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IMPACT OF A VISCOPLASTIC ROD AGAINST AN ELASTIC OBSTACLE

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Dedicated to Professor Dang Dinh Ang on the occasion of his 70th birthday

1. Introduction

In this paper we consider an analytical model for the calculation of the velocity distribution in a viscoplastic rod of finite length L undergoing an impact on an elastic obstacle. The rod is assumed to translate along its longitudinal axis and to strike against an elastic obstacle in the normal direction. We also assume that the material is incompressible and that the phenomenon is one-dimensional. This model can thus be described as that of the impact of a rod on an elastic spring.

The deformation of the rod takes place according to Bingham's law

$$\sigma^\star(x^\star,t^\star)+\sigma_0=\mu\frac{\partial u^\star}{\partial x^\star}(x^\star,t^\star),\quad \text{if } |\sigma^\star|>\sigma_0, \tag{1.1}$$

where $\sigma^\star$ is the stress, σ_0 the yield stress, μ the coefficient of viscosity and $u^\star$ the velocity in the x-direction.

When the impact takes place, the rod is divided into two parts separated by $x^\star = s^\star(t^\star)$. All the points in the region $s^\star(t^\star) \le x^\star \le L$ have the same $u^\star(t^\star)$. The functions $u^\star(t^\star)$ and $s^\star(t^\star)$ are related by the following equation

$$\frac{du^\star}{dt^\star}=\frac{\sigma_0}{\rho(L-s^\star(t^\star))}, \tag{1.2}$$

where ρ is the density of the rod ((1.2) is the equation of motion for the rigid part); in the viscoplastic region $0 < x^\star < s^\star(t^\star)$, the velocity $u^\star(x^\star,t^\star)$ satisfies the equation

$$\rho\frac{\partial u^\star}{\partial t^\star}=\mu\frac{\partial^2 u^\star}{\partial x^{\star 2}}. \tag{1.3}$$

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Moreover, at the interface between the two regions, the stress reaches the yield value, one has, by (1.1)

$$\frac{\partial u^\star}{\partial x^\star}(s^\star(t^\star), t^\star) = 0. \tag{1.4}$$

Since the obstacle is an elastic spring, the normal impact pressure on the end of impact, according to Hooke's law, is proportional to the displacement of the obstacle

$$\sigma_n^\star(0,t^\star) = -k^\star[\int_0^{t^\star} u^\star(0,\tau)d\tau + a_0], \;\; if \; \int_0^{t^\star} u^\star(0,\tau)d\tau + a_0 \leq 0$$

where $\sigma_n^\star(0,t^\star)$ is the normal impact pressure on the end of impact, $k^\star$ the rigidity of spring, a_0 the initial position of the end of impact, the quantity $\int_0^{t^\star} u^\star(0,\tau)d\tau + a_0$ is the change in lenght of the spring (if $\int_0^{t^\star} u^\star(0,\tau)d\tau + a_0 \leq 0$).

It is physically obvious that $a_0 < 0$. The problem will be considered only in the time interval of the impact, that is

$$\int_0^{t^\star} u^\star(0,\tau)d\tau + a_0 \leq 0 \text{ for } 0 < t^\star \leq T^\star \tag{1.5}$$

and (1.5) can be written

$$\sigma^\star(0,t^\star) = k^\star \left[\int_0^{t^\star} u^\star(0,\tau)d\tau + a_0\right]. \tag{1.6}$$

Introducing the dimensionless variables

$$\begin{aligned} x &= x^\star/L,\; t = t^\star/T, \\ u(x,t) &= u^\star(Lx,Tt)T/L,\; s(t) = s^\star(Tt)/L, \\ \sigma(x,t) &= \sigma^\star(Lx,Tt)/\sigma_0 \end{aligned} \tag{1.7}$$

and using the following notations

$$R = \frac{\rho L^2}{\mu T},\; S = \frac{\sigma_0 T}{\mu},\; Q = \frac{k^\star L}{\sigma_0},\; a = \frac{a_0}{L} \tag{1.8}$$

we obtain, after some simple computations, the dimensionless system

$$u_t(x,t) = \frac{1}{R} u_{xx}(x,t) \text{ for } 0 < x < s(t), 0 < t < T', \tag{1.9}$$

$$\frac{du}{dt}(s(t),t) = \frac{S}{R(1-s(t))}, \tag{1.10}$$

$$u_x(s(t),t) = 0, \tag{1.11}$$

$$u_x(0,t) = S\left\{1 + Q\left[\int_0^t u(0,\tau)d\tau + a\right]\right\} \tag{1.12}$$

where $T' = T^*/T$.

In view of (1.12), the condition (1.5) becomes

$$u_x(0,t) - S \leq 0, \quad 0 < t < T'. \tag{*}$$

Assuming continuity of the solution and of all its derivatives up to the boundary and letting $x \uparrow s(t)$ in (1.9), we obtain (using(1.10), (1.11))

$$u_t(s(t),t) = \frac{S}{R(1-s(t))}, \tag{1.13}$$

$$u_{xx}(s(t),t) = \frac{S}{1-s(t)}. \tag{1.14}$$

We assume given the initial values

$$u(x,0) = \varphi(x),\ 0 < x < b, \tag{1.15}$$

$$s(0) = b,\ 0 < b < 1. \tag{1.16}$$

To make (1.11), (1.12) and (1.14) consistent, we must require that $\varphi(x)$ is C^2 on $(0,b)$ and satisfies

$$\varphi'(0) = S(1+Qa),\ \varphi'(b) = 0,\ \varphi''(b) = \frac{S}{1-b} \tag{1.17}$$

Note that the present problem (1.9)-(1.16) is related to the one posed by Barenblatt and Ishlinskii in 1962, on the impact of a viscoplastic rod against a fixed rigid obstacle (see [1]). In 1975 A. Fasano and M. Primicerio [2] considered the same problem with the velocity at the end of impact not jumping to zero as in [1], but going monotonically to zero according to a given function. In our problem, the above boundary condition is replaced by a pressure load depending on the displacement of the end of impact. For simplicity we assume that $b > 0$. The case $b = 0$ will be studied elsewhere.

The remainder of the paper is organized as follows. In section 2, we reformulate the problem as a system of integral equations. Section 3 is devoted to the problem of existence and uniqueness. In the final section we give a numerical example.

2. Integral equation formulation

We shall reformulate the problem as a system of integral equations. That can be solved by successive approximation, using the contraction principle.

Put $v = u_t$. The equations and the conditions for v can be derived easily from the equations and the conditions for u by differentiation. It can be shown that v satisfies

$$v_t(x,t) = \frac{1}{R} v_{xx}(x,t), \tag{2.1}$$

$$v(s(t),t) = \frac{S}{R(1-s(t))}, \tag{2.2}$$

$$v_x(s(t),t) = -\frac{S\dot{s}(t)}{1-s(t)}, \tag{2.3}$$

$$v(x,0) = \frac{1}{R}\varphi''(x) \stackrel{def}{=} \psi(x), \tag{2.4}$$

$$v_x(0,t) = SQ\left[\int_0^t v(0,\tau)d\tau + \varphi(0)\right], \tag{2.5}$$

$$\psi(b) = \frac{S}{R(1-b)}, \ \psi'(0) = SQ\varphi(0). \tag{2.6}$$

Assume for now that $s(t)$ is C^1 on $(0, T')$ and that $\varphi(x)$ is C^3 on $(0, b)$. Once $v(x,t)$ is found by solving the system (2.1)-(2.6), we can calculate $u(x,t)$ from the formula

$$u(x,t) = R\int_0^x \int_{s(t)}^{\xi} v(\eta,t)d\eta d\xi + u(0,t) \tag{2.7}$$

Let $k = R^{1/2}$. We define the Green's functions

$$\begin{cases} K(x,t;\xi,\tau) = \dfrac{k}{2\sqrt{\pi(t-\tau)}} \exp\left[-\dfrac{k^2(x-\xi)^2}{4(t-\tau)}\right], \\ G(x,t;\xi,\tau) = K(x,t;\xi,\tau) - K(x,t;-\xi,\tau), \\ N(x,t;\xi,\tau) = K(x,t;\xi,\tau) + K(x,t;-\xi,\tau), \end{cases} \tag{2.8}$$

for $0 < x < s(t),\ 0 < \xi < s(\tau),\ 0 < \tau < t$.

Thus, let $v(\xi,\tau)$, $s(\tau)$ be a solution of the system (2.1)-(2.6) with (x,t) replaced by (ξ,τ). Integrating the following identity

$$(Nv_\xi - N_\xi v)_\xi - k^2(Nv)_\tau = 0 \tag{2.9}$$

over the region $\{(\xi,\tau) \,:\, 0 \le \xi \le s(\tau),\ \epsilon \le \tau \le t-\epsilon\}$, applying Green's formula and letting $\epsilon \to 0^+$, we obtain

$$\begin{aligned} v(x,t) = &\int_0^b \psi(\xi)N(x,t;\xi,0)d\xi - \frac{1}{k^2}\int_0^t v_1(\tau)N(x,t;0,\tau)d\tau \\ &+ \frac{1}{k^2}\int_0^t v_x(s(\tau),\tau)N(x,t;s(\tau),\tau)d\tau \\ &- \frac{1}{k^2}\int_0^t v_0(\tau)N_\xi(x,t;s(\tau),\tau)d\tau \\ &+ \int_0^t v_0(\tau)N(x,t;s(\tau),\tau)\dot{s}(\tau)d\tau, \end{aligned} \tag{2.10}$$

where we have put

$$\begin{cases} v_0(t) = v(s(t),t), \\ v_1(t) = v_x(0,t). \end{cases} \tag{2.11}$$

Taking the x-derivative of both sides of (2.10), we obtain

$$\begin{aligned} v_x(x,t) = &\int_0^b \psi(\xi)N_x(x,t;\xi,0)d\xi - \frac{1}{k^2}\int_0^t v_1(\tau)N_x(x,t;0,\tau)d\tau \\ &+ \frac{1}{k^2}\int_0^t v_x(s(\tau),\tau)N_x(x,t;s(\tau),\tau)d\tau \\ &- \frac{1}{k^2}\int_0^t v_0(\tau)N_{\xi x}(x,t;s(\tau),\tau)d\tau \\ &+ \int_0^t v_0(\tau)N_x(x,t;s(\tau),\tau)\dot{s}(\tau)d\tau. \end{aligned} \tag{2.12}$$

Integrating by parts we get

$$\begin{aligned} v_x(x,t) = &\int_0^b \psi'(\xi)G(x,t;\xi,0)d\xi - \frac{1}{k^2}\int_0^t v_1(\tau)N_x(x,t;0,\tau)d\tau \\ &+ \frac{1}{k^2}\int_0^t v_x(s(\tau),\tau)N_x(x,t;s(\tau),\tau)d\tau \\ &+ \int_0^t \dot{v}_0(\tau)G(x,t;s(\tau),\tau)d\tau, \end{aligned} \tag{2.13}$$

where we have used the identities $N_{\xi x} = k^2 G_\tau$, $N_x = -G_\xi$. Now, let $x \uparrow s(t)$. Using a lemma of [3] we have

$$\frac{1}{2}v_x(s(t),t) = \int_0^b \psi'(\xi)G(s(t),t;\xi,0)d\xi - \frac{1}{k^2}\int_0^t v_1(\tau)N_x(s(t),t;0,\tau)d\tau$$

$$+\frac{1}{k^2}\int_0^t v_x(s(\tau),\tau)N_x(s(t),t;s(\tau),\tau)d\tau+\int_0^t \dot{v}_0(\tau)G(s(t),t;s(\tau),\tau)d\tau, \quad (2.14)$$

where $v_1(t)$ is calculated from (2.5) and (2.10). In order to define v_1, we note that (2.10) can by (2.2), (2.3) be written

$$\begin{aligned} v(x,t)=\int_0^b \psi(\xi)N(x,t;\xi,0)d\xi-\frac{1}{k^2}\int_0^t v_1(\tau)N(x,t;0,\tau)d\tau \\ -\frac{1}{k^2}\int_0^t v_0(\tau)N_\xi(x,t;s(\tau),\tau)d\tau. \end{aligned} \quad (2.15)$$

Letting $x\to 0$, we obtain

$$\begin{aligned} v(0,t)=\int_0^b \psi(\xi)N(0,t;\xi,0)d\xi-\frac{1}{k^2}\int_0^t v_1(\tau)N(0,t;0,\tau)d\tau \\ -\frac{1}{k^2}\int_0^t v_0(\tau)N_\xi(0,t;s(\tau),\tau)d\tau. \end{aligned} \quad (2.16)$$

Combining (2.5) and (2.16) gives

$$\begin{aligned} v_1(t)=SQ\Bigg[&\int_0^t\int_0^b \psi(\xi)N(0,t';\xi,0)d\xi dt' \\ &-\frac{1}{k^2}\int_0^t\int_0^{t'} v_1(\tau)N(0,t';0,\tau)d\tau dt' \\ &-\frac{1}{k^2}\int_0^t\int_0^{t'} v_0(\tau)N_\xi(0,t';s(\tau),\tau)d\tau dt'+\varphi(0)\Bigg]. \end{aligned} \quad (2.17)$$

From (2.8) we get

$$\begin{aligned} -\frac{1}{k^2}\int_0^t\int_0^{t'} v_1(\tau)N(0,t';0,\tau)d\tau dt' &= -\frac{1}{k\sqrt{\pi}}\int_0^t\int_0^{t'}\frac{v_1(\tau)}{\sqrt{t'-\tau}}d\tau dt' \\ &= -\frac{2}{k\sqrt{\pi}}\int_0^t v_1(\tau)\int_0^{t'}\frac{1}{2\sqrt{t'-\tau}}dt'd\tau \\ &= -\frac{2}{k\sqrt{\pi}}\int_0^t v_1(\tau)\sqrt{t-\tau}d\tau. \end{aligned} \quad (2.18)$$

We finally have

$$v_1(t) = SQ\left[\int_0^t\int_0^b \psi(\xi)N(0,t';\xi,0)d\xi dt' - \frac{2}{k\sqrt{\pi}}\int_0^t v_1(\tau)\sqrt{t-\tau}d\tau \right. \\ \left. -\frac{1}{k^2}\int_0^t\int_0^{t'} v_0(\tau)N_\xi(0,t';s(\tau),\tau)d\tau dt' + \varphi(0)\right]. \tag{2.19}$$

Define

$$r(t) = \dot{s}(t) \tag{2.20}$$

so that

$$s(t) = b + \int_0^t r(\tau)d\tau, \tag{2.21}$$

and recall (2.2) and (2.3). Then (2.14) and (2.19) can be written as

$$r(t) = -\frac{2(1-s(t))}{S}\left[\int_0^b \psi'(\xi)G(s(t),t;\xi,0)d\xi \right. \\ -\frac{1}{k^2}\int_0^t v_1(\tau)N_x(s(t),t;0,\tau)d\tau - \frac{S}{R}\int_0^t \frac{r(\tau)}{1-s(\tau)}N_x(s(t),t;s(\tau),\tau)d\tau \\ \left. +\frac{S}{R}\int_0^t \frac{r(\tau)}{(1-s(\tau))^2}G(s(t),t;s(\tau),\tau)d\tau\right], \tag{2.22}$$

$$v_1(t) = SQ\left[\int_0^t\int_0^b \psi(\xi)N(0,t';\xi,0)d\xi dt' - \frac{2}{k\sqrt{\pi}}\int_0^t v_1(\tau)\sqrt{t-\tau}d\tau \right. \\ \left. -\frac{S}{R}\int_0^t\int_0^{t'} \frac{1}{1-s(\tau)}N_\xi(0,t';s(\tau),\tau)d\tau dt' + \varphi(0)\right]. \tag{2.23}$$

Thus, the solution of the problem (1.9)-(1.16) can be found by solving the integral equations (2.22) and (2.23), where $s(t)$ is defined by (2.21).

3. Existence and uniqueness results

In this section, we state some results on existence and uniqueness of solutions. First, it can be shown that there exist $M > 0$ and $\delta > 0$ such that the right-hand sides of (2.22) and (2.23) define a contraction on the closed ball of radius M, center 0 in a space of continuous functions on $[0,\delta]$. We therefore obtain the following theorem.

THEOREM 1. *Assume that $\varphi(x)$ is C^3 on $(0,b)$ and satisfies (1.16). Then there* *exists one and only one solution $(v(x,t),\ s(t))$ of the system (2.1)-(2.6) (or $(u(x,t),\ s(t))$ of the system (1.9)-(1.12)) for $0 \le t < \delta$, δ sufficiently small.*

Applying Theorem 1 and using the condition (*) in Section 2 we obtain the following global existence result:

THEOREM 2. *Under the conditions of Theorem 1 and the additional conditions*

(i) $\varphi(0) < 0$,

(ii) $\sup_{0\le x\le b} |\psi(x) - \frac{S}{R(1-b)}| < \frac{S}{2R(1-b)}$,

there are $T^\star$, ϵ, $C > 0$ such that if $-\epsilon < \varphi(0) < 0$, $0 < a < \epsilon$ and $\sup_{0\le x\le b} |\psi'(x)| < CT_0^{-1}\min\{\frac{b}{2}, \frac{1-b}{2}\}$, then the system (2.1)-(2.6) (or (1.9)-(1.12)) has a unique solution $(v(x,t),\ s(t))$ on the domain $0 < x < s(t)$, $0 \le t \le T^\star$ satisfying

$$\begin{aligned} &0 < T^\star \le T_0, \\ &u_x(0,t) - S \le 0 \quad \text{for every } 0 \le t < T^\star, \\ &u_x(0,T^\star) - S = 0. \end{aligned}$$

Here $T_0 = \frac{2R(1-b)}{S}\left[-\varphi(0) + \sqrt{\varphi(0)^2 - \frac{Sa}{R(1-b)}}\right]$.

PROOF. Let A be the set of all numbers $T > 0$ such that (2.1)-(2.5) has a unique solution $(v(x,t), s(t))$ in $[0,T]$ satisfying

$$\begin{aligned} &(a) \qquad u_x(0,t) - S < 0, \\ &(b) \qquad \left|v(0,t) - \frac{S}{R(1-b)}\right| < \frac{S}{2R(1-b)}, \\ &(c) \qquad |\dot{s}(t)| < T_0^{-1}\min\{b/2, (1-b)/2\} \end{aligned} \tag{3.1}$$

for every $0 \le t < T$.

We put:

$$T^* = \sup A.$$

We first prove that $T^* \le T_0$. From (2.5), (2.7) we have

$$u_x(0,t) - S = \varphi'(0) - S + SQ\int_0^t [\int_0^\theta v(0,\tau)d\tau + \varphi(0)]d\theta, \quad t \in [0,T^*). \tag{3.2}$$

In view of (3.2), (3.1)(b), (3.1)(a) we get:

$$\frac{S}{4R(1-b)}t^2 + \varphi(0)t + a < 0, \quad t \in [0, T^*) \tag{3.3}$$

where $S > 0$ and $\varphi'(0) = S(1 + Qa)$.

From (3.3) we get

$$0 < T^* \leq T_0 = \frac{2R(1-b)}{S}\left[-\varphi(0) + \sqrt{\varphi(0)^2 - \frac{Sa}{R(1-b)}}\right]. \tag{3.4}$$

Now we want to prove that

$$\begin{aligned} u_x(0,t) - S \leq 0, \quad & t \in [0, T^*), \\ u_x(0, T^*) - S = 0. & \end{aligned}$$

From (3.1)(c), (3.4) and (2.21) we get

$$b/2 \leq s(t) \leq (1+b)/2, \quad t \in [0, T^*). \tag{3.5}$$

In view of (2.22), (3.4), (3.5), we get after some computations

$$|r(t)| \leq C\{\sup_{0 \leq x \leq b} |\psi'(x)| + \sqrt{T_0} \sup_{0 \leq t \leq T^*} (|v_1(t)| + |r(t)|)\} \tag{3.6}$$

for every $0 \leq t \leq T^*$.

Similarly, from (2.23) it follows that

$$|v_1(t)| \leq CT_0\{\sup_{0 \leq x \leq b} |\psi(x)| + \sup_{0 \leq t \leq T^*} |v_1(t)| + \sqrt{T_0}\} + |\varphi(0)| \tag{3.7}$$

for every $0 \leq t \leq T^*$.

Letting $x \downarrow 0$ in (2.16) and estimating the result thus obtained we get

$$\begin{aligned} \left|v(0,t) - \frac{S}{R(1-b)}\right| \leq \sup_{0 \leq x \leq b}\left|\psi(x) + \frac{S}{R(1-b)}\right| & \\ + C\sqrt{T_0} \sup_{0 \leq t \leq T^*} (|v_1(t)| + |v_0(t)|) & \end{aligned} \tag{3.8}$$

From (2.2) and (3.1)(b), (3.1)(c) we infer that

$$\sup_{0 \leq t \leq T^*} |v_0(t)| \leq \frac{2S}{R(1-b)}.$$

Hence (3.8) give:

$$\left|v(0,t) - \frac{S}{R(1-b)}\right| \leq \sup_{0\leq x\leq b}\left|\psi(x) + \frac{S}{R(1-b)}\right| + C\sqrt{T_0}\{\sup_{0\leq t\leq T^*}|v_1(t)| + \frac{2S}{R(1-b)}\}. \tag{3.9}$$

By (3.6)-(3.9), there is an $\varepsilon > 0$ such that if $0 < a < \varepsilon$ and $-\varepsilon < \varphi(0) < 0$ then

$$|r(t)| < C \sup_{0\leq x\leq b}|\psi'(x)|$$
$$\left|v(0,t) - \frac{S}{R(1-b)}\right| < \frac{S}{2R(1-b)} \quad t\in[0,T^*] \tag{3.10}$$

If

$$\sup_{0\leq x\leq b}|\psi'(x)| < C^{-1}T_0^{-1}\min\{b/2,(1-b)/2\}$$

then from (3.10) we get

$$|r(t)| < T_0^{-1}\min\{b/2,(1-b)/2\}, \quad t\in[0,T^*]. \tag{3.11}$$

Now, assume by the contrary that

$$u_x(0,T^*) - S < 0. \tag{3.12}$$

From (3.11), (3.12) we can apply the local existence result to get a $\delta > 0$ such that (2.1)-(2.6) has a unique solution on $[0, T^* + \delta]$ for every $t \in [0, T^* + \delta]$. Hence $T^* + \delta \in A$, which is a contradiction. Thus

$$u_x(0,T^*) - S = 0.$$

This completes the proof of the Theorem.

4. A numerical example

We have used a numerical method based on (2.22) and (2.23) for solving a test problem. In our numerical experiment, we have used $a = -0.01$, $b = 0.5$, $R = S = 1$, and $Q = -1/a$; the initial velocity in the zone of viscoplastic flow

$$\varphi(x) = \frac{4}{3}x^3 - x^2 - 1.$$

Figure 1 shows plots of the moving boundary $s(t)$ for various values of t. Once

$s(t)$ is known, we used (2.10), (2.7) and (1.1) to calculate velocity and stress fields on the rod. Figure 2 and 3 show velocity and stress profiles for fixed t.

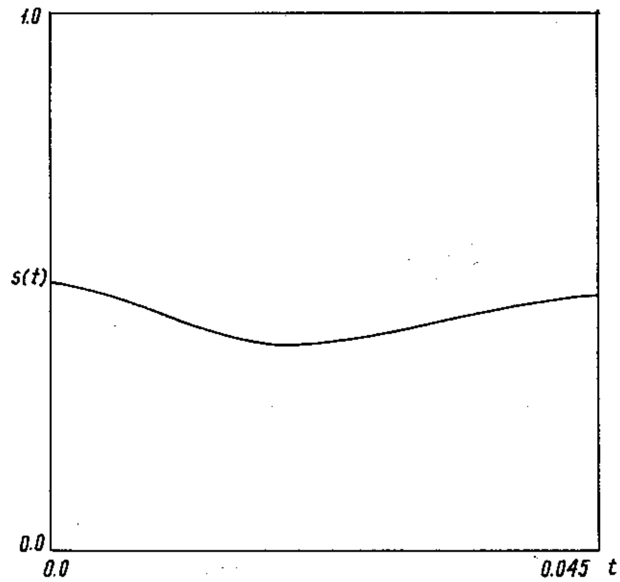

Fig. 1: Moving boundary

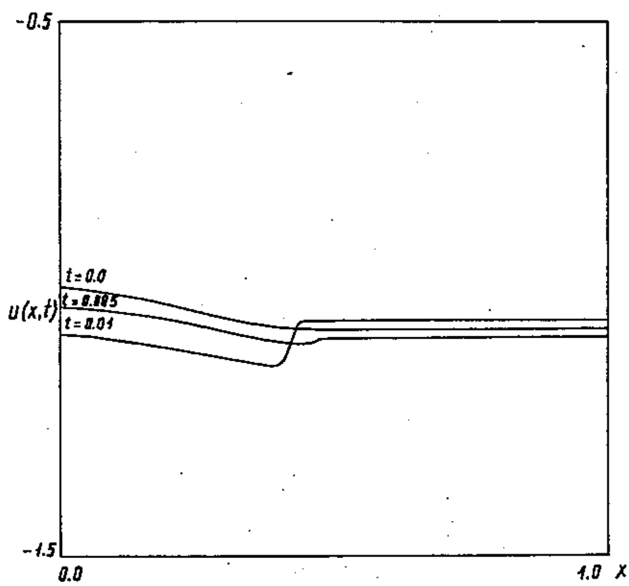

Fig. 2: Velocity profiles for fixed

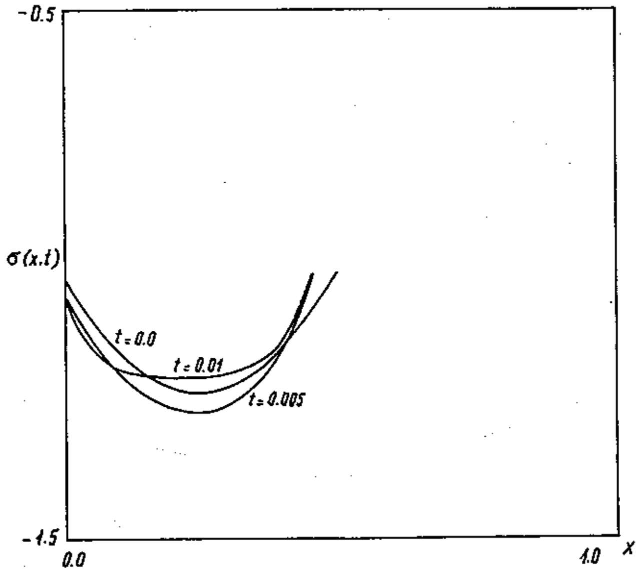

Fig. 3

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