a$, $\varphi(z) = \lim_{i\to\infty} c_i \log|h_i|$ uniformly on every compact subset in D_a, where $\{h_i\}_{i\geq 1}$ are holomorphic functions on D_a, $0 < c_i < 1$, for $i \geq 1$. Then $H(\Omega_\varphi(\mathbf{D}_a))$ has properties $(\widetilde{\Omega}, \overline{\Omega})$ if and only if $H(\mathbf{D}_a)$ has the same properties.*

Proof. It suffices to prove the case $(\widetilde{\Omega})$ because the case $(\overline{\Omega})$ can be proved in a similar way. Assume that $H(\mathbf{D}_a)$ has property $(\widetilde{\Omega})$. By [3] we can choose an exhaustion of $\mathbf{D}_a$ by compact polydiscs $\{D_q\}_{q\geq 1}$ of the form

$$D_q = \Big\{y = (y_j) \in \mathbf{D}_a : \sup_j \Big|\frac{y_j}{b_j^{(q)}}\Big| \leq 1\Big\}.$$

Since $\varphi(z) \neq -\infty$ for $z \in \mathbf{D}_a$, without loss of generality we may assume that for each $q \geq 1$,

$$h_q(z) \neq 0 \quad \text{for} \quad z \in \mathbf{D}_q\,.$$

By the convexity of $\mathbf{D}_q$ we can write

$$h_q^{c_q}(z) = e^{g_q(z)} \quad \text{for} \quad z \in \mathbf{D}_q\,,$$

where $g_q \in H(\mathbf{D}_q)$.

For each $q \geq 1$ choose $\hat{g}_q \in H(\mathbf{D}_a)$ such that

$$\big\|\hat{g}_q - g_q\big\|_{\mathbf{D}_q} < \frac{1}{q}\,.$$

Since $\{|e^{g_q(z)}|\}_{q\geq 1}$ converges uniformly on all compact sets of $\mathbf{D}_a$ to e^φ, it follows that $\{|e^{\hat{g}_q(z)}|\}_{q\geq 1}$ also converges uniformly on all compact subsets of $\mathbf{D}_a$ to e^φ. Take a sequence of positive numbers $\{\delta_q\} \downarrow 0$ such that $2\delta_{q+1} < \delta_q$. Write each $f \in H(\Omega_\varphi(\mathbf{D}_a))$ in the form

$$f(z,\lambda) = \sum_{n\geq 0} h_n(z)\lambda^n,$$

where

$$h_n(z) = \frac{1}{2\pi i} \int\limits_{|\lambda| = e^{-(\varphi(z)+\delta_q)}} \frac{f(z,\lambda)}{\lambda^{n+1}}\, d\lambda.$$

By Cauchy's theorem, $h_n(z)$ is not dependent on δ_q. We have

$$\sup\Big\{|h_n(z)\lambda^n| : (z,\lambda)\in K_q,\ n\geq 0\Big\}$$
$$\leq \sup\Big\{\Big|h_n(z)e^{-n(\varphi(z)+\delta_q)}\Big| : z\in \mathbf{D}_q,\ n\geq 0\Big\} \leq \|f\|_q\,,$$

where

$$K_q = \Big\{(z,\lambda) : z\in\mathbf{D}_q,\ |\lambda|\leq e^{-(\varphi(z)+\delta_q)}\Big\},$$
$$\|f\|_q = \sup\ \Big\{|f(z,\lambda)| : z\in\mathbf{D}_q,\ |\lambda|\leq e^{-(\varphi(z)+\delta_q)}\Big\}.$$

On the other hand,

$$\begin{aligned}\|f\|_p &\leq \sum_{n\geq 0}\sup_{K_p}|h_n(z)\lambda^n| \\ &\leq \sum_{n\geq 0}\sup_{z\in D_p}\Big|h_n(z)e^{-n(\varphi(z)+\delta_p)}\Big| \\ &\leq \Big(\sum_{n\geq 0}e^{-n(\delta_p-\delta_q)}\Big)\sup_{\mathbf{D}_q}\Big|h_n(z)e^{-n(\varphi(z)+\delta_q)}\Big|.\end{aligned}$$

Hence, the topology of $H(\Omega_\varphi(\mathbf{D}_a))$ can be defined by the system of semi-norms $\big\{|||\cdot|||_q\big\}$, where

$$|||f|||_q = \sup\Big\{|||h_n|||_q : n\geq 0\Big\},$$
$$|||h_n|||_q = \sup\Big\{\Big|h_n(z)e^{-n(\varphi(z)+\delta_q)}\Big| : z\in\mathbf{D}_q\Big\}.$$

Since $H(\mathbf{D}_a)\in(\widetilde{\Omega})$, we have

$$\forall p\ \exists q,\ d>0\ \forall k\ \exists C>0\ \forall r>0 : U_q\subset Cr^dU_k+\frac{1}{r}U_p\,, \tag{4}$$

where

$$U_p = \Big\{h\in H(\mathbf{D}_a) : \|h\|_p = \sup_{z\in\mathbf{D}_p}|h(z)|\leq 1\Big\}.$$

We show that

$$\forall p\ \exists q,\ \varepsilon>0\ \forall k>0\ \exists D>0\ \forall r>0 : W_q\subset Dr^\varepsilon W_k+\frac{1}{r}W_p\,, \tag{5}$$

where

$$W_p = \Big\{ f \in H(\Omega_\varphi(\mathbf{D}_a)) : |||f|||_p \le 1 \Big\}.$$

First observe that (5) holds for all $q \ge p$, $\varepsilon > 0$, $D > 0$, $k > 0$ and $0 < r \le 1$, because $W_q \subset W_p$.

Now let $f \in W_q$, $r < 1$. Write

$$\begin{aligned} |||h_n|||_p &= \sup_{\mathbf{D}_p} \Big| h_n(z) e^{-n(\varphi(z)+\delta_p)} \Big| \\ &\le \sup_{\mathbf{D}_q} \Big| h_n(z) e^{-n(\varphi(z)+\delta_q)} \Big| \, e^{-n(\delta_p - \delta_q)} \\ &\le e^{-n(\delta_p-\delta_q)} \le \frac{1}{r} \end{aligned}$$

if $n \ge \alpha = \dfrac{\log r}{\delta_p - \delta_q} \cdot$ For each $n \ge 1$ choose j_n such that

$$\Big| \frac{e^{n\varphi(z)}}{\big|e^{n\hat{g}_{j_n}(z)}\big|} - 1 \Big| < 1$$

and

$$\Big| \frac{\big|e^{n\hat{g}_{j_n}(z)}\big|}{e^{n\varphi(z)}} - 1 \Big| < 1 \quad \text{for} \quad z \in \mathbf{D}_k\,.$$

Since

$$\begin{aligned} \Big\| \frac{h_n}{e^{n\hat{g}_{j_n}}} \Big\|_q &= \sup \Big\{ \Big| \frac{h_n(z)}{e^{n\hat{g}_{j_n}(z)}} \Big| : z \in \mathbf{D}_q \Big\} \\ &= \sup \; \Big\{ \Big| h_n(z) e^{-n(\varphi(z)+\delta_q)} \Big| : z \in \mathbf{D}_q \Big\} \times \\ &\qquad \times \sup \Big\{ \Big| \frac{e^{n(\varphi(z)+\delta_q)}}{e^{n\hat{g}_{j_n}(z)}} \Big| : z \in \mathbf{D}_q \Big\} \\ &\le 2e^{n\delta_q} \le 2e^{\delta_q \frac{\log r}{\delta_p-\delta_q}} \\ &\le 2r^{\frac{\delta_q}{\delta_p-\delta_q}} \end{aligned}$$

for all $0 \le n \le \alpha$, and by (4), we have

$$h_n = 2Cr^{\left(d + \frac{\delta_q}{\delta_p - \delta_q}\right)} e^{n\hat{g}_{j_n}} u_n + 2r^{\frac{2\delta_q - \delta_p}{\delta_p-\delta_q}} e^{n\hat{g}_{j_n}} .v_n\,,$$

$u_n \in U_k$, $v_n \in U_p$ for $0 \le n \le \alpha$.

For $k > q$ and $0 \le n \le \alpha$ we have

$$\begin{aligned}\big|\big\|e^{n\hat{g}_{j_n}}u_n\big\|\big|_k &= \sup_{z\in\mathbf{D}_k}\left|e^{n\hat{g}_{j_n}(z)}u_n(z).e^{-n(\varphi(z)+\delta_k)}\right| \\ &\le \sup_{z\in D_k}\left|e^{n\left[(\hat{g}_{j_n}(z)-\varphi(z))-\delta_k\right]}\right| \\ &\le \sup\left\{\left|\frac{e^{n(\hat{g}_{j_n}(z))}}{e^{n\varphi(z)}}\right| : z\in D_k\right\} \\ &= \sup\left\{\left|\frac{\left|e^{n\hat{g}_{j_n}(z)}\right|}{e^{n\varphi(z)}}\right| : z\in D_k\right\} \\ &\le 2.\end{aligned}$$

This shows that $\dfrac{e^{n\hat{g}_{j_n}}u_n}{2} \in W_k$ for $0 \le n \le \alpha$. On the other hand,

$$\begin{aligned}|\|e^{n\hat{g}_{j_n}}v_n\||_p &= \sup_{\overline{D}_p}\left|e^{n\hat{g}_{j_n}(z)}v_n(z)e^{-n(\varphi(z)+\delta_p)}\right| \\ &\le 2 \quad \text{for} \quad 0 \le n \le \alpha.\end{aligned}$$

Hence, we can write

$$f(z,\lambda) = \sum_{n=0}^{\infty} h_n(z)\lambda^n = \sum_{n=0}^{\alpha} h_n(z)\lambda^n + g(z,\lambda),$$

where

$$g(z,\lambda) = \sum_{n=\alpha+1}^{\infty} h_n(z)\lambda^n.$$

We have

$$|\|g\||_p = \sup\left\{|\|h_n\||_p : n \ge \alpha+1\right\} \le \frac{1}{r}$$

and

$$h_n = 4Cr^{\left(d+\frac{\delta_q}{\delta_p-\delta_q}\right)}\frac{e^{n\hat{g}_{j_n}}u_n}{2} + 2r^{\frac{2\delta_q-\delta_p}{\delta_p-\delta_q}}e^{n\hat{g}_{j_n}}v_n$$

for $0 \le n \le \alpha$, where $\dfrac{e^{n\hat{g}_{j_n}}u_n}{2} \in W_k$ and

$$\left|\left\|2r^{\frac{2\delta_q-\delta_p}{\delta_p-\delta_q}}e^{n\hat{g}_{j_n}}v_n\right\|\right|_p \le \frac{4}{r^{\frac{\delta_p-2\delta_q}{\delta_p-\delta_q}}}$$

with $\dfrac{\delta_p - 2\delta_q}{\delta_p - \delta_q} > 0$. Thus

$$f \in Dr^{\varepsilon} W_k + \frac{1}{r} W_p \quad \text{for} \quad r > 0,$$

where

$$\varepsilon = \frac{1}{\eta}\Big(d + \frac{\delta_q}{\delta_p - \delta_q}\Big), \quad \eta = \min\Big(\frac{\delta_p - 2\delta_q}{\delta_p - \delta_q}\ , 1\Big).$$

This shows that $H(\Omega_\varphi(\mathbf{D}_a)) \in (\widetilde{\Omega})$.

Conversely, we note that $H(\mathbf{D}_a)$ is the quotient space of $H(\Omega_\varphi(\mathbf{D}_a))$ under the continuous linear surjection $T : H(\Omega_\varphi(\mathbf{D}_a)) \longrightarrow H(\mathbf{D}_a)$ given by

$$T(f) = g,$$

where

$$g(z) = f(z, 0) \quad \text{for} \quad z \in \mathbf{D}_a, \quad f \in H(\Omega_\varphi(\mathbf{D}_a)).$$

Hence, if $H(\Omega_\varphi(\mathbf{D}_a))$ has property $(\widetilde{\Omega})$ then $H(\mathbf{D}_a) \in (\widetilde{\Omega})$. □

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