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ON ANTISIMPLE PRIMITIVE RADICAL OF Γ_N-RINGS

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Abstract. Let M be a Γ-ring and let $M_2 = \begin{pmatrix} R & \Gamma \\ M & L \end{pmatrix}$, where R (resp. L) is the right (resp. left) operator ring of M. In this paper we show that the class of all primitive subdirectly irreducible Γ-rings is special, and then we establish relationships between the antisimple primitive radicals of the Γ-ring M, the right operator ring R of M, the matrix $\Gamma_{n,m}$-ring $M_{m,n}$, the M-ring Γ and the ring M_2.

1. Introduction

The notion of antisimple radical of a ring was first defined by Andrunakievich [1]. This notion was extended to Γ-rings in [4] and the relationships between the antisimple radicals of a Γ-ring M, the right operator ring R of M and the matrix $\Gamma_{n,m}$ -ring $M_{m,n}$ were established. In this paper, we give examples to show that antisimple primitive radical is different from antisimple radical and that it lies strictly between the Jacobson radical and the Brown-McCoy radical. For any Γ_N-ring M, there are five related rings: the Γ-ring M, the right (left) operator ring $R(L)$ of Γ-ring M, the matrix $\Gamma_{n,m}$-ring $M_{m,n}$, the M-ring Γ and the ring M_2. The relationships between the antisimple primitive radicals of the Γ-ring M, the right operator ring R of M, the matrix $\Gamma_{n,m}$-ring $M_{m,n}$, the M-ring Γ and the ring M_2 are also established. These results have some connections with a problem raised by Szász [11, Problem 55].

Let M and Γ be additive abelian groups. If, for all $x, y, z \in M, \alpha, \beta, \nu \in \Gamma$ the following hold:

(1) there exists a unique element $x\alpha y \in M$,

(2) $x\alpha(y\beta z) = (x\alpha y)\beta z$;

(3) $x\alpha(y+z) = x\alpha y + x\alpha z; x(\alpha+\beta)y = x\alpha y + x\beta y; (x+y)\alpha z = x\alpha z + y\alpha z$;

then M is called a Γ-ring. A weak Γ_N-ring is a pair (M, Γ) such that

(4) M is a Γ-ring, and Γ is a M-ring;

(5) $x\alpha(y\beta z) = x(\alpha y\beta)z = (x\alpha y)\beta z; (\alpha x\beta)y\mu = \alpha(x\beta y)\mu = \alpha x(\beta y\mu)$.

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In addition, if

(6) $x\mu y = 0$ for all $x, y \in M$ implies $\mu = 0$,

then the pair (M, Γ) is called a Γ_N-ring. The notion of ideals (resp. prime ideals) of a Γ-ring M is well-known in the literature, see [2, 3, 9].

Let M be a Γ-ring, and suppose that $x \in M$ and $\gamma \in \Gamma$. We will denote by $[\gamma, x]$ the endomorphism ring of M defined by $y[\gamma, x] = y\gamma x$ for all $y \in M$. The subring R of the ring of right endomorphisms of M generated by the set $\{[\gamma, x] : \gamma \in \Gamma, x \in M\}$ is called the right operator ring of M. The left operator ring L of M is defined similarly. For the subsets $N \subseteq M, \Theta \subseteq \Gamma$, we denote by $[\Theta, N]$ the set of all finite sums $\sum_i [\gamma_i, x_i]$ in R, where $\gamma_i \in \Theta, x_i \in N$. In particular, $R = [\Gamma, M]$. M is said to have right unity if there exists an element $\sum_{i=1}^{n} [\delta_i, a_i] \in R$ such that $\sum_{i=1}^{n} x\delta_i a_i = x$ for all $x \in M$. In this case, $\sum_{i=1}^{n} [\delta_i, a_i]$ is the unity of the ring R. Left unity is defined similarly.

If $A \subseteq M, Q \subseteq R$, we define

$$A^{*'} = \{r \in R : Mr \subseteq A\},$$
$$A^{+'} = \{y \in L : yM \subseteq A\},$$
$$Q^{*} = \{x \in M : [\beta, x] \in Q \text{ for all } \beta \in \Gamma\}.$$

It is easily see that if $A \trianglelefteq M$ and $Q \trianglelefteq R$, then $A^{*'} \trianglelefteq R, A^{+'} \trianglelefteq L$ and $Q^{*} \trianglelefteq M$. Here "$A \trianglelefteq M$" means that A is an ideal of M.

Now let (M, Γ) be a weak Γ_N-ring, and $A \subseteq M$. Define

$$\Gamma(A) = \{\gamma \in \Gamma : M\gamma M \subseteq A\}.$$

Then $\Gamma(A) \trianglelefteq \Gamma$ if $A \trianglelefteq M$, then . Similarly, if $\Phi \subseteq \Gamma$, we define

$$M(\Phi) = \{x \in M : \Gamma x \Gamma \subseteq \Phi\}.$$

Then, $M(\Phi) \trianglelefteq M$ if $\Phi \trianglelefteq \Gamma$. Moreover, if $A \subseteq M$ and $\Phi \subseteq \Gamma$, then $A \subseteq M(\Gamma(A))$ and $\Phi \subseteq \Gamma(M(\Phi))$.

It is easy to see that if $A \trianglelefteq M$, then $(M/A, \Gamma/\Gamma(A))$ is a Γ_N-ring with the operations

$$(x + A)(\gamma + \Gamma(A))(y + A) = x\gamma y + A$$
$$(\gamma + \Gamma(A))(x + A)(\mu + \Gamma(A)) = \gamma x\mu + \Gamma(A).$$

Let (M, Γ) be a Γ_N-ring with the right operator ring R and the left operator ring L. The set

$$M_2 = \begin{pmatrix} R & \Gamma \\ M & L \end{pmatrix} = \{\begin{pmatrix} r & \gamma \\ m & l \end{pmatrix} | r \in R, \gamma \in \Gamma, m \in M, l \in L\}$$

forms a ring under the usual matrix multiplication and addition. Moreover, if $I \trianglelefteq M$, then

$$I_2 = \begin{pmatrix} I^{*'} & \Gamma(I) \\ I & I^{+'} \end{pmatrix} \trianglelefteq M_2 \text{ and } \begin{pmatrix} [\Gamma, I] & \Gamma I \Gamma \\ I & [I, \Gamma] \end{pmatrix} \trianglelefteq M_2.$$

2. Antisimple primitive radicals of Γ-rings

J. Luh calls a Γ-ring M (right) primitive if the right operator ring R of M is primitive and $M\Gamma x = 0$ implies $x = 0$. Left primitive can be defined similarly. It is easy to prove that primitive Γ-rings are prime Γ-rings. Following S. Kyuno [8],[9], we call a Γ-ring M, subdirectly irreducible (abbreviated as SBI) if the intersection of all nonzero ideals of M is not zero. The uniquely determined minimal ideal of a SBI Γ-ring M is called the heart of M, and it is denoted by $H(M)$.

DEFINITION 2.1. Let $\mathcal{K}$ be the class of all SBI Γ-rings, whose Jacobson radical is $\{0\}$. The upper radical determined by the class $\mathcal{K}$ is called the antisimple primitive radical of Γ-rings and it is denoted by $\mathcal{A}_s(-)$. A Γ-ring M is said to be primitive subdirectly irreducible (abbreviated as PSBI) if M is a primitive and subdirectly irreducible Γ-ring. An ideal I of M is called a PSBI ideal if M/I is a PSBI Γ-ring.

Radical classes of Γ-rings, special radical, and the upper radical $\mu\mathcal{M}$ determined by a class $\mathcal{M}$ of Γ-rings are defined exactly as those in ordinary ring theory, see [5].

LEMMA 2.2. *Let $\mathcal{R}$ be a hereditary radical of Γ-rings, M a SBI Γ-ring with heart H. Then M is $\mathcal{R}$-semisimple if and only if H is $\mathcal{R}$-semisimple. Moreover, H is a simple Γ-ring or a zero Γ-ring.*

LEMMA 2.3. *Let M be a semiprime Γ-ring and $A \trianglelefteq M$. If R' denotes the right operator ring of the Γ-ring A, then $R' \cong [\Gamma, A]$ (see [7, Lemma 2.3]).*

PROPOSITION 2.4. *For any Γ-ring $M, M \in \mathcal{K}$ if and only if M is a PSBI Γ-ring.*

PROOF. It is clear that if M is PSBI, then $M \in \mathcal{K}$. Conversely, if $M \in \mathcal{K}$, we have $J(M) = 0$ and M is a SBI Γ-ring with heart H. By Lemma 2.2, H

is J-semisimple, and then $H\Gamma H \neq 0$ and H is a simple Γ-ring. Hence M is a prime Γ-ring and H is a PSBI Γ-ring. Since $0 \neq [\Gamma, H] \trianglelefteq R = [\Gamma, M]$ and $[\Gamma, H]$ is a primitive ring by Proposition 2.3, it follows from [10, Lemma 1] that R is a primitive ring, hence M is a PSBI Γ-ring.

Following Heyman and Roos [6], we call a class $\mathcal{M}$ of Γ-rings special if $\mathcal{M}$ satisfies the following conditions

(a) Each $M \in \mathcal{M}$ is a prime Γ-ring;

(b) For every $M \in \mathcal{M}$, if $I \trianglelefteq M$ then $I \in \mathcal{M}$;

(c) If $0 \neq I \trianglelefteq \circ M$ (essential ideal of M) and $I \in \mathcal{M}$, then $M \in \mathcal{M}$.

The upper radical determined by a special class of Γ-rings is called a special radical of Γ-rings.

THEOREM 2.5. *The class $\mathcal{K}$ of all PSBI Γ-rings is a special class of Γ-rings.*

PROOF. Clearly, $\mathcal{K}$ consists of prime Γ-rings. Suppose that $M \in \mathcal{K}$ with heart $H(M)$ and $A \trianglelefteq M$. If $A = 0$, then $A \in \mathcal{K}$. Suppose $A \neq 0$. We will prove that $A \in \mathcal{K}$ with heart $H(M)$. Suppose that $0 \neq I \trianglelefteq A$. Then $H(M) \subseteq I$ since

$$H(M) = H(M)\Gamma H(M)\Gamma H(M) \subseteq I^*\Gamma I^*\Gamma I^* \subseteq I,$$

where $I^* = I + I\Gamma M + M\Gamma I + M\Gamma I\Gamma M$. Hence A is a SBIΓ-ring with heart $H(M)$. On the other hand, since $R = [\Gamma, M]$ is a primitive ring and $0 \neq [\Gamma, A] \trianglelefteq R$, it follows that $[\Gamma, A]$ is a primitive ring by [10, Lemma 1]. Thus, $A \in \mathcal{K}$ by Proposition 2.3.

Next we check condition (c) for $\mathcal{K}$. Suppose that M is a Γ-ring, $0 \neq I \trianglelefteq \circ M$ and $I \in \mathcal{K}$ with heart $H(I)$. Then, M is a prime Γ-ring. For, if $P, Q \trianglelefteq M$ such that $P\Gamma Q = 0$, then $(P \cap I)\Gamma(Q \cap I) = 0$. By the primality of I we have $P \cap I = 0$ or $Q \cap I = 0$. Since $I \trianglelefteq \circ M$, we get that $P = 0$ or $Q = 0$. Next, we prove that M is a SBI Γ-ring. Let A be an arbitrary nonzero ideal of M. Since $I \trianglelefteq \circ M$, $0 \neq A \cap I \trianglelefteq I$. Hence $H(I) \subseteq A$, so M is SBI. Finally, M is a primitive Γ-ring by a similar argument as that of the proof of Proposition 2.4. This completes the proof of the theorem.

From [5], Proposition 2.7 and Theorem 2.8 we get the following results

THEOREM 2.6. *For any Γ-ring $M, \mathcal{A}_s(M) = \cap\{I \trianglelefteq M | M/I$ is a PSBI Γ-ring$\}$.*

THEOREM 2.7. *If I is an ideal of Γ-ring M, then $\mathcal{A}_s(I) = I \cap \mathcal{A}_s(M)$.*

THEOREM 2.8. *A Γ-ring M is a subdirect sum of PSBI Γ-rings if and only if $\mathcal{A}_s(M) = 0$.*

PROOF. The theorem is immediate from Theorem 2.6 and [9, Lemma 2].

The next result characterizes the antisimple primitive radical of Γ-rings. Its proof is identical to that of the correspoding result for the case of antisimple rings or antisimple Γ-rings (see [11, Proposition 12.4] or [4, Proposition 2.6]). Hence we will omit them.

PROPOSITION 2.9. *The following are equivalent for a Γ-ring M:*

(a) $\mathcal{A}_s(M) = M$;

(b) Every homomorphic images of M is a subdirect sum of SBI Γ-rings $\{M_i : i \in I\}$ *such that* $\mathcal{A}_s(H(M_i)) = H(M_i)$ *for each* $i \in I$;

(c) M does not contain any primitive ideal P such that M/P has a minimal ideal;

(d) No ideal of M can be mapped homomorphically onto a nonzero simple primitive Γ-ring.

The following example shows that antisimple primitive radical is different from antisimple radical.

EXAMPLE 2.10. Let R be the ring of all power series in non-commutating indeterminates x and y over a field F. Let I be the ideal of all power series with constant term 0. Then I is a radical ring in the sense of Jacobson and $x \notin (x - yx^2y)_I$ for the ideal generated by $x - yx^2y$ in I. By Zorn's lemma, there exists an ideal A of I maximal with respect to the condition $A \supseteq (x - yx^2y)_I$ and $x \notin A$. Write $S = I/A, \overline{x} = x + A$ and $\overline{y} = y + A$. Let T be the ideal generated by $\overline{x}$ in S. By [11, example 32.7], T is a simple radical ring in the sense of Jacobson and $T^2 = T$. Hence T is a prime SBI ring but not a primitive SBI ring. In fact, we have $\mathcal{A}_s(T) = T$.

The antisimple primitive radical is a new radical which is also different from the Jacobson radical and the Brown-McCoy radical.

THEOREM 2.11. $J(-) < \mathcal{A}_s(-) < B(-)$, *where* $B(-)$ *is the Brown-McCoy radical in the sense of Booth [7].*

PROOF. First, by the definitions, it is easy to see that $J(-) \leq \mathcal{A}_s(-) \leq B(-)$. To prove that $J(-) < \mathcal{A}_s(-)$ we need only to prove that there exists a $\mathcal{A}_s$-radical ring which is not a J-radical ring. For this let K be a field of characteristic zero, and α an automorphism of infinite order of K. Let R be the set of all polynomials $a_0 + za_1 + \ldots + z^n a_n$ in an indeterminate z over K with coefficients a_i from the field K. Let equality and addition of these polynomials be defined as usual. Let the multiplication be given by $kz = zk^{\alpha}$ for every $k \in K$. Let $T = xR$ be an ideal of R. Then by [11, example 32.1], T is a right primitive ring and not a simple ring. Furthermore, T is a radical ring for the upper radical determined by the class of all right primitive simple rings. We can prove that T has no ideal which can be mapped homomorphically onto a simple ring. Hence by Proposition 2.9(d), T is a $\mathcal{A}_s$-radical ring.

Now we will prove that $\mathcal{A}_s(-) < B(-)$. Let R be the ring of all finite rank linear transformations of a vector space of countable infinite dimension. Then R is a B-radical ring and a simple primitive ring. Hence R is $\mathcal{A}_s$-semisimple ring, and of course not an $\mathcal{A}_s$-radical ring.

3. Antisimple primitiveradical of operator rings

In this section we consider right operator rings. Analogous results for left operator rings can be obtained by using similar arguments.

PROPOSITION 3.1. *A Γ-ring M is PSBI if and only if the right operator ring R of M is a PSBI ring and $M\Gamma x = 0$ implies $x = 0$. Furthermore, $H(R) = [\Gamma, H(M)]$ and $H(M) = MH(R)$.*

PROOF. Suppose that M is a PSBI Γ-ring with heart $H(M)$. Then R is a primitive ring. For every nonzero ideal I of R, $MI \supseteq H(M)$ and $[\Gamma, H(M)] \subseteq [\Gamma, H(M)I] \subseteq I$. By the primeness of M, we have that $M\Gamma x = 0$ implies $x = 0$, and hence $[\Gamma, H(M)] \neq 0$. Thus R is a PSBI ring.

Conversely, suppose that R is a PSBI ring with heart $H(R)$ and $M\Gamma x = 0$ implies $x = 0$. For any nonzero ideal P of M, $0 \neq [\Gamma, P] \trianglelefteq R$ and $0 \neq MH(R) \subseteq M[\Gamma, P] \subseteq P$. Thus M is a PSBI Γ-ring. By the above proof, it is easy to see that $H(R) = [\Gamma, H(M)]$ and $H(M) = MH(R)$.

The next lemmas helps us to establish the relationship between PSBI ideals of a Γ-ring M and that of the right operator ring R.

LEMMA 3.2. *If A is an ideal of the Γ-ring M, R and $[\Gamma, M/A]$ are the right operator rings of Γ-ring M and Γ-ring M/A, respectively, then $[\Gamma, M/A] \cong R/A^{*'}$ under the mapping $\sum_i [\gamma_i, x_i + A] \to \sum_i [\gamma_i, x_i] + \mathcal{A}^{*'}$.*

PROOF. Straightforward.

COROLLARY 3.3. *Let M be a Γ-ring with right operator ring R, P an ideal of R and $[\Gamma, M/P^*]$ be the right operator rings of Γ-ring M/P^*. If M has right* *unity or P is a prime ideal of R, then $[\Gamma, M/P^*] \cong R/P$ under the mapping $\sum_i[\gamma_i, x_i + P^*] \to \sum_i[\gamma_i, x_i] + P$.*

PROOF. This result may easily be verified by direct computation.

THEOREM 3.4. *Let M be a Γ-ring with left and right operator rings L and R, respectively. Then the mapping $P \to P^*$(resp. $P \to P^+$) defines a one-to-one correspondence between the PSBI ideals of M and that of R (resp. L). Moreover, $(P^*)^{*'} = P$(resp. $(P^+)^{+'} = P$).*

PROOF. This follows immeddiately from Proposition 3.1 and Lemma 3.2.

THEOREM 3.5. *Let M be a Γ-ring with right operator ring R. Then $\mathcal{A}_s(R) = [\mathcal{A}_s(M)]^{*'}$.*

PROOF. By Theorem 2.6 and 3.4, we have

$$\begin{aligned}\mathcal{A}_s(R) &= \cap\{I^{*'} | I \text{ is a PSBI ideal of } M\}\\ &= (\cap\{I | I \text{ is a PSBI ideal of } M\})^{*'},\\ &= [\mathcal{A}_s(M)]^{*'}.\end{aligned}$$

4. Primitive subdirectly irreducible matrix Γ-rings

For the definition of the matrix Γ-ring of a given Γ-ring M, we refer to [8]. The next theorem indicates a way to construct new PSBI Γ-rings from given ones.

THEOREM 4.1. *M is a PSBI Γ-ring if and only if $M_{m,n}$ is a PSBI $\Gamma_{n,m}$-ring. Furthermore, $H(M_{m,n}) = (H(M))_{m,n}$.*

PROOF. Suppose that M is PSBI Γ-ring. Then, by Proposition 3.1, $R = [\Gamma, M]$ is a PSBI ring and $M\Gamma x = 0$ implies $x = 0$. Denote the right operator ring of $M_{m,n}$ by $[\Gamma_{n,m}, M_{m,n}]$. Recall that $[\Gamma_{n,m}, M_{m,n}] \cong R_n$ (see [9, p.376]). Since PSBI is a Morita invariant property, $[\Gamma_{n,m}, M_{m,n}]$ is a PSBI ring. Also, if $M_{m,n}\Gamma_{n,m}(x_{i,j}) = 0$ for $(x_{i,j}) \in M_{m,n}$, then for all $m \in M, \gamma \in \Gamma$, we have that

$$0 = (m\epsilon_{ik})(\gamma\epsilon_{kj})(x_{st}) = \begin{pmatrix} 0 & \dots & 0 \\ \dots & \dots & \dots \\ m\gamma x_{j1} & \dots & m\gamma x_{jn} \\ \dots & \dots & \dots \\ 0 & \dots & 0 \end{pmatrix} (i).$$

Therefore, $M\Gamma x_{ij} = 0$ for all $1 \le i \le m, 1 \le j \le n$, and consequently $x_{ij} = 0$ and $(x_{ij}) = 0$. Hence $M_{m,n}$ is a PSBI $\Gamma_{n,m}$-ring by Proposition 3.1.

Conversely, suppose that $M_{m,n}$ is a PSBI $\Gamma_{n,m}$-ring. Then $[\Gamma_{n,m}, M_{m,n}] \cong R_n$ is a PSBI ring. Hence R is a PSBI ring. Also, if $M\Gamma x = 0, x \in M$, then $M_{m,n}\Gamma_{n,m}xe_{11} = 0$, and consequently $xe_{11} = 0$, i.e. $x = 0$. Hence M is a PSBI Γ-ring.

LEMMA 4.2. [8, Lemma 4]. *If $I \trianglelefteq M$, then the matrix $\Gamma_{n,m}$-ring $(M/I)_{m,n}$ is isomorphic to the $\Gamma_{n,m}$-ring $M_{m,n}/I_{m,n}$.*

LEMMA 4.3. [9, Theorem 2]. *Let M be an arbitrary Γ-ring. Then the prime ideals of the matrix $\Gamma_{n,m}$-ring $M_{m,n}$ are precisely the sets $P_{n,m}$, where P is a prime ideal of the Γ -ring M.*

As a consequence of Lemma 4.3 and Theorem 4.1 we get

THEOREM 4.4. *Let M be an arbitrary Γ-ring. Then the PSBI ideals of the matrix $\Gamma_{n,m}$-ring $M_{m,n}$ are precisely the sets $P_{n,m}$, where P is a PSBI ideal of the Γ-ring M.*

THEOREM 4.5. *If M is a Γ-ring, then $\mathcal{A}_s(M_{m,n}) = (\mathcal{A}_s(M))_{m,n}$.*

PROOF. By Theorem 4.4 and 2.6, we have that

$$\begin{aligned} \mathcal{A}_s(M_{m,n}) &= \cap\{(I)_{m,n} | I \text{ is a PSBI ideal of } M\} \\ &= (\cap\{I | I \text{ is a PSBI ideal of } M\})_{m,n}, \\ &= [\mathcal{A}_s(M)]_{m,n} \end{aligned}$$

5. Primitive subdirectly irreducible property of the M-ring Γ and M_2

Let (M, Γ) be Γ_N-ring, we shall establish the relationships between PSBI property of the Γ-ring M, the M-ring Γ and the ring M_2.

The proof of the following lemma may be easily verified by direct computation.

LEMMA 5.1. *Let (M, Γ) be a Γ_N-ring. Then the left operator ring L' of the M-ring Γ is isomorphic to $[\Gamma, M]/K$, where $K = \{x \in [\Gamma, M] | \ x\Gamma = 0\}$.*

THEOREM 5.2. *Let (M, Γ) be a Γ_N-ring. Then the Γ-ring M is PSBI if and only if the M-ring Γ is PSBI. Furthermore, $H(M) = MH(\Gamma)M$ and $H(\Gamma) = \Gamma H(M)\Gamma$.*

PROOF. Suppose that the Γ-ring M is PSBI. By the primeness of M and Lemma 5.1, we can prove that the left operator ring L' of the M-ring Γ is isomorphic to $R = [\Gamma, M]$. Thus, L' is primitive and if $\gamma M\Gamma = 0, \gamma \in \Gamma$. It follows that $(M\gamma M)\Gamma M = 0$, hence $M\gamma M = 0$ which implies $\gamma = 0$. By Proposition 3.1, Γ is a (left) primitive M-ring. Let Φ be any nonzero ideal of the M-ring Γ. Then $0 \neq M\Phi M \trianglelefteq M$, and hence $M\Phi M \supseteq H(M)$. Therefore, $\Phi \supseteq \Gamma M\Phi M\Gamma \supseteq \Gamma H(M)\Gamma \neq 0$. This proves that the M-ring Γ is PSBI. The converse can be proved similarly.

The following lemma will be useful in the characterization of the PSBI property of M_2.

LEMMA 5.3. *Let (M, Γ) be a Γ_N-ring and A be an ideal of M_2. Then $\{x \in M :$ there exist $r \in R, \gamma \in \Gamma, s \in L$ such that $\begin{pmatrix} r & \gamma \\ x & s \end{pmatrix} \in A\} = 0$ implies $A = 0$.*

PROOF. Suppose that $\begin{pmatrix} r & \gamma \\ x & s \end{pmatrix} \in A$. It is sufficient to show that $\begin{pmatrix} r & \gamma \\ x & s \end{pmatrix} = 0$. By assumption, $x = 0$. Since $A \trianglelefteq M_2$, for any $m, n \in M$,

$$\begin{pmatrix} 0 & 0 \\ m & 0 \end{pmatrix}\begin{pmatrix} r & \gamma \\ 0 & s \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ mr & [m,\gamma] \end{pmatrix} \in A,$$

$$\begin{pmatrix} r & \gamma \\ 0 & s \end{pmatrix}\begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix} = \begin{pmatrix} [\gamma, n] & 0 \\ sn & 0 \end{pmatrix} \in A$$

and

$$\begin{pmatrix} 0 & 0 \\ m & 0 \end{pmatrix}\begin{pmatrix} r & \gamma \\ 0 & s \end{pmatrix}\begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ m\gamma n & 0 \end{pmatrix} \in A.$$

By the assumption, we have that $mr = 0, sn = 0$ and $m\gamma n = 0$, for every $m, n \in M$, whence $r = 0, s = 0$ and $\gamma = 0$. This completes the proof.

THEOREM 5.4. *Let (M, Γ) be a Γ_N-ring having a right unity. Then the ring $M_2 = \begin{pmatrix} R & \Gamma \\ M & L \end{pmatrix}$ is a PSBI ring if and only if M is a PSBI Γ-ring. Furthermore,*

$$H(M_2) = \begin{pmatrix} [\Gamma, H(M)] & \Gamma(H(M)) \\ H(M) & [H(M), \Gamma]. \end{pmatrix}$$

PROOF. Suppose that M_2 is a PSBI ring and let $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Hence, by [10, Lemma 1] and since $eM_2e \cong R$, we have that R is a primitive ring, whence M is a primitive Γ-ring by Proposition 3.1 and [3, Theorem 3.5]. On the other hand, for any $0 \neq I \trianglelefteq M$,

$$M_2 \trianglerighteq \begin{pmatrix} [\Gamma, I] & \Gamma I \Gamma \\ I & [I, \Gamma] \end{pmatrix} \supseteq H(M_2) \neq 0.$$

Thus, $\cap\{I | 0 \neq I \trianglelefteq M\} \neq 0$ because otherwise, we have

$$\cap\{\begin{pmatrix} [\Gamma, I] & \Gamma I \Gamma \\ I & [I, \Gamma] \end{pmatrix} \mid 0 \neq I \trianglelefteq M\} = 0,$$

a contradiction. Hence M is a PSBI Γ-ring.

Conversely, suppose that M is a PSBI Γ-ring with heart $H(M)$. By [3, Theorem 3.5] and the primeness of M, M_2 is a prime ring. Again by [10, Lemma 1] and since R is a primitive ring, M_2 must be primitive. For any nonzero ideal A of M_2, let

$$I = \{x \in M : \text{ there exist } r \in R, \gamma \in \Gamma, s \in L \text{ such that } \begin{pmatrix} r & \gamma \\ x & s \end{pmatrix} \in A\}.$$

It is easily verified that $I \trianglelefteq M$. Then, by Lemma 5.3, $I \neq 0$ and $I \supseteq H(M)$. Thus, $\cap\{A : 0 \neq A \trianglelefteq M_2\} \neq 0$ and M_2 is a PSBI ring.

Finally, since

$$\begin{pmatrix} [\Gamma, H(M)] & \Gamma(H(M)) \\ H(M) & [H(M), \Gamma] \end{pmatrix} \trianglelefteq M_2,$$

we get

$$H(M_2) \subseteq \begin{pmatrix} [\Gamma, H(M)] & \Gamma(H(M)) \\ H(M) & [H(M), \Gamma] \end{pmatrix}.$$

Conversely, since

$$\begin{pmatrix} 0 & 0 \\ H(M) & 0 \end{pmatrix} \subseteq H(M_2),$$

we have

$$\begin{pmatrix} 0 & \Gamma \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ H(M) & 0 \end{pmatrix} = \begin{pmatrix} [\Gamma, H(M)] & 0 \\ 0 & 0 \end{pmatrix} \subseteq H(M_2)$$

$$\begin{pmatrix} 0 & 0 \\ H(M) & 0 \end{pmatrix} \begin{pmatrix} 0 & \Gamma \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & [H(M), \Gamma] \end{pmatrix} \subseteq H(M_2)$$

and

$$\begin{pmatrix} 0 & \Gamma \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ H(M) & 0 \end{pmatrix} \begin{pmatrix} 0 & \Gamma \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \Gamma H(M) \Gamma \\ 0 & 0 \end{pmatrix} \subseteq H(M_2).$$

This concludes the proof.

6. Antisimple primitive radical of the M-ring Γ and the ring M_2

In this section we study the relationships of PSBI ideals and the antisimple primitive radicals of a Γ-ring M to the corresponding ideals and radicals of the M-ring Γ and ring M_2.

LEMME 6.1. *Let* (M,Γ) *be a weak* Γ_N*-ring. If* $P \trianglelefteq M$, *and* $R, [\Gamma/\Gamma(P), M/P]$ *are the right operator rings of the* Γ*-ring* M *and the* $\Gamma/\Gamma(P)$*-ring* M/P, *respectively, then* $[\Gamma/\Gamma(P), M/P] \cong R/P^{*'}$ *under the mapping* $\sum_i[\gamma_i+\Gamma(P), x_i+P] \to \sum_i[\gamma_i, x_i] + P^{*'}$.

COROLLARY 6.2. *Keep the notions of Lemma 6.1. If* Q *is a prime ideal of* R, *then* $[\Gamma/\Gamma(Q^*), M/Q^*] \cong R/Q$ *under the mapping* $\sum_i[\gamma_i + \Gamma(Q^*), x_i + Q^*] \to \sum_i[\gamma_i, x_i] + Q$.

PROOF. The proof of Lemma 6.1 and Corollary 6.2 can be easily verified by direct computation.

We notice that analogous results can be obtained for the left operator ring.

If (M,Γ) is a Γ_N-ring, it can be easily seen from Lemma 6.1, that an ideal P of M is PSBI if and only if M/P is a PSBI $\Gamma/\Gamma(P)$-ring.

THEOREM 6.3. *Let* (M,Γ) *be a weak* Γ_N*-ring. Then the mapping* $P \to \Gamma(P)$ *defines a one-to-one correspondence between the sets of PSBI ideals of the* Γ*-ring* M *and those of the* M*-ring* Γ.

PROOF. It follows from Lemma 6.1 and Theorem 5.1.

As immediate consequences of Theorem 6.3 we get the following results.

COROLLARY 6.4. *Let* (M,Γ) *be a weak* Γ_N*-ring. Then* $\mathcal{A}_s(\Gamma) = \Gamma(\mathcal{A}_s(M))$.

COROLLARY 6.5. *Let* (M,Γ) *be a weak* Γ_N*-ring. Then the* Γ*-ring* M *is antisimple primitive radical if and only if the* M*-ring* Γ *is antisimple primitive radical.*

THEOREM 6.6. *Let* (M,Γ) *be a* Γ_N*-ring having right unity, and let* R *and* L *denote, respectively, the right and left operator rings of the* Γ*-ring* M. *Then a subset* P_2 *of* M_2 *is a PSBI ideal of* M_2 *if and only if*

$$P_2 = \begin{pmatrix} P^{*'} & \Gamma(P) \\ P & P^{+'} \end{pmatrix},$$

where P *is a PSBI ideal of* M.

PROOF. Suppose first that P is a PSBI ideal of the Γ-ring M. Then M/P is a PSBI $\Gamma/\Gamma(P)$-ring, whence

$$(M/P)_2 = \begin{pmatrix} [\Gamma/\Gamma(P), M/P] & \Gamma/\Gamma(P) \\ M/P & [M/P, \Gamma/\Gamma(P)] \end{pmatrix} \cong \begin{pmatrix} R/P^{*'} & \Gamma/\Gamma(P) \\ M/P & L/P^{+'} \end{pmatrix}$$

is a PSBI ring by Theorem 5.4. It follows that $M_2/\begin{pmatrix} P^{*'} & \Gamma(P) \\ P & P^{+'} \end{pmatrix}$ is a PSBI ring. This means P_2 is a PSBI ideal of M_2.

Conversely, suppose that the subset P_2 is a PSBI ideal of M_2, then by [3, Theorem 3.6], $P_2 = \begin{pmatrix} P^{*'} & \Gamma(P) \\ P & P^{+'} \end{pmatrix}$, where P is a prime ideal of M. But $M_2/P_2 \cong \begin{pmatrix} R/P^{*'} & \Gamma/\Gamma(P) \\ M/P & L/P^{+'} \end{pmatrix}$ is a PSBI ring. Hence, by Theorem 5.4, P is a PSBI ideal of M. This concludes the proof.

As consequences of Theorem 3.5, Corollary 6.5 and Theorem 6.6, we obtain the following results.

COROLLARY 6.7. *Let (M, Γ) be a Γ_N-ring having right unity. Then*

$$\mathcal{A}_s(M_2) = \begin{pmatrix} \mathcal{A}_s(R) & \mathcal{A}_s(\Gamma) \\ \mathcal{A}_s(M) & \mathcal{A}_s(L) \end{pmatrix}.$$

COROLLARY 6.8. *Let (M, Γ) be a Γ_N-ring having right unity. Then M_2 is an antisimple primitive radical ring if and only if M is an antisimple primitive radical Γ-ring.*

REMARK. The author does not know whether the assumption in Theorem 5.4 that the Γ-ring M has a right unity can be removed.

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