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# EXTENDING HOLOMORPHIC MAPS THROUGH PLURI-POLAR SETS IN HIGH DIMENSION

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**Abstract.** In this paper we prove that if a complex space $X$ has the strictly holomorphic 1-extension property through polar sets, then $X$ also has the strictly holomorphic n-extension property through pluri-polar sets for all $n \geq 2$. Moreover, some results of Suzuki and Järvi are deduced from the above-mentioned theorem.

## Introduction

Let $S$ be a pluri-polar set in a complex manifold $Z$ and $X$ be a complex space. If $f : Z \setminus S \to X$ is a holomorphic map, the question as to whether one can find a holomorphic extension $f : Z \to X$ of the map $f$ has been studied by various authors (see [2], [7], [8], [9]). For example, if $X$ is a Siegel domain of type 2 in $C^n$, then every map $f$ has a holomorphic extension to $Z$ (see Sibony [7]) or if $X$ is a convex domain in $C^n$, a holomorphic extension through hypersurfaces of every map $f$ exists iff $X$ is hyperbolic (see P. K. Ban [1]).

In the case $X$ is a Riemann surface and $Z\setminus S$ is the punctured disc $0 < |z| < 1$ in $\mathbf{C}$, the problem was investigated by Royden [4]. Recently, Järvi [2] generalized Royden's results for the case of compact subsets of capacity zero in a domain $Z \subset \mathbf{C}$. The above-mentioned extension problem for high dimension, i.e. $Z$ is an arbitrary complex manifold, was also investigated by Suzuki [8]. However his proof is not correct.

The aim of the present note is to study the above-mentioned problem for high dimension. In particular, we will give a correct proof of Suzuki's result . At the same time, a generalization of Järvi's results to the case of an arbitrary complex manifold will be obtained.

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## 1. Some definitions and main results

DEFINITION. Let $X$ be a complex space. We say that $X$ has the holomorphic $n$-extension property through pluri-polar sets if for every complex manifold $Z$ of dimension n, the restriction map $R : H(Z, X) \to H(Z \setminus S, X)$ is surjective for every pluri-polar closed set $S$ in $Z$.

Here $H(Z, X)$ denotes the space of holomorphic maps from $Z$ into $X$ equipped with the compact-open topology. If the restriction map $R$ is a homeomorphism, then we say that $X$ has the strictly holomorphic n-extension property.

Shortly, the homeomorphism $R$ is denoted by $R : H(Z, X) \cong H(Z \setminus S, X)$.

In this section we prove the following two theorems.

THEOREM 1.1. *Let $X$ be a complex space having the strictly holomorphic 1-extension property through polar set. Then $X$ also has the strictly holomorphic n-extension property through pluri-polar sets for $n \geq 2$.*

THEOREM 1.2. *Every Siegel domain $D$ of type 2 in a Banach space $B$ has the strictly holomorphic n-extension property through pluri-polar sets for all $n \geq 1$.*

Here, by a Siegel domain of type 2 in $B$ we mean a domain $D$ in $B$ of the form

$$D = \{(u, v) \in B = (A \oplus iA) \times W : \text{ Im } u - F(w, w) \in V\},$$

where $A$ is a real Banach space, $F : W \times W \to A \oplus iA$ is a continuous $V$-Hermitian map and $V$ is an open convex cone in $A$

## 2. Proof of Theorem 1.1

First observe that $X$ satisfies the weak disc condition. This means that if $\{f_n\} \subset H(\Delta, X)$, where $\Delta$ is the open unit disc in $\mathbf{C}$, converges to $f$ in $H(\Delta^*, X)$, $\Delta^* = \Delta \setminus \{0\}$, then $f$ can be extended holomorphically to $\Delta$ and $f_n \to f$ in $H(\Delta, X)$. By [6], $X$ has the Hartogs holomorphic extension property.

Given $n \geq 1$. Since the problem is local, without loss of generality we may assume that $Z = U \times W$, where $U$ is an open polydisc in $\mathbf{C}^{n-1}$ and $W$ is an open set in $\mathbf{C}$.

For every pluri-polar set $S$ in $Z$, we put

$$S' = \{z \in U : z \times W \subset S\},$$

and

$$S'' = \{w \in W : U \times w \subset S\}.$$

We claim that $S'$ and $S''$ are pluri-polar in $U$ and $W$, respectively. It suffices to check this for $S'$ because the check for $S''$ is similar.

Given $z_0 \in S'$. Take $w_0 \in W$ such that $(z_0, w_0) \notin S$. Since $S$ is pluri-polar we can find a neighbourhood $U_0 \times W_0$ of $(z_0, w_0)$ in $U \times W$ and a plurisubharmonic function $\varphi$ on $U_0 \times W_0$ such that $\varphi|_{S \cap (U_0 \times W_0)} = -\infty$ and $\varphi(z_0, w_0) \neq -\infty$. If we put $\alpha(z) = \varphi(z, w_0)$ for $z \in U_0$, then $\alpha$ is plurisubharmonic on $U_0$ such that $\alpha(z) = \varphi(z, w_0) = -\infty$ for every $z \in U_0 \cap S'$ and $\alpha(z_0) = \varphi(z_0, w_0) \neq -\infty$.

We put

$$S^w = \{z \in U : (z, w) \in S\} \text{ for each } w \in W$$

and

$$S_z = \{w \in W : (z, w) \in S\} \text{ for each } z \in U.$$

Then $S^w$ is pluri-polar for $w \notin S''$, and, similarly, $S_z$ is pluri-polar for $z \notin S'$. Indeed, reasoning as above, it suffices to check this for $S^w$, where $w \notin S''$. From the relation $w \notin S''$ there exists $a \in U$ such that $(a, w) \notin S$. By a theorem of Josefson [3] we can find a plurisubharmonic function $\varphi$ on $U \times W$ such that $\varphi|_S = -\infty$ and $\varphi(a, w) \neq -\infty$. Define a plurisubharmonic function $\alpha$ on $U$ by $\alpha(z) = \varphi(z, w)$ for $z \in U$. It is obvious that $\alpha \neq -\infty$ and $\alpha|_{S^w} = -\infty$.

Now consider a sequence $\{f_k\}$ of holomorphic maps from $U \times W \setminus S$ into $X$ which converges to $f$ in $H(U \times W \setminus S, X)$. For each $w \notin S''$ and each $k \geq 1$ consider the holomorphic map

$$f_k^w : U \setminus S^w \to X$$

given by

$$z \mapsto f_k^w(z) = f_k(z, w).$$

By the inductive hypothesis, $f_k^w \to f^w$ in $H(U, X)$. Similarly, for each $z \notin S'$, the sequence of holomorphic maps $\{f_{k,z}\} \subset H(W \setminus S'', X)$, given by $f_{k,z}(w) = f_k(z, w)$, converges to $f_z$ in $H(W, X)$. Thus, we can define the maps

$$f_1 : U \setminus S' \times W \to X \quad \text{by} \quad f_1(z, w) = f_z(w)$$

and

$$f_2 : U \times W \setminus S'' \to X \ \text{ by } \ f_2(z,w) = f^w(z).$$

We shall show that $f_1$ and $f_2$ are holomorphic. By Shiffman [5] and by the holomorphic extendability of $X$, it suffices to prove that $f_1$ ( reps.$f_2$) is holomorphic in $z \in U \setminus S'$ (resp. in $w \in W \setminus S''$). We will prove this statement only for $f_1$ because the proof for $f_2$ is analogous. Fix $w \in W$. Let $\{z_p\} \subset U \setminus S', \{z_p\} \to z \in U \setminus S'$. Since $S$ is closed, it follows that

$$P = \big(\bigcup_{p=1}^{\infty} S_{z_p}\big) \cup S_z$$

is a pluri-polar closed set in $W$. On the other hand, inductive hypothesis yields

$$H(W,X) \cong H(W \setminus P, X).$$

Since $f_{z_p} \to f_z$ in $H(W \setminus P, X)$, we see that $f_{z_p} \to f_z$ in $H(W,X)$. Hence

$$f_1(z_p, w_p) = f_{z_p}(w_p) \to f_z(w) = f_1(z,w)$$

for every $\{w_p\} \to w$ in $W$. Thus $f_1$ is continuous on $U \setminus S' \times W$. Similarly, $f_2$ is continuous on $U \times W \setminus S''$. Since $U \times W \setminus S \subset (U \setminus S') \times (W \setminus S'')$ and $U \times W \setminus S$ is dense in $U \times W$, we have

$$f_1|_{(U\setminus S')\times(W\setminus S'')} = f_2|_{(U\setminus S')\times(W\setminus S'')}.^{*}$$

Moreover $f_z(w) = f_1(z,w) = f_2(z,w) = f^w(z)$ for $z \in U \setminus S'$ and $w \in W \setminus S''$. This means $f_1$ is separately holomorphic on $(U \setminus S') \times (W \setminus S'')$. By Shiffman [5] $f_1$ is holomorphic on $(U \setminus S') \times (W \setminus S'')$. Similarly, $f_2$ is also holomorphic on $(U \setminus S') \times (W \setminus S'')$.

By the continuity of $f_1$ and $f_2$ on $U \setminus S' \times W$ and $U \times W \setminus S''$, respectively, it follows that $f_1$ and $f_2$ are separately holomorphic on $U \setminus S' \times W$ and $U \times W \setminus S''$ respectively. Thus from (*) we can define a function $\overline{f}$ on $(U \setminus S' \times W) \cup (U \times W \setminus S'')$ by

$$\overline{f}|_{U\setminus S'\times W} = f_1 \ \textit{and} \ \overline{f}|_{U\times W\setminus S''} = f_2$$

which is separately holomorphic. Again by Shiffman [5] $\overline{f}$ is holomorphic. Since by [6].

$$^\wedge\big[(U \setminus S' \times W) \cup (U \times W \setminus S''\big] = U \times W,$$

we see that $\overline{f}$ is extended to a holomorphic map $\hat{f}$ from $U \times W$ into $X$.

It remains to show that $\hat{f}_k \to \hat{f}$ in $H(U \times W, X)$. First we shall see that

$\hat{f}_k|_{U\setminus S'} \to \hat{f}|_{U\setminus S'}$ in $H(W,X)$. Let $\{z_k\} \subset U \setminus S', \{w_k\} \subset W$ with $z_k \to z \in U \setminus S', \{w_k\} \to w \in W$.

As before put

$$P = \{\bigcup_{k=1}^{\infty} S_{z_k}\} \cup S_z.$$

Since $f_{k,z_k} \to f_z$ in $H(W\setminus P, X)$, we have $\hat{f}_{k,z_k} \to \hat{f}_k$ in $H(W,X)$. Hence $\hat{f}_k \to \hat{f}$ in $H(U \setminus S' \times W, X)$ and $\hat{f}_k \to \hat{f}$ in $H(U \times W, X)$.

The proof of theorem 1.1 is complete.

## 3. Proof of Theorem 1.2

First we will show that $D$ is convex in $B$. Indeed, for every $(u_1, w_1), (u_2, w_2) \in D$ and for every $\theta \in [0,1]$ we have

$$\begin{aligned} &Im\big[\theta u_1 + (1-\theta)u_2\big] - F\big(\theta w_1 + (1-\theta)w_2, \theta w_1 + (1-\theta)w_2\big) \\ &\quad = \theta\big[Im\ u_1 - F(w_1,w_1)\big] + (1-\theta)\big[Im\ u_2 - F(w_2,w_2)\big] + \\ &\qquad \theta(1-\theta)F(w_1 - w_2, w_1 - w_2) \in V. \end{aligned}$$

Hence $\big(\theta u_1 + (1-\theta)u_2, \theta w_1 + (1-\theta)w_2\big) \in D$, i.e. $D$ is convex in $B$.

Write

$$V = \big\{x \in A : x_\alpha^*(x) > 0, \alpha \in I\big\}.$$

Since $V$ is a cone, we have

$$\bigcap_{\alpha \in I} Ker\ x_\alpha^* = \{0\}.$$

Let $(u_1, w_1), (u_2, w_2)$ be two points of $D$ such that $u_1 \neq u_2$. Without loss of generality, we may assume that $Re\ u_1 \neq Re\ u_2$. Choose $\alpha \in I$ such that $x_\alpha^*(Re\ u_1) \neq x_\alpha^*(Re\ u_2)$. Let $z_\alpha^* \in (A \oplus iA)^*$ be given by $z_\alpha^*(x+iy) = x_\alpha^*(x) + ix_\alpha^*(y)$. Then

$$f(u,w) = \frac{z_\alpha^*(u)}{z_\alpha^*(u)+1}, (u,w) \in D,$$

is a bounded holomorphic function on $D$ with $f(u_1, w_1) \neq f(u_2, w_2)$.

Assume that there exists a non-constant complex line $L$ in $D$. By We may assume that $L$ has the form $(u_0, \lambda w_0), \lambda \in \mathbf{C}$. Since $\bigcap_{\alpha \in I} Ker z_\alpha^* = \{0\}$ and $(i(Im\ u_0 - k^2 F(w_0, w_0)), 0) \in D$ for each, $k \geq 1$ we can find $z_{\alpha_k}^*$ such that $|z_{\alpha_k}^*(y_k)| \to \infty$, where $y_k = i(Im\ u_0 - k^2 F(w_0, w_0))$. For each $k \geq 1$, consider the bounded holomorphic function $f_k$ on $D$ defined by

$$f_k(u,w) = \frac{z^*_{\alpha_k}(u)}{z^*_{\alpha_k}(u)+i}$$

Put

$$\begin{aligned} T_k(u,w) =&(u - Re\ u_0 - 2iF(w,kw_0) + ik^2F(w_0,w_0), w - kw_0) \\ z_k =&\big(iIm\frac{u_0}{2} - Re\frac{u_0}{2}, \frac{kw_0}{2}\big). \end{aligned}$$

Since $T_k$ is a biholomorphism from $D$ onto $D$, we can define $g_k = f_k \circ T_k$ for $k \geq 1$. By the inequality

$$\sup |\frac{z^*_{\alpha_k}(z_k)}{z^*_{\alpha_k}(z_k)+i}| < 1$$

there exists a sequence $\{\sigma_k\}$ of automorphisms of $\Delta$ such that

$$\sigma_k\big(\frac{z^*_{\alpha_k}(z_k)}{z^*_{\alpha_k}(z_k)+i}\big) = 0, \text{ for } k \geq 1$$

and

$$|\sigma_k\big(\frac{z^*_{\alpha_k}(y_k)}{z^*_{\alpha_k}(y_k)+1}\big)| \to 1 \text{ as } k \to \infty.$$

Fix $q \in D$. Since $T_k(\frac{u_0}{2}, \frac{kw_0}{2}) = z_k$ for $k \geq 1$ and by the tautness of $\Delta$, it follows that

$$\sup_{k\geq 1} |\sigma_k \circ f_k \circ T_k(q)| < 1.$$

Again there exists a sequence $\{\beta_k\}$ of automorphisms of $\Delta$ such that

$$\beta_k\sigma_k g_k(q) = 0 \text{ for } k \geq 1$$

and

$$|\beta_k\sigma_k g_k(u_0, kw_0)| \to 1 \text{ as } k \to \infty.$$

This is impossible. Thus, $D$ is a convex domain containing no complex lines.

Now we show that $D$ has the strictly 1-extension property. Given $\{f_k\} \subset H(\Delta \setminus S, D)$ with $f_k \to f$ in $H(\Delta \setminus S, D)$. Let $z_0 \in S$. Since $S$ is polar, dim $S = 0$ and hence there exists a neighbourhood $U_0$ of $z_0$ in $S$ such that $\partial U_0 = \emptyset$. Let $U$ be a neighbourhood of $z_0$ in $Z$ such that $U \cap S = U_0$. Take a $\mathbf{C}^\infty$-function $\varphi$ in a neighbourhood of $\overline{U}$ such that $0 \leq \varphi \leq 1, \varphi|_{S\cap U} = 0$ and $\varphi = 1$ in a neighbourhood of $\partial U$.

By Sard theorem, there exists $0 < r < 1$ such that $\varphi^{-1}(r)$ is a $\mathbf{C}^\infty$-curve. Put

$$W = \{z \in U : \varphi(z) < r\}.$$

For each $k \geq 1$ consider the holomorphic function $g_k$ on $W$ given by

$$g_k(z) = \frac{1}{2\pi i} \int_{\partial W} \frac{f_k(t)}{t - z} dt.$$

Since $x^*_\alpha g_k$ is holomorphic on $W$ for all $\alpha \in I$, we have $x^*_\alpha g_k = x^*_\alpha f_k|_W$ for $\alpha \in I$ and $k \geq 1$, where $\{x^*_\alpha\} \subset B^*$ are choosen such that $D = \cap\{Re\ x^*_\alpha < \epsilon_\alpha\}$. We may assume that $0 \in D$, and hence $\epsilon_\alpha > 0$ for all $\alpha \in I$. We have

$$\bigcap_\alpha ker\ x^*_\alpha = \{0\},$$

and hence

$$g_k|_{W\setminus S} = f_k|_{W\setminus S} \text{ for all } k \geq 1.$$

Thus $f_k$ is extended holomorphically to $z_0 \in S$. Since $z_0$ is arbitrary, $f_k$ can be extended to a holomorphic function $\hat{f}_k$ on $Z$. Moreover $\hat{f}_k \to \hat{f}$ in $H(Z, B)$. We will see that $\hat{f}(\Delta) \subset D$. Indeed, for the above $z_0$ and $W$ we have $f(\partial W) \subset D$, and hence $Re\ x^*_\alpha f(\partial W) < \epsilon_\alpha$ for $\alpha \in I$. By the maximum principle for harmonic functions, it follows that

$$Re\ x^*_\alpha f(z_0) < \epsilon_\alpha \text{ for } \alpha \in I.$$

Hence $f(z_0) \in D$. Thus, $f(W) \subset D$.

## 4. Some applications

In this section we will combine Theorem 1.1 with the result of Järvi [2] and Suzuki [8] in order to extend their result for high dimension.

THEOREM 4.1. (cf. Järvi [2]) Let $X$ be a compact hyperbolic Riemann surface. Then $X$ has the strictly holomorphic n-extension property through pluri-polar sets for every $n \geq 1$.

PROOF. By Järvi [2] the restriction map $R : H(Z, X) \to H(Z \setminus S, X)$ is surjective for every complex curve $Z$ and every polar set $S$ in $Z$. By the complete hyperbolicity of $X$, it follows that $H(Z \setminus S, X) \cong H(Z, X)$. Theorem 1.1 yields $H(Z \setminus S, X) \cong H(Z, X)$ for every complex manifold $Z$ and every pluri-polar set $S$ in $Z$.

THEOREM 4.2. (cf. Suzuki [8]) Let $M$ be a compact complex manifold such that

(i) $M$ has a universal cover $\Omega$ which is biholomorphic to a bounded domain in $\mathbf{C}^n$.

(ii) Every biholomorphism of the universal cover $\Omega \to M$ can be extended holomorphically to a neighbourhood of $\overline{\Omega}$.

Then $M$ has the strictly holomorphic n-extension property through pluripolar sets for all $n \geq 1$.

PROOF. First note that $M$ is hyperbolic. By the result of Suzuki [8] the theorem holds for $n = 1$. Theorem 1.1 implies that Theorem 4.2 holds for all $n \geq 1$.

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