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ON HAMILTON CYCLES IN CUBIC $(10, n)$-METACIRCULANT GRAPHS

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Abstract. We prove in this paper that every connected cubic $(10, n)$-metacirculant graph has a Hamilton cycle if n is a positive integer such that $\varphi(n)$ is not divisible by 5, where $\varphi(n)$ is the number of integers z satisfying $0 \leq z < n$ and $gcd(z, n) = 1$.

1. Introduction

The problem of the existence of a Hamilton cycle in *vertex-transitive graphs* apeals the attention of researchers for many years. At present, there is only a very limited supply of connected vertex-transitive non-hamiltonian graphs. These graphs are the Petersen graph, the Coxeter graph and the two graphs obtained from these by replacing each vertex by a triangle. C. Thomassen (see [4]) has conjectured that there are only finitely many such graphs. None of these four graphs is a Cayley graph, so that it may be conjectured that every connected Cayley graph on a finite group has a Hamilton cycle. This has been shown true at least for abelian groups [5] and some other special groups [6, 14].

Among vertex-transitive graphs, (m, n)-*metacirculant graphs* introduced recently in [1] are interesting because they were defined as a logical generalization of the Petersen graph with the purpose of providing a class of vertex-transitive graphs in which there might be some new non-hamiltonian connected vertex-transitive graphs. It has been asked [1] whether all connected (m, n)-metacirculant graphs, other than the Petersen graph, have a Hamilton cycle.

For $n = p^t$ with p a prime, connected (m, n)-metacirculant graphs, other than the Petersen graph, have been proved to have a Hamilton cycle [2]. Connected cubic (m, n)-metacirculant graphs, other than the Petersen graph, also have been proved to be hamiltonian for m odd [7], m = 2 [3, 7], and m divisible by 4 [8, 11]. Thus, the remaining values of m, for which we still do not know whether all connected cubic (m, n)-metacirculant graphs have a Hamilton cycle, are of the form $m = 2\mu$ with $\mu \geq 3$ an odd positive integer.

Recently, some results on Hamilton cycles in cubic (m, n)-metacirculant graphs for the case $m = 6$ have been obtained in [12, 13]. In this paper,

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we investigate the problem of the existence of a Hamilton cycle in cubic (m,n)-metacirculant graphs for the case $m = 10$. Namely, we will prove here that a connected cubic $(10,n)$-metacirculant graph has a Hamilton cycle if n is a positive integer such that $\varphi(n)$ is not divisible by 5, where $\varphi(n)$ is the number of integers z satisfying $0 \le z < n$ and $gcd(z,n) = 1$.

2. Preliminaries

In this paper we consider only finite undirected graphs without loops or multiple edges. If G is a graph, then V(G) and E(G) denote its vertex-set and its edge-set, respectively. If n is a positive integer, then we write Z_n for the ring of integers modulo n and Z_n^* for the multiplicative group of units in Z_n.

Let m and n be two positive integers, $\alpha \in Z_n^*, \mu = \lfloor m/2 \rfloor$ and let $S_0, S_1, \dots, S_\mu$ be subsets of Z_n satisfying the following conditions:

(1) $0 \notin S_0 = -S_0$;

(2) $\alpha^m S_r = S_r$ for $0 \le r \le \mu$;

(3) If m is even, then $\alpha^\mu S_\mu = -S_\mu$.

Then we define the (m,n)-*metacirculant graph* $G = MC(m,n,\alpha,S_0,S_1,\dots,S_\mu)$ to be the graph with vertex-set $V(G) = \{v_j^i : i \in Z_m;\ j \in Z_n\}$ and edge-set $E(G) = \{v_j^i v_h^{i+r} : 0 \le r \le \mu;\ i \in Z_m;\ h,j \in Z_n \text{ and } (h-j) \in \alpha^i S_r\}$, where superscripts and subscripts are always reduced modulo m and modulo n, respectively.

The above construction is designed to allow the permutations ρ with $\rho(v_j^i) = v_{j+1}^i$ and τ with $\tau(v_j^i) = v_{\alpha j}^{i+1}$ to be automorphisms of G. Thus, (m,n)-metacirculant graphs are vertex-transitive.

We will use the following results of [9, 10].

LEMMA 1. [9] *Let $G = MC(m,n,\alpha,S_0,S_1,\dots,S_\mu)$ be a cubic (m,n)-metacirculant graph such that $m > 2$ is even, $S_0 = \emptyset, S_i = \{s\}$ with $0 \le s < n$ for some $i \in \{1,2,\dots,\mu-1\}$, $S_j = \emptyset$ for all $i \ne j \in \{1,2,\dots,\mu-1\}$ and $S_\mu = \{k\}$ with $0 \le k < n$. Then*

(i) If G is connected, then either i is odd and gcd(i,m) = 1 or i is even, μ is odd and gcd(i,m) = 2.

(ii) If i is odd and gcd(i,m) = 1, then G is isomorphic to the cubic (m,n)-metacirculant graph $G' = MC(m,n,\alpha',S'_0,S'_1,\dots,S'_\mu)$ with $\alpha' = \alpha^i, S'_0 = \emptyset, S'_1 = \{s\}, S'_2 = \dots = S'_{\mu-1} = \emptyset$ and $S'_\mu = \{k\}$.

(iii) If i is even, μ is odd, gcd(i,m) = 2 and $i = 2^r i'$ with $r \ge 1$ and i' odd, then G is isomorphic to the cubic (m,n)-metacirculant graph $G'' = MC(m,n,\alpha'', S''_0, S''_1,\dots,S''_\mu)$ with $\alpha'' = \alpha^{i'}, S''_0 = S''_1 = \dots = S''_{2^r-1} = \emptyset, S''_{2^r} = \{s\}, S''_{2^r+1} = \dots = S''_{\mu-1} = \emptyset$ and $S''_\mu = \{k\}$.

LEMMA 2. [9] *(i) Let $G = MC(m,n,\alpha,S_0,S_1,\dots,S_\mu)$ be a cubic (m,n)-metacirculant graph such that $m > 2$ is even, $S_0 = \emptyset, S_1 = \{s\}, S_2 = \dots = S_{\mu-1} = \emptyset$*

and $S_\mu = \{k\}$. *Then G is connected if and only if gcd(p,n) = 1, where p is* $[k - s(1 + \alpha + \alpha^2 + \cdots + \alpha^{\mu-1})]$ *reduced modulo n.*

(ii) *Let* $G = MC(m, n, \alpha, S_0, S_1, \ldots, S_\mu)$ *be a cubic (m, n)-metacirculant graph such that* $m > 2$ *is even,* $\mu = \lfloor m/2 \rfloor$ *is odd,* $S_0 = S_1 = \cdots = S_{2^r-1} = \emptyset$ *with* $r \geq 1, S_{2^r} = \{s\}, S_{2^r+1} = \cdots = S_{\mu-1} = \emptyset$ *and* $S_\mu = \{k\}$. *Then G is connected if and only if gcd(q,n) = 1, where q is* $[k(1 + \alpha + \alpha^2 + \cdots + \alpha^{2^r-1}) - s(1 + \alpha + \alpha^2 + \cdots + \alpha^{\mu-1})]$ *reduced modulo n.*

LEMMA 3. [10] *Let* $G = MC(m, n, \alpha, S_0, S_1, \ldots, S_\mu)$ *be a connected cubic (m, n)-metacirculant graph such that m is even, greater than 2 and not divisible by 4,* $S_0 = S_1 = \cdots = S_{2^r-1} = \emptyset$ *with* $r \geq 1, S_{2^r} = \{s\}$ *with* $0 \leq s < n, S_{2^r+1} = \cdots = S_{\mu-1} = \emptyset$ *and* $S_\mu = \{k\}$ *with* $0 \leq k < n$. *Let* $\overline{n} = gcd(\alpha - 1, n)$ *and* $\overline{\overline{n}} = gcd(1 - \alpha + \alpha^2 - \cdots + \alpha^{\mu-1}, n)$. *Then G has a Hamilton cycle if each of the following conditions is met:*

(i) *Either* $gcd(n/(\overline{n}\,\overline{\overline{n}}), \mu\overline{n} - 1) = 1$; *or*

(ii) $\overline{\overline{n}} = 1$.

Now we recall the definition of a brick product of a cycle with a path defined in [3]. This product plays a role in the proof of the theorem in the next section. Let C_n with $n \geq 3$ and P_m with $m \geq 1$ be the graphs with vertex-sets $V(C_n) = \{u_1, u_2, \ldots, u_n\}, V(P_m) = \{v_1, v_2, \ldots, v_{m+1}\}$ and edge-sets $E(C_n) = \{u_1u_2, u_2u_3, \ldots, u_nu_1\}, E(P_m) = \{v_1v_2, v_2v_3, \ldots, v_mv_{m+1}\}$, respectively. The *brick product* $C_n^{[m+1]}$ of C_n with P_m is defined as follows [3]. The vertex-set of $C_n^{[m+1]}$ is the cartesian product $V(C_n) \times V(P_m)$. The edge-set of $C_n^{[m+1]}$ consists of all pairs of the form $(u_i, v_h)(u_{i+1}, v_h)$ and $(u_1, v_h)(u_n, v_h)$, where i = 1, 2, ..., n–1 and h = 1, 2, ..., m + 1, together with all pairs of the form $(u_i, v_h)(u_i, v_{h+1})$, where $i + h \equiv 0 \pmod 2$, i = 1, 2, ..., n and h = 1, 2, ..., m.

The following result has been proved in [3].

LEMMA 4. [3] *Consider the brick product* $C_n^{[m]}$ *with n even. Let* $C_{n,1}$ *and* $C_{n,m}$ *denote the two cycles in* $C_n^{[m]}$ *on the vertex-sets* $\{(u_i, v_1) : i = 1, 2, \ldots, n\}$ *and* $\{(u_i, v_m) : i = 1, 2, \ldots, n\}$, *respectively. Let F denote an arbitrary perfect matching joining the vertices of degree 2 in* $C_{n,1}$ *with the vertices of degree 2 in* $C_{n,m}$. *If X is a graph obtained by adding the edges of F to* $C_n^{[m]}$, *then X has a Hamilton cycle.*

3. Main result

The purpose of this section is to prove the following result.

THEOREM 5. *Let G be a connected cubic (10, n)-metacirculant graph. Then G possesses a Hamilton cycle if n is a positive integer such that* $\varphi(n)$ *is not* *divisible by 5, where $\varphi(n)$ is the number of integers z satisfying $0 \le z < n$ and gcd(z,n) = 1.*

PROOF. Let n be a positive integer such that $\varphi(n)$ is not divisible by 5 and $G = MC(10, n, \alpha, S_0, \dots, S_5)$ be a connected cubic $(10, n)$-metacirculant graph. Assume first that $S_0 \neq \emptyset$. Then $n > 1$ and the order of G equal to $10n$ is greater than 10 which is the order of the Petersen graph. It follows that G is not isomorphic to the Petersen graph and by [7] it possesses a Hamilton cycle in this case. Thus, we may assume from now on that $S_0 = \emptyset$. Since G is a cubic $(10, n)$-metacirculant graph, this implies that only the following may happen:

(i) $S_0 = \emptyset, S_i = \{s\}$ with $0 \le s < n$ for some $i \in \{1, 2, 3, 4\}, S_j = \emptyset$ for all $i \neq j \in \{1, 2, 3, 4\}$ and $S_5 = \{k\}$ with $0 \le k < n$;

(ii) $S_0 = \cdots = S_4 = \emptyset$ and $|S_5| = 3$.

Since G is connected, the possibility (ii) cannot occur. So only (i) may happen. By Lemma 1, without loss of generality, we may assume that $G = MC(10, n, \alpha, S_0, \dots, S_5)$ has one of the following forms:

1. $S_0 = \emptyset, S_1 = \{s\}, S_2 = S_3 = S_4 = \emptyset$ and $S_5 = \{k\}$;
2. $S_0 = S_1 = \emptyset, S_2 = \{s\}, S_3 = S_4 = \emptyset$ and $S_5 = \{k\}$;
3. $S_0 = S_1 = S_2 = S_3 = \emptyset, S_4 = \{s\}$ and $S_5 = \{k\}$.

Therefore, since G is connected, by Lemma 2

$$gcd\big(k, s(1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4), n\big) = 1. \tag{3.1}$$

On the other hand, by the definition of $(10, n)$-metacirculant graphs, we have

I. $\alpha^{10} s \equiv s \pmod n$

$$\iff (\alpha^5 + 1)(\alpha - 1)(1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4)s \equiv 0 \;(\bmod\, n); \tag{3.2}$$

II. $\alpha^5 k \equiv -k \pmod n$

$$\iff (\alpha^5 + 1)k \equiv 0 \;(\bmod\, n). \tag{3.3}$$

Let $z = n/gcd(\alpha^5 + 1, n)$. Then z is a divisor of both k and $(\alpha - 1)(1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4)s$. Therefore, by (3.1) z is a divisor of $\alpha - 1$. Thus, $\alpha^{10} - 1 = (\alpha^5 + 1)(\alpha - 1)(1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4) \equiv 0 \pmod n$, i.e., the order of α in Z_n^* is a divisor of 10. But it is well-known that $|Z_n^*| = \varphi(n)$. So by the hypothesis of our theorem, $\alpha^2 \equiv 1 \pmod n$. We consider separately two cases.

Case 1. $G = MC(10, n, \alpha, S_0, \dots, S_5)$ with $S_0 = \emptyset, S_1 = \{s\}, S_2 = S_3 = S_4 = \emptyset$ and $S_5 = \{k\}$.

An edge of G of the type $v_j^i v_{j+\alpha^i s}^{i+1}$ is called an S_1-edge, and of the type $v_j^i v_{j+\alpha^i k}^{i+5}$ an S_5-edge. A cycle C in G is called an S_1-cycle if every edge of C is an S_1-edge. Consider S_1-cycles in G. Since every vertex of G is incident with just two S_1-edges, any S_1-cycle B_j in G can be represented in the form

$$B_j = P(v_y^0)P(v_{y+z}^0)P(v_{y+2z}^0)\cdots,$$

where z is $5s + 5\alpha s$ and $P(v_h^0) = v_h^0 v_{h+s}^1 v_{h+s+\alpha s}^2 v_{h+2s+\alpha s}^3 v_{h+2s+2\alpha s}^4 \cdots v_{h+4s+4\alpha s}^8 v_{h+5s+4\alpha s}^9$. Further, it is clear that all S_1-cycles in G are isomorphic to each other and have an even length ℓ. Moreover, two vertices v_f^i and v_g^{i+2} of G are vertices at distance 2 apart in the same S_1-cycle B_j if and only if $g = f + s + \alpha s$ in Z_n.

If G has only one S_1-cycle, then this cycle is trivially a Hamilton cycle of G. Therefore, we assume that G has at least two S_1-cycles. Let v_f^i and v_g^{i+2} with i even be two vertices at distance 2 apart in the same S_1-cycle B_j. Then the vertices of G adjacent to v_f^i and v_g^{i+2} by S_5-edges are $v_{f'}^{i+5}$ and $v_{g'}^{i+7}$, respectively, where $f' = f + \alpha^i k = f + k$ and $g' = g + \alpha^{i+2} k = g + k$. Since $g = f + s + \alpha s$ in Z_n, we have $g' = g + k = f + s + \alpha s + k = f' + s + \alpha s$ in Z_n. Thus, $v_{f'}^{i+5}$ and $v_{g'}^{i+7}$ are vertices at distance 2 apart in the same S_1-cycle $B_{j'}$. Moreover, since i is even, the superscripts i + 5 and i + 7 of respectively $v_{f'}^{i+5}$ and $v_{g'}^{i+7}$ are odd.

Let $C_\ell^{[r]}$ be the brick product of a cycle C_ℓ with a path P_{r-1}, where C_ℓ is isomorphic to S_1-cycles of G and r is the number of distinct S_1-cycles in G. Denote by $C_{\ell,1}$ and $C_{\ell,r}$ the two cycles in $C_\ell^{[r]}$ on the vertex-sets $\{(u_i, v_1) : i = 1, 2, \ldots, \ell\}$ and $\{(u_i, v_r) : i = 1, 2, \ldots, \ell\}$, respectively. Using the property of G proved in the preceding paragraph and the fact that G is a connected cubic graph, it is not difficult to see that G is isomorphic to a graph X obtained from $C_\ell^{[r]}$ by adding the edges of a perfect matching joining the vertices of degree 2 in $C_{\ell,1}$ with the vertices of degree 2 in $C_{\ell,r}$. By Lemma 4, X has a Hamilton cycle. Therefore, G has a Hamilton cycle in Case 1.

Case 2. $G = MC(10, n, \alpha, S_0, \ldots, S_5)$ with $S_{2^r} = \{s\}$ (r = 1 or 2), $S_5 = \{k\}$ and $S_j = \emptyset$ for all $j \neq 2^r$ and 5.

An edge of G of the type $v_j^i v_{j+\alpha^i s}^{i+2^r}$ is called an S_{2^r}-edge, and of the type $v_j^i v_{j+\alpha^i k}^{i+5}$ an S_5-edge. A cycle C in G is called an S_{2^r}-cycle if every edge of C is an S_{2^r}-edge.

Since $\alpha^2 \equiv 1 \pmod{n}, (\alpha + 1)(\alpha - 1) \equiv 0 \pmod{n}$. On the other hand, $gcd(1 - \alpha + \alpha^2 - \alpha^3 + \alpha^4, \alpha - 1, n) = 1$ because gcd(α,n)=1. Therefore, $\overline{\overline{n}} = gcd(1 - \alpha + \alpha^2 - \alpha^3 + \alpha^4, n)$ is a divisor of $gcd(\alpha + 1, n)$. Since $1 - \alpha + \alpha^2 - \alpha^3 + \alpha^4 = u(\alpha + 1) + 5$ for some integer u, it follows that $\overline{\overline{n}}$ is a divisor of 5. Thus, $\overline{\overline{n}} = 1$ or 5.

If $\overline{\overline{n}} = 1$, then G has a Hamilton cycle by Lemma 3(ii). If $\overline{\overline{n}} = 5$, then $n = 5^a x$ with $a \geq 1$ and $\alpha + 1 = 5^b y$ with $b \geq 1$. Since G is connected, by Lemma 2(ii),

$$\begin{aligned} &gcd\big([k(1 + \alpha + \cdots + \alpha^{2^r - 1}) - s(1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4)], n\big) \\ &= gcd\big([k(\alpha + 1)(1 + \alpha^2 + \alpha^4 + \cdots + \alpha^{2^r - 2}) - s(1 + \alpha + \\ &\quad \alpha^2 + \alpha^3 + \alpha^4)], n\big) = 1. \end{aligned} \tag{3.4}$$

On the other hand, since $\alpha^2 \equiv 1 \pmod n$, from (3.3) we have

$$k(\alpha+1) \equiv k(\alpha^5+1) \equiv 0 \text{ (mod } n). \tag{3.5}$$

From (3.4) and (3.5) it follows that

$$gcd(s,n) = 1. \tag{3.6}$$

Let $G' = MC(10, n, \alpha', S'_0, \dots, S'_5)$ be a cubic $(10,n)$-metacirculant graph such that $\alpha' = \alpha, S'_{2^r} = \{1\}, S'_5 = \{0\}$ and $S'_j = \emptyset$ for all $j \neq 2^r$ and 5. Further, let $V(G') = \{x^i_j : i \in Z_{10};\ j \in Z_n\}$. Then since (3.6) holds, it is not difficult to verify that the mapping

$$\psi : V(G') \to V(G) : \begin{cases} x^i_j \mapsto v^i_{js} & \text{if i is even,} \\ x^i_j \mapsto v^i_{js+k} & \text{if i is odd} \end{cases}$$

is an isomorphism of G' and G. Therefore, without loss of generality we may assume that G is a connected cubic $(10,n)$-metacirculant graph $MC(10, n, \alpha, S_0, \dots, S_5)$ such that $n = 5^a x$ with $a \geq 1, \alpha + 1 = 5^b y$ with $b \geq 1, S_{2^r} = \{1\}, S_5 = \{0\}$ and $S_j = \emptyset$ for all $j \neq 2^r$ and 5. It is not difficult to see that such a graph has just 10 disjoint S_{2^r}-cycles, namely, $A^0, A^1, A^2, A^3, A^4, B^0, B^1, B^2, B^3$ and B^4 which contain $v^0_0, v^{2^r}_0, v^{2\cdot 2^r}_0, v^{3\cdot 2^r}_0, v^{4\cdot 2^r}_0, v^5_0, v^{5+2^r}_0, v^{5+2\cdot 2^r}_0, v^{5+3\cdot 2^r}_0$ and $v^{5+4\cdot 2^r}_0$, respectively. Moreover, for each S_{2^r}-cycle A^ℓ or B^ℓ, $\ell = 0,1,2,3,4$, each element of Z_n appears as a subscript of one and only one vertex of this cycle.

Let ρ and τ be the automorphisms of G defined by $\rho(v^i_j) = v^i_{j+1}$ and $\tau(v^i_j) = v^{i+1}_{\alpha j}$. Set $\beta = \rho\tau^{2^r}$. Then

$$\beta(v^i_j) = \rho\tau^{2^r}(v^i_j) = \rho(v^{i+2^r}_{\alpha^{2^r} j}) = \rho(v^{i+2^r}_j) = v^{i+2^r}_{j+1}. \tag{3.7}$$

So β maps every vertex of A^ℓ, $\ell = 0,1,2,3,4$, to the vertex following it in A^ℓ. Further, since $\alpha + 1 = 5^b y$ with $b \geq 1$, $\alpha \equiv 4 \pmod 5$. Therefore,

$$\beta(B^0) = B^2,\ \beta(B^2) = B^4,\ \beta(B^4) = B^1,\ \beta(B^1) = B^3 \text{ and } \beta(B^3) = B^0. \tag{3.8}$$

From (3.7) and (3.8) it is not difficult to see that G is isomorphic to the graph $\overline{H}$ such that

$$V(\overline{H}) = \{\overline{u}^i_j,\ \overline{w}^i_j : i \in Z_5,\ j \in Z_n\}, \text{ and}$$
$$E(\overline{H}) = \overline{F} \cup \overline{E}_0 \cup \overline{E}_1 \cup \overline{E}_2 \cup \overline{E}_3 \cup \overline{E}_4,$$

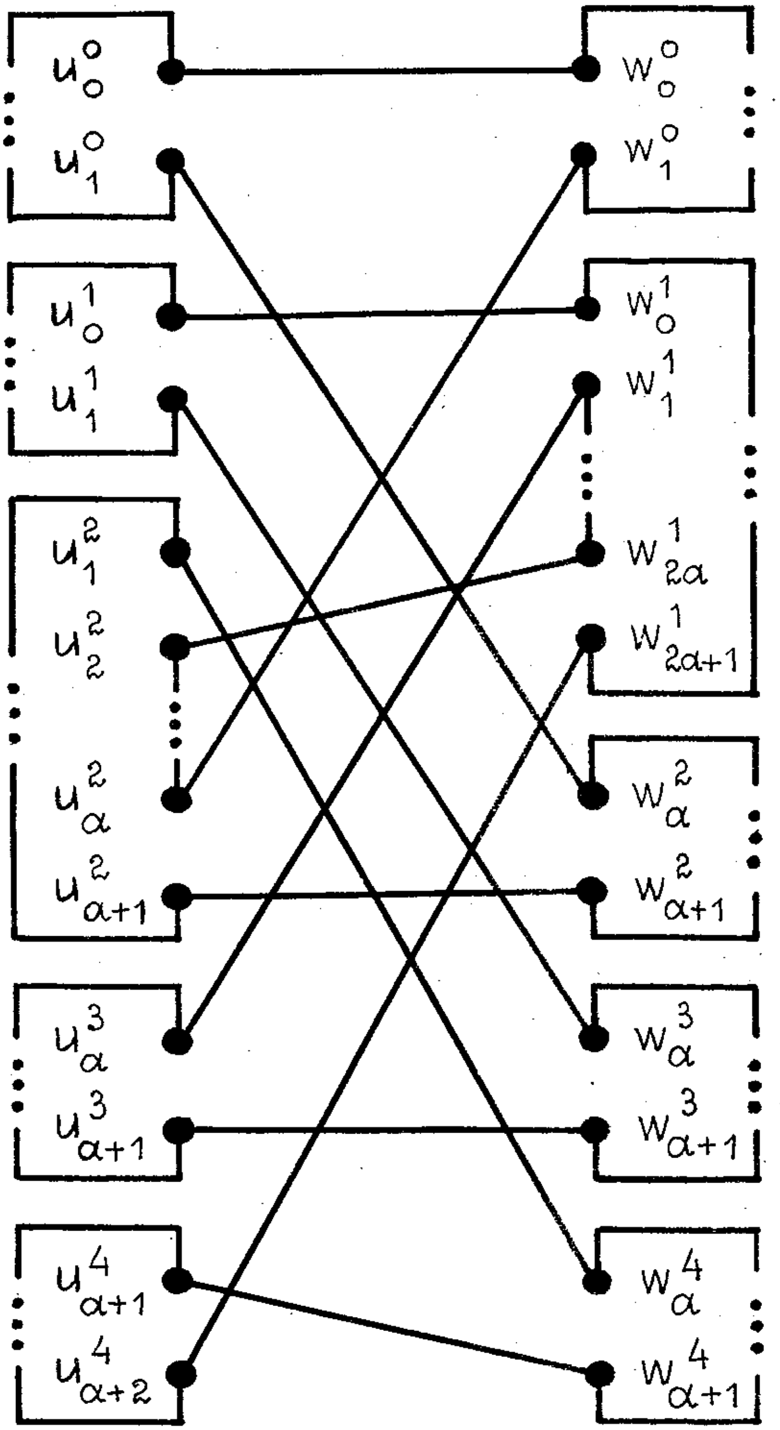
u_0^0
u_1^0
u_0^1
u_1^1
u_1^2
u_2^2
u_α^2
$u_{\alpha+1}^2$
u_α^3
$u_{\alpha+1}^3$
$u_{\alpha+1}^4$
$u_{\alpha+2}^4$
w_0^0
w_1^0
w_0^1
w_1^1
$w_{2\alpha}^1$
$w_{2\alpha+1}^1$
w_α^2
$w_{\alpha+1}^2$
w_α^3
$w_{\alpha+1}^3$
w_α^4
$w_{\alpha+1}^4$

Fig. 1

where

$$\begin{aligned}
\overline{F} &= \{\overline{u}_j^i\overline{u}_{j+1}^i,\ \overline{w}_j^i\overline{w}_{j+\alpha}^i :\ i \in Z_5,\ j \in Z_n\},\\
\overline{E}_0 &= \{\overline{u}_j^i\overline{w}_j^i :\ i \in Z_5,\ j \in Z_n \text{ and } j \equiv 0 \pmod 5\},\\
\overline{E}_1 &= \{\overline{u}_j^i\overline{w}_j^{i+2} :\ i \in Z_5,\ j \in Z_n \text{ and } j \equiv 1 \pmod 5\},\\
\overline{E}_2 &= \{\overline{u}_j^i\overline{w}_j^{i+4} :\ i \in Z_5,\ j \in Z_n \text{ and } j \equiv 2 \pmod 5\},\\
\overline{E}_3 &= \{\overline{u}_j^i\overline{w}_j^{i+1} :\ i \in Z_5,\ j \in Z_n \text{ and } j \equiv 3 \pmod 5\},\\
\overline{E}_4 &= \{\overline{u}_j^i\overline{w}_j^{i+3} :\ i \in Z_5,\ j \in Z_n \text{ and } j \equiv 4 \pmod 5\},
\end{aligned}$$

and all superscripts and subscripts are always reduced modulo 5 and modulo n, respectively. In its turn, $\overline{H}$ is isomorphic to the graph H with

$$\begin{aligned}
V(H) &= \{u_j^i,\ w_j^i :\ i \in Z_5,\ j \in Z_n\}, \text{ and}\\
E(H) &= F \cup E_0 \cup E_1 \cup E_2 \cup E_3 \cup E_4,
\end{aligned}$$

where

$$\begin{aligned}
F &= \{u_j^i u_{j+1}^i,\ w_j^i w_{j+1}^i :\ i \in Z_5,\ j \in Z_n\},\\
E_0 &= \{u_j^i w_{\alpha j}^i :\ i \in Z_5,\ j \in Z_n \text{ and } j \equiv 0 \pmod 5\},\\
E_1 &= \{u_j^i w_{\alpha j}^{i+2} :\ i \in Z_5,\ j \in Z_n \text{ and } j \equiv 1 \pmod 5\},\\
E_2 &= \{u_j^i w_{\alpha j}^{i+4} :\ i \in Z_5,\ j \in Z_n \text{ and } j \equiv 2 \pmod 5\},\\
E_3 &= \{u_j^i w_{\alpha j}^{i+1} :\ i \in Z_5,\ j \in Z_n \text{ and } j \equiv 3 \pmod 5\},\\
E_4 &= \{u_j^i w_{\alpha j}^{i+3} :\ i \in Z_5,\ j \in Z_n \text{ and } j \equiv 4 \pmod 5\},
\end{aligned}$$

and all superscripts and subscripts are always reduced modulo 5 and modulo n, respectively. Since $\alpha \equiv 4 \pmod 5$, the graph H has a Hamilton cycle shown in Figure 1. Therefore, G also has a Hamilton cycle in Case 2.

The proof of the theorem is complete. □

REFERENCES

[1] B. Alspach and T.D. Parsons, *A construction for vertex-transitive graphs*, Canad. J. Math. **34** (1982), 307–318.

[2] B. Alspach, *Hamilton cycles in metacirculant graphs with prime power cardinal blocks*, Annals of Discrete Math. **41** (1989), 7–16.

[3] B. Alspach and C.-Q. Zhang, *Hamilton cycles in cubic Cayley graphs on dihedral groups*, Ars Combin. **28** (1989), 101–108.

[4] J. C. Bermond, "Hamiltonian graphs," In: Selected topics in graph theory (eds. L. W. Beineke and R. J. Wilson), Academic Press, London, 1978.

[5] C. C. Chen and N. F. Quimpo, "On strongly hamiltonian abelian group graphs," In: Combinatorial Mathematics VIII (ed. K. McAvaney), Lecture Notes in Math. 884 (1980) 23–34.

[6] K. Keating and D. Witte, *On Hamilton cycles in Cayley graphs in groups with cyclic commutator subgroup*, Annals of Discrete Math. **27** (1985), 89–102.

[7] Ngo Dac Tan, *On cubic metacirculant graphs*, Acta Math. Vietnamica **15** (1990), 57–71.

[8] Ngo Dac Tan, *Hamilton cycles in cubic* $(4, n)$*-metacirculant graphs*, Acta Math. Vietnam. **17** (1992), 83–93.

[9] Ngo Dac Tan, *Connectedness of cubic metacirculant graphs*, Acta Math. Vietnamica **18** (1993), 3–17.

[10] Ngo Dac Tan, *On Hamilton cycles in cubic* (m, n)*-metacirculant graphs*, Austral. J. Combin. **8** (1993), 211–232.

[11] Ngo Dac Tan, *Hamilton cycles in cubic* (m, n)*-metacirculant graphs with* m *divisible by 4*, Graphs and Combin. **10** (1994), 67–73.

[12] Ngo Dac Tan, *Sufficient conditions for the existence of Hamilton cycles in cubic (6,n)-metacirculant graphs, I* (Submitted).

[13] Ngo Dac Tan, *Sufficient conditions for the existence of Hamilton cycles in cubic (6,n)-metacirculant graphs, II,* (Submitted).

[14] D. Witte and J. E. Gallian, *A survey: Hamiltonian cycles in Cayley graphs*, Discrete Math. **51** (1984), 293–304.

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