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SURFACE TEMPERATURE DETERMINATION FROM BOREHOLE MEASUREMENTS: A FINITE SLAB MODEL

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Abstract. The authors consider the problem of determining the temperature at one surface of a slab of finite thickness from temperatures measured at interior points. The problem is reduced to solving a Cauchy problem for the heat equation, which is then regularized using the Tikhonov method. Error estimastes are given.

Dedicated to Professor Dang Dinh Ang on the occasion of his 70th birthday

1. Introduction

The problem of determining surface temperatures from measurements at inner points of a domain has been considered by several authors in view of its many applications, e.g. in geophysics, where the problem is to find temperatures at the surface from borehole measurements. The model most commonly used is that of a one dimensional space domain represented by a half-line $x > 0$ (see e. g. Carasso [3], Engl, Manselli [8], Le-Navarro [17] and [18], Talenti and Vessella [5]). While this simplified model, in many instances, yields useful results, it is not realistic when applied to the earth. Indeed, the conductivity of the earth is not constant throughout but, instead, varies from point to point. It is the purpose of this paper to take account of this lack of homogeneity. As a first approximation, we can consider it as a series of superposed layers, each with constant conductivity. Thus we are led to consider a slab of finite thickness $0 < x < a$ with constant conductivity, assumed to be equal to 1.

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In the case the space domain is $0 < x < \infty$, and the temperature at infinity is assumed to be 0, it is sufficient for surface temperature determination to specify the temperature at one interior point. In the case of a slab of finite thickness $0 < x < a$ with a assumed to be > 2 for notational convenience, we shall have to measure the temperature at two points in the interior, which we take to be $x = 1$ and $x = 2$. As we shall see, with temperature measurements at two points, uniqueness of solution is guaranteed. However, the problem is ill-posed. Hence a regularization is in order, in fact, we shall apply the Tikhonov regularization method. Error estimates will be given.

The remainder of this paper is organized as follows. In section 2, we shall determine the flux at $x = 1$. The determination of the surface temperature $u(0,t)$ is then reduced to solving a Cauchy problem for the heat equation in the slab $0 < x < 1$, which is the object of section 3.

The results of this paper were announced in the Abstracts of The Third International Congress On Industrial And Applied Mathematics ([19])

2. Flux determination at $x = 1$

The problem is to determine $u_x(1,t) \equiv w(t)$. We shall derive the integral equation that $w(t)$ satisfies. It turns out that the integral equation is of the Volterra type (see (7)) and can be solved by iteration on each finite interval $0 < t < T$. We shall determine conditions on the measured data $f(t)$ and $g(t)$ such that the solution $w(t)$ is in $L^2(R^2)$, in which case we can use Fourier integral transform techniques.

Consider the equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0 \quad \text{in} \quad 1 < x < a, t > 0$$

where $a > 2$ and let

$$\begin{aligned} u(2,t) &= g(t), \qquad t > 0 \\ u(1,t) &= f(t), \qquad t > 0 \\ u_x(1,t) &= w(t), \qquad t > 0 \end{aligned}$$

We shall assume throughout that $u(x,0) = 0$, which is simply a matter of convenience and does not affect the generality. The functions w is unknown and f, g are given.

Put

$$G(x,t;\xi,\tau)=\frac{1}{2\sqrt{\pi(t-\tau)}}\left[exp\left(-\frac{(x-\xi)^2}{4(t-\tau)}\right)-exp\left(-\frac{(x+\xi-4)^2}{4(t-\tau)}\right)\right].$$

Let $1<x<2, t>0$. Integrating the identity

$$\frac{\partial}{\partial\xi}\left(G\frac{\partial u}{\partial\xi}-u\frac{\partial G}{\partial\xi}\right)-\frac{\partial}{\partial\tau}(uG)=0$$

over the domain $(1,2)\times(0,t-\varepsilon)$ for small $\varepsilon>0$ and letting $\varepsilon\to 0$, we have, after some rearrangements.

$$\int_0^t G(x,t;1,\tau)w(\tau)d\tau=-u(x,t)-\int_0^t G_\xi(x,t;2,\tau)g(\tau)d\tau$$
$$+\int_0^t G_\xi(x,t;1,\tau)g(\tau)d\tau \qquad (1)$$

In order to get an integral equation in $w(t)$, we shall take the x-derivative of each side of (1) and then, let $x\to 1+$. First, taking the x-derivative of the left hand side of (1), we have

$$\frac{\partial}{\partial x}\int_0^t G(x,t;1,\tau)d\tau=-\frac{1}{4\sqrt{\pi}}\int_0^t\frac{x-1}{(t-\tau)^{3/2}}exp\left(-\frac{(x-1)^2}{4(t-\tau)}\right)w(\tau)d\tau$$
$$+\frac{1}{4\sqrt{\pi}}\int_0^t\frac{x-3}{(t-\tau)^{3/2}}exp\left(-\frac{(x-3)^2}{4(t-\tau)}\right)w(\tau)d\tau$$
$$\equiv I_1+I_2 \qquad (2)$$

It is easy to see that

$$\lim_{x\to 1+}I_2=-\frac{1}{2\sqrt{\pi}}\int_0^t\frac{1}{(t-\tau)^{3/2}}exp\left(-\frac{1}{t-\tau}\right)w(\tau)d\tau \qquad (3)$$

If w is in $L^2(R^+)$, then using arguments similar to those in [20], we have

$$\lim_{x\to 1+}I_1=-\frac{1}{2}w(t)$$

and for continuous $w(t)$,

$$\lim_{x\to 1+}I_1=-\frac{1}{2}w(t)$$

Now, taking the x-derivative of each term in the right hand side of (1) and letting $x \to 1+$, we have

$$\lim_{x\to 1+} \frac{\partial}{\partial x} u(x,t) = w(t) \tag{4}$$

$$\begin{aligned}&\lim_{x\to 1+} \frac{\partial}{\partial x} \int_0^t G_\xi(x,t;2,\tau) g(\tau) d\tau \\ &= \frac{1}{2\sqrt{\pi}} \int_0^t (t-\tau)^{-3/2} \left[1 - \frac{1}{2(t-\tau)}\right] exp\left(-\frac{1}{4(t-\tau)}\right) g(\tau) d\tau,\end{aligned} \tag{5}$$

$$\begin{aligned}&\lim_{x\to 1+} \frac{\partial}{\partial x} \int_0^t G_\xi(x,t;1,\tau) f(\tau) d\tau \\ &= \frac{1}{4\sqrt{\pi}} \int_0^t (t-\tau)^{-3/2} \left[1 + \left(1 - \frac{1}{t-\tau}\right) exp\left(-\frac{1}{t-\tau}\right)\right] f(\tau) d\tau.\end{aligned} \tag{6}$$

From (1)-(6), we arrive at the equation

$$\begin{aligned}&\frac{1}{2} w(t) - \frac{1}{2\sqrt{\pi}} \int_0^t (t-\tau)^{-3/2} exp\left(-\frac{1}{t-\tau}\right) w(\tau) d\tau \\ &= -\frac{1}{2\sqrt{\pi}} \int_0^t (t-\tau)^{-3/2} \left[1 - \frac{1}{2(t-\tau)}\right] exp\left(-\frac{1}{4(t-\tau)}\right) g(\tau) d\tau \\ &+ \frac{1}{4\sqrt{\pi}} \int_0^t (t-\tau)^{-3/2} \left[1 + \left(1 - \frac{1}{t-\tau}\right) exp\left(-\frac{1}{t-\tau}\right)\right] f(\tau) d\tau.\end{aligned} \tag{7}$$

Eq. (7) is a linear Volterra integral equation of the second kind. For $f(t)$ and $g(t)$ integrable on $[0,T]$ for each $T > 0$ (which will be assumed), the right hand side of (7) is a continuous function. Hence, for each $T > 0$, Eq. (7) can be solved by successive approximation on the space $C[0,T]$ of continuous functions on $[0,T]$, its unique solution being denoted by $w_T(t)$. Since $T > 0$ is arbitrary, we can define a $w(t)$ continuous on R^+, such that for each $T > 0, w(t) = w_T(t)$ for $0 \le t \le T$.

We have just pointed out that Eq. (7) admits a unique solution, continuous on R^+. For our later analysis, it is important that $w(t)$ be in $L^2(R^+)$, for then we can apply the isometric properties of Fourier transforms on $L^2(R)$ for our error estimastes. We first set some notations and rewrite Eq. (7) in a convenient form. We let

$$K(t) = \begin{cases} \sqrt{2} t^{-3/2} exp(-\frac{1}{t}), & t > 0 \\ 0, & t \le 0 \end{cases}$$

$$F(t) = \begin{cases} \frac{1}{2\sqrt{\pi}} \int_0^t (t-\tau)^{-3/2} \left[1 + \left(1 - \frac{1}{2(t-\tau)}\right) exp\left(-\frac{1}{t-\tau}\right)\right] f(\tau)d\tau \\ \quad -\frac{1}{\sqrt{\pi}} \int_0^t (t-\tau)^{-3/2} \left[1 - \frac{1}{2(t-\tau)}\right] exp\left(-\frac{1}{4(t-\tau)}\right) g(\tau)d\tau, & t > 0 \\ 0, & t \leq 0 \end{cases}$$

and set $w(t) = 0$ if $t \leq 0$. Then we have

$$w(t) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} K(t-\tau)w(\tau)d\tau = F(t), t \in R. \tag{8}$$

Taking the Fourier - transform of both sides of (8), we have formally

$$[1 - \hat{K}(p)]\hat{w}(p) = \hat{F}(p) \qquad p \in R \tag{9}$$

where

$$\begin{aligned} \hat{K}(P) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} K(t)e^{-ipt}dt \\ &= exp\left[-\sqrt{\frac{|p|}{2}} + i.sgn(p)\sqrt{\frac{|p|}{2}}\right], \quad p \in R \end{aligned}$$

and

$$\begin{aligned} \hat{F}(P) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(t)e^{-ipt}dt \\ &= \frac{1}{2\pi}\left[\int_0^{\infty}\int_0^t \alpha(t-\tau)f(\tau)e^{-ipt}d\tau dt \right. \\ &\quad \left. - \int_0^{\infty}\int_0^t \beta(t-\tau)f(\tau)e^{-ipt}d\tau dt\right] \end{aligned} \tag{10}$$

in which

$$\alpha(t) = \begin{cases} \frac{1}{\sqrt{2}} t^{-3/2}\left[1 + \left(1 - \frac{1}{2t}\right) exp\left(-\frac{1}{t}\right)\right], & t > 0 \\ 0, & t \leq 0 \end{cases}$$

$$\beta(t) = \begin{cases} \sqrt{2} t^{-3/2}\left[1 - \frac{1}{2t}\right] exp\left(-\frac{1}{4t}\right), & t > 0 \\ 0, & t \leq 0 \end{cases}$$

Note that $\hat{F}(p)$ is in $L^2(R)$ provided $f(t)$ and $g(t)$ are in $L^2(R)$, which we shall assume. In fact, in order for (9) to have a solution $\hat{w}(p)$ in $L^2(R)$, we shall assume further that $\frac{\hat{F}(p)}{|p|^{1/2}}$ is in $L^2(R)$, which is satisfied if

$$\frac{\hat{f}(p)}{|p|^{1/2}} \quad \text{and} \quad \frac{\hat{g}(p)}{|p|^{1/2}} \quad \text{are in} \quad L^2(R). \tag{11}$$

Assuming (11), Eq. (8), or equivalently Eq. (7), has a unique solution in $L^2(R)$ and this solution is stable with respect to variations in f and g provided these functions are equipped with the norm

$$\|f\| = |f|_2 + |\frac{\hat{f}(p)}{|p|^{1/2}}|_2,$$

$$\|g\| = |g|_2 + |\frac{\hat{g}(p)}{|p|^{1/2}}|_2.$$

Where $|.|_2$ is the L^2-norm. These are unwieldy norms. If f and g are simply in L^2 then the problem is ill-posed. We regularize (9) into

$$\varepsilon\hat{w}_\varepsilon(p) + |1 - \hat{K}(p)|^2 \hat{w}_\varepsilon(p) = (1 - \overline{\hat{K}(p)})\hat{F}(p) \quad , \varepsilon > 0. \tag{12}$$

Let w_0 be the solution of (8) corresponding to f replaced by f_0 and g replaced by g_0. Suppose

$$|f - f_0|_2 \le \varepsilon \qquad |g - g_0|_2 \le \varepsilon. \tag{13}$$

Assume w_0 is in $H^s(R)$. Then for small ε

$$|w - w_0|_2 \le C_0 \left[ln \left(\frac{1}{\varepsilon} \right) \right]^{-s} \tag{14}$$

where C_0 is some constant. The details (which are similar to those in the next section) are omitted.

3. Determination of surface temperature

We have shown in the previous section that the problem of surface temperature determination reduces to solving a Cauchy problem for the heat equation on the slab $0 < x < 1$. We point out that the problem is ill-posed and that it has been the object of numerous publications. We single out the paper of Hao and Gorenflo [15], describing a class of functions in which the problem is well-posed and a modification method to solve the problem stably. In [8] Engl and Manselli consider a mixed problem in the slab $0 < x < 1$ for which $u_x(0,t) = 0$ and $u(1,t) = f(t)$ are specified. In [21], Knabner and Vessella derived stability estimates for solutions of the Cauchy problem for the heat equation on the slab $0 < x < 1$ with $u(0,t) = f(t)$ and $u(0,t) = 0$ given. For other related papers, the reader is referred to the paper of Hao and Gorenflo (loc. cit.) which contains a huge bibliography. Returning to our problem, we shall find $v(t)$ $(= u(0,t))$ from $u(1,t) = f(t), u_x(1,t) = w(t)$ using integral equation and Fourier transform techniques.

The derivation of the integral equation in $v(t)$ is similar to that for $w(t)$ of the previous section. For the reader's convenience, we sketch the main steps.

Consider the equation

$$u_{xx} - u_t = 0 \qquad 0 < x < 1, \ t > 0 \tag{15}$$

together with the Cauchy data at $x = 1$

$$u(1,t) = f(t)$$
$$u_x(1,t) = w(t)$$

We have to find $u(0,t) = v(t)$. Letting

$$N(x,t;\xi,\tau) = \frac{1}{2\sqrt{\pi(t-\tau)}}\left[exp\left(-\frac{(x-\xi)^2}{4(t-\tau)}\right) - exp\left(-\frac{(x+\xi)^2}{4(t-\tau)}\right)\right]$$

and using Green's identity, we find as in the previous section :

$$\frac{1}{2\sqrt{\pi}}\int_0^t \frac{x}{(t-\tau)^{3/2}} exp\left(-\frac{x^2}{4(t-\tau)}\right) v(\tau)d\tau = u(x,t) - \int_0^t N(x,t;1,\tau)w(\tau)d\tau + \int_0^t \frac{\partial N}{\partial \xi}(x,t;1,\tau)f(\tau)d\tau. \tag{16}$$

Letting $x \to 1-$, we have

$$\frac{1}{2\sqrt{\pi}}\int_0^t \frac{1}{(t-\tau)^{3/2}} exp\left(-\frac{1}{4(t-\tau)}\right) v(\tau)d\tau$$
$$= f(t) + \frac{1}{2\sqrt{\pi}}\int_0^t \frac{1}{(t-\tau)^{1/2}} exp\left(-\frac{1}{t-\tau}\right) w(\tau)d\tau$$
$$- \frac{1}{2\sqrt{\pi}}\int_0^t \frac{1}{(t-\tau)^{3/2}} exp\left(-\frac{1}{t-\tau}\right) f(\tau)d\tau. \tag{17}$$

Let $G_1(t)$ be the right-hand side of (17) which is defined for $t > 0$ and depends continuously on $f(t)$ and $w(t)$ in the L^2-sense and let

$$k(t) = \begin{cases} \frac{1}{\sqrt{2}t^{3/2}} exp\left(-\frac{1}{4t}\right), & t > 0 \\ 0, & t \leq 0 \end{cases}$$

$$G(t) = \begin{cases} G_1(t), & t > 0 \\ 0, & t \leq 0. \end{cases}$$

Then we have from (17)

$$\begin{aligned} (k \star v)(t) &\equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} k(t-\tau)v(\tau)d\tau \\ &= \frac{1}{2\sqrt{\pi}} \int_0^t (t-\tau)^{-3/2} exp\left(-\frac{1}{4(t-\tau)}\right) v(\tau)d\tau \\ &= G(t), \qquad t \in R \end{aligned} \tag{18}$$

which is a convolution equation in $v(t)$.

Now, we are able to regularize this equation by constructing a family $(v_\beta)_{\beta>0}$ of regularized solutions and pick a regularized solution that is "close" to the exact solution of (18). We note that, by regularized solution, we mean a function that is stable with respect to variations in the right hand side of (18). We have

THEOREM 1. *Suppose the exact solution v_0 of (18) corresponding to G is in $H^s(R), s > 0$, i.e.,*

$$(1+|t|^2)^{s/2})\hat{v}_0(t) \in L^2(R)$$

and let

$$|G - G_0| < \delta$$

Then there exists a regularized solution v_δ of (18) such that

$$|v_\delta - v_0|_2 \leq \frac{C}{\left[ln\frac{1}{\delta}\right]^s} \qquad \text{for small } \delta > 0$$

where C is any constant greater than

$$2(2(s+1))^s\sqrt{2}.max(|t^s\hat{k}|_\infty + |t^s\hat{v}_0|_2, \left(|\hat{v}_0|_2^2 + 1\right)^{1/2}.$$

Furthermore, if $\frac{\hat{v}_0}{|\hat{k}|} \in L^2(R)$ then we have

$$|v_\delta - v_0|_2 \leq C\sqrt{(\delta)}$$

where δ is any constant greater than $2\left(1 + |\frac{\hat{v}_0}{\hat{k}}|_2\right)$.

PROOF. We have

$$\hat{k}(t) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} k(x)e^{-ixt}dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} k(x)(cosxt - i.sinxt)dx$$
$$= -exp\left(-\sqrt{\frac{|t|}{2}} + i.sgn(t)\sqrt{\frac{|t|}{2}}\right)$$

and note that the function

$$|\hat{k}(t) = exp\left(-\sqrt{\frac{|t|}{2}}\right)$$

is decreasing in $t > 0$.

For every $\beta > 0$, the function

$$\psi(t) = \frac{\overline{\hat{k}(t)}}{\beta + |\hat{k}|^2} \tag{19}$$

belongs to $L^2(R)$. Defining

$$v_\beta(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x)e^{itx}dx$$

we have $v_\beta \in L^2(R)$ and by (20), v_β satisfies the equation

$$\beta\hat{v}_\beta(t) + |\hat{k}(t)|^2\hat{v}_\beta(t) = \overline{\hat{k}(t)}\hat{G}(t), \qquad t \in R. \tag{20}$$

On the other hand, we have from (18)

$$\hat{k}(t).\hat{v}_0(t) = \hat{G}_0(t). \tag{21}$$

From (20) and (21), we have

$$\beta(\hat{v}_\beta(t) - \hat{v}_0(t)) + |\hat{k}(t)|^2(\hat{v}_\beta(t) - \hat{v}_0(t))$$
$$= -\beta\hat{v}_0(t) + \overline{\hat{k}(t)}(\hat{G}(t) - \hat{G}_0(t)), \qquad t \in R. \tag{22}$$

Multiplying both sides of (3.8) by the conjugate of $\hat{v}_\beta(t) - \hat{v}_0(t)$ and then intergrating over R, we have

$$\beta|\hat{v}_\beta - \hat{v}_0|_2^2 + |\hat{k}(\hat{v}_\beta - \hat{v}_0)|_2^2$$
$$= -\beta \int_{-\infty}^{\infty} \hat{v}_0\overline{\hat{v}_\beta(t) - \hat{v}_0(t)}dt + \int_{-\infty}^{\infty} \overline{\hat{k}(t)}(\hat{G}(t) - \hat{G}_0(t))\overline{\hat{v}_\beta(t) - \hat{v}_0(t)}dt. \tag{23}$$

Let $\beta = \delta < 1$ and note that

$$|\hat{G} - \hat{G}_0|_2 = |G - G_0|_2 < \delta.$$

Then, we have

$$\delta|\hat{v}_\delta - \hat{v}_0|_2^2 + |\hat{k}(\hat{v}_\delta - \hat{v}_0)|_2^2 \le \frac{\delta}{2}|\hat{v}_0|_2^2 + \frac{\delta}{2}|\hat{v}_\delta - \hat{v}_0|_2^2 + \frac{\delta^2}{2} + \frac{1}{2}|\hat{v}_\delta - \hat{v}_0|_2^2.$$

Therefore

$$\delta|\hat{v}_\delta - \hat{v}_0|_2^2 + |\hat{k}(\hat{v}_\delta - \hat{v}_0)|_2^2 \le \delta\left(|\hat{v}_0|_2^2 + 1\right).$$

In particular

$$|\hat{v}_\delta - \hat{v}_0|_2^2 \le |\hat{v}_0|_2^2 + 1 \tag{24}$$

$$|\hat{k}(\hat{v}_\delta - \hat{v}_0)|_2^2 \le \delta\left(|\hat{v}_0|_2^2 + 1\right).$$

Now, multiplying both sides of the identity

$$\delta(\hat{v}_\delta(t) - \hat{v}_0(t)) + |\hat{k}(t)|^2(\hat{v}_\delta(t) - \hat{v}_0(t)) = -\delta\hat{v}_0(t) + \overline{\hat{k}(t)}(\hat{G}(t) - \hat{G}_0(t))$$

by $|t|^{2s}\overline{(\hat{v}_\delta(t) - \hat{v}_0(t))}$ and then integrating over R, we have

$$\begin{aligned}
&\delta|t^s(\hat{v}_\delta - \hat{v}_0)|_2^2 + |t^s\hat{k}(\hat{v}_\delta - \hat{v}_0)|_2^2 \\
&= -\delta\int_{-\infty}^{\infty}|t|^{2s}\hat{v}_0(t)\overline{\hat{v}_\delta(t) - \hat{v}_0(t)}dt \\
&\quad + \int_{-\infty}^{\infty}|t|^{2s}\overline{\hat{k}(t)}(\hat{G}(t) - \hat{G}_0(t))\overline{\hat{v}_\delta(t) - \hat{v}_0(t)}dt \\
&\le \delta|t^s\hat{v}_0|_2|t^s(\hat{v}_\delta - \hat{v}_0)|_2 + |t^s\hat{k}|_\infty|\hat{G} - \hat{G}_0|_2|t^s(\hat{v}_\delta - \hat{v}_0)|_2
\end{aligned} \tag{25}$$

where

$$|t^s\hat{k}|_\infty = \sup_{t\in R}|t|^s|\hat{k}(t)| = exp(s\ln 8s^2 - 2s) < \infty.$$

In particular

$$\delta|t^s(\hat{v}_\delta - \hat{v}_0)|_2^2 \le \delta|t^s(\hat{v}_\delta - \hat{v}_0)\left(|t^s\hat{k}|_\infty + |t^s\hat{v}_0|_2\right)$$

i.e.

$$|t^s(\hat{v}_\delta - \hat{v}_0)|_2^2 \le |t^s\hat{k}|_\infty + |t^s\hat{v}_0|_2. \tag{26}$$

Letting

$$A = \max\left(|t^s\hat{k}|_\infty + |t^s\hat{v}_0|_2, (|\hat{v}_0|_2^2 + 1)^{1/2}\right)$$

we have, from (24) and (26),

$$|\hat{v}_\delta - \hat{v}_0|_2 \le A$$

and

$$|t^s(\hat{v}_\delta - \hat{v}_0)|_2 \le A \tag{27}$$

Now, we have

$$|v_\delta - v_0|_2^2 = |\hat{v}_\delta - \hat{v}_0|_2^2 = \int_{-\infty}^{\infty} |\hat{v}_\delta(t) - \hat{v}_0(t)|^2 dt$$

For any $t_\delta > 0$, we have

$$\begin{aligned}\int_{|t|\le t_\delta} |\hat{v}_\delta(t) - \hat{v}_0(t)|^2 dt &\le \int_{|t|\le t_\delta} \frac{|\hat{k}(t)|^2}{|\hat{k}(t_\delta|^2} |\hat{v}_\delta(t) - \hat{v}_0(t)|^2 dt \\ &\le exp(\sqrt{2t_\delta}) \int_{-\infty}^{\infty} |\hat{k}(t)(\hat{v}_\delta(t) - \hat{v}_0(t))|^2 dt \\ &\le exp(\sqrt{2t_\delta}) A^2 \delta. \end{aligned} \tag{28}$$

On the other hand,

$$\begin{aligned}\int_{|t|>t_\delta} |\hat{v}_\delta(t) - \hat{v}_0(t)|^2 dt &\le \frac{1}{t_\delta^{2s}} \int_{-\infty}^{\infty} |t^s(\hat{v}_\delta(t) - \hat{v}_0(t))|^2 dt \\ &\le \frac{A^2}{t_\delta^{2s}}. \end{aligned} \tag{29}$$

Let t_δ be the positive solution of the equation

$$\delta exp\left(\sqrt{2t_\delta}\right) = \frac{1}{t_\delta^{2s}}$$

or equivalently

$$t_\delta^{2s} exp\left(\sqrt{2t_\delta}\right) = \frac{1}{\delta}. \tag{30}$$

The function $h(y) = y^{2s} exp\left(\sqrt{2y}\right)$ is strictly increasing in $y > 0$ and $h(R^+) = R^+$, so that Eq. (30) has a unique solution t_δ and $t_\delta \to +\infty$ as $\delta \to 0$. Indeed, we have

$$2s \ln t_\delta + \sqrt{2t_\delta} = \ln\left(\frac{1}{\delta}\right)$$

For δ sufficiently small, we have

$$2(s+1)t_\delta > 2s\ln t_\delta + \sqrt{2t_\delta} = \ln\left(\frac{1}{\delta}\right).$$

Therefore

$$\frac{1}{t_\delta^{2s}} < \left(\frac{2(s+1)}{\ln(\frac{1}{\delta})}\right)^{2s}. \tag{31}$$

By (28)-(31), we have

$$|v_\delta - v_0|_2^2 \le \frac{2A^2}{t_\delta^{2s}} \le \left(\frac{C}{\left(\ln(\frac{1}{\delta})\right)^s}\right)^2$$

where $C = (2(s+1))^s.\sqrt{2}A$ as desired.

Now, if we assume that $\frac{\hat{v}_0}{|\hat{k}|} \in L^2(R)$ then by multiphying both sides of (22) by the conjugate of $\hat{v}_\beta(t) - \hat{v}_0(t)$ and then integrating over R, we have

$$\begin{aligned}&\beta|\hat{v}_\beta - \hat{v}_0|_2^2 + |\hat{k}(\hat{v}_\beta - \hat{v}_0)|_2^2 \\ &= -\beta\int_{-\infty}^{\infty} \hat{v}_0\overline{\hat{v}_\beta(t) - \hat{v}_0(t)}dt + \int_{-\infty}^{\infty}\overline{\hat{k}(t)}(\hat{G}(t) - \hat{G}_0(t))\overline{\hat{v}_\beta(t) - \hat{v}_0(t)}dt \\ &\le \beta|\frac{\hat{v}_0}{\hat{k}}|_2|\hat{k}(\hat{v}_\beta - \hat{v}_0)|_2 + |\hat{G} - \hat{G}_0|_2|\hat{k}(\hat{v}_\beta - \hat{v}_0)|_2.\end{aligned}$$

Letting $\beta = \delta$ and noting that $|\hat{G} - \hat{G}_0|_2 = |G - G_0|_2 < \delta$ we have

$$\delta|\hat{v}_\delta - \hat{v}_0|_2^2 + |\hat{k}(\hat{v}_\delta - \hat{v}_0)|_2^2 \le \delta|\hat{k}(\hat{v}_\delta - \hat{v}_0)|_2\left(1 + \frac{\hat{v}_0}{\hat{k}}|_2\right). \tag{32}$$

In particular

$$|\hat{k}(\hat{v}_\delta - \hat{v}_0)|_2^2 \le \delta|\hat{k}(\hat{v}_\delta - \hat{v}_0)|_2\left(1 + \frac{\hat{v}_0}{\hat{k}}|_2\right)$$

i.e.

$$|\hat{k}(\hat{v}_\delta - \hat{v}_0)|_2 \le \delta\left(1 + \frac{\hat{v}_0}{\hat{k}}|_2\right). \tag{33}$$

From (32) and (33), we have

$$\delta|\hat{v}_\delta - \hat{v}_0|_2^2 \le \delta^2\left(1 + \frac{\hat{v}_0}{\hat{k}}|_2\right)$$

i.e.

$$|\hat{v}_\delta - \hat{v}_0|_2^2 \le \delta\left(1 + \frac{\hat{v}_0}{\hat{k}}|_2\right)$$

Since $|v_\delta - v_0|_2 = |\hat{v}_\delta - \hat{v}_0|_2$, we have

$$|\hat{v}_\delta - \hat{v}_0|_2 \leq C\sqrt{\delta}$$

where $C = 1 + |\frac{\hat{v}_0}{k}|_2$.

This completes the proof of Theorem 1.

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