ACTA MATHEMATICA VIETNAMICA
Volume 20, Number 2, 1995, pp. 181-192

A NEW COMPUTATION OF TWO-DIMENSIONAL FINITE-PART INTEGRALS

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Dedicated to Professor Dang Dinh Ang on the occasion of his 70th birthday

Introdution

In several engineerning problems in fracture mechanics and eletromagnetic theory (Cabdulkhaev [2], Lee-Advani-Lee [4], Mayrkhofer [5], Mayrhofer-Fischer [6], Monegato [7] [8], Theocaris [10], Theocaris-Ioakimidis-Kazantzakis [11]) we have to deal with the evaluation of the two-dimensional finite-part integrals of the form

$$\begin{aligned} I(X,Y) &= \oint_S \frac{u(x,y)dxdy}{[(x-X)^2+(y-Y)^2]^{3/2}} \\ &= \int_0^{2\pi} \oint_0^{R(\theta)} \frac{u(X+r\cos(\theta), Y+r\sin(\theta))drd\theta}{r^2}; \end{aligned} \tag{1}$$

$$x = X + r\cos(\theta)\ , \quad y = Y + r\sin(\theta)\ , \quad r = [(x-X)^2+(y-Y)^2]^{3/2}\ ,$$

where S is a two-dimensional domain, and $\theta \in [0, 2\pi] \to R(\theta)$ is the equation of the boundary of S, and

$$\oint_0^{R(\theta)} \frac{u(X+r\cos(\theta), Y+r\sin(\theta))drd\theta}{r^2}$$

denotes the one-dimensional finite-part in the Hadamard sense.

The two-dimensional finite-part integral (1) does not exist as an ordinary integral or as a Cauchy-type principal value integral.

In this paper we propose a new method for numerically evaluating the two-dimensional finite-part integrals of the form (1), and an error analysis. A variant of this method is used recently for computing a tomographic reconstruction from projection [1].

Received May 30, 1995

1. A computation of the two-dimensional finite-part integrals.

Consider the two-dimensional finite-part integral

$$I(X,Y) = \oint_S \frac{u(x,y)dxdy}{[(x-X)^2+(y-Y)^2]^{3/2}} \tag{2}$$

where S denotes a rectangle, (X,Y) is a point in the interior of S, and $u : S \to R$ is a continuous function. When u is defined on a closed bounded domain $\Omega \subseteq S$, we set $u = 0$ on $S \setminus \Omega$; hence we may assume that $\Omega = S$.

Setting

$$\begin{aligned} x &= X + r\cos(\theta) \ , \quad y = Y + r\sin(\theta), \\ r &= [(x-X)^2+(y-Y)^2]^{1/2}, \theta \in [0, 2\pi]; \end{aligned} \tag{3}$$

we get

$$I(X,Y) = \int_0^{2\pi} \oint_0^{R(\theta)} \frac{u(X+r\cos(\theta), Y + r\sin(\theta))drd\theta}{r^2} \tag{4}$$

where $\theta \in [0,2\pi] \to R(\theta)$ is the equation of the boundary of S, and the singular integral

$$\oint_0^{R(\theta)} \frac{u(X+r\cos(\theta), Y+r\sin(\theta))dr}{r^2}$$

denotes the one-dimensional finite-part integral in the Hadamard sense.

Consider the rectangle $S = [a,b] \times [c,d]$, and let $x_i = hi,\ y_j = hj,\ 0 \le i \le N,\ 0 \le j \le M,\ h = (b-a)/N = (d-c)/M$, be partitions of the intervals $[a,b]$ and $[c,d]$ respectively; then the set $\{(x_i, y_j) : 0 \le i \le N,\ 0 \le j \le M\ \}$ forms a grid partitioning the rectangle S.

We set

$$\xi_i = x_i - h/2\ ,\ 1 \le i \le N\ ;\ \vartheta_j = y_j - h/2\ ,\ 1 \le j \le M\ ; \tag{5}$$

and we assume that $X \in [\xi_1, \xi_N[\ ,\ Y \in [\vartheta_1, \vartheta_N[$. Then we get two integers I, J such that

$$1 \le I < N\ ,\ 1 \le J < M\ ,\ X \in [\xi_I, \xi_{I+1}[\ ,\ Y \in [\vartheta_J, \vartheta_{J+1}[\ . \tag{6}$$

We have the inclusion

$$[\xi_I, \xi_{I+1}[\times[\vartheta_J, \vartheta_{J+1}[\ \subseteq\ [x_{I-1}, x_{I+1}[\times[y_{J-1}, y_{J+1}[, \tag{7}$$

and the distance from the point (X,Y) to the boundary $\delta(S_{I,J})$ of the rectangle

$S_{I,J} = [x_{I-1}, x_{I+1}[\times[y_{J-1}, y_{J+1}[$ is

$$dist((X,Y), \delta(S_{I,J})) \in [h/2, 2h2^{1/2}] \tag{8}$$

Now, the function

$$(x,y) \in S \setminus S_{I,J} \longrightarrow \frac{u(x,y)}{[(x-X)^2+(y-Y)^2]^{3/2}}$$

is regular, hence the integral

$$\int_{S\setminus S_{I,J}} \frac{u(x,y)dxdy}{[(x-X)^2+(y-Y)^2]^{3/2}}$$

is regular. We get

$$I(X,Y) = \int_{S\setminus S_{I,J}} \frac{u(x,y)dxdy}{[(x-X)^2+(y-Y)^2]^{3/2}} + \oint_{S_{I,J}} \frac{u(x,y)dxdy}{[(x-X)^2+(y-Y)^2]^{3/2}} \tag{9}$$

By bicubic spline approximation (see Prenter [9] p.131) we may assume that the function $(x,y) \in S_{I,J} \longrightarrow u(x,y)$ is a cubic polynomial in two variables x, y. Then the singular integral

$$\oint_{S_{I,J}} \frac{u(x,y)dxdy}{[(x-X)^2+(y-Y)^2]^{3/2}}$$

is a linear combination of the singular integrals

$$\oint_{S_{I,J}} \frac{x^m y^n dxdy}{[(x-X)^2+(y-Y)^2]^{3/2}}\ ,\ 0 \le m,n \le 3. \tag{10}$$

Setting

$$R_{i,j} = [x_i, x_{i+1}[\times[y_j, y_{j+1}[\ ,\ 0 \le i < N\ ,\ 0 \le j < M\ , \tag{11}$$

we have $[a,b[\times[c,d[= \bigcup\{R_{i,j},\ 0 \le i < N,\ 0 \le j < M\}$, and

$$\int_{S\setminus S_{I,J}} \frac{u(x,y)dxdy}{[(x-X)^2+(y-Y)^2]^{3/2}} =$$

$$\sum\left\{\int_{R_{i,j}} \frac{u(x,y)dxdy}{[(x-X)^2+(y-Y)^2]^{3/2}}\ ;\ R_{i,j} \subseteq S\setminus S_{I,J}\right\} \tag{12}$$

By bicubic spline approximation ([9], p.131) we may assume that the function $(x,y) \in R_{i,j} \longrightarrow u(x,y)$ is a cubic polynomial in two variables x, y. Then the regular integral

$$\int_{R_{i,j}} \frac{u(x,y)dxdy}{[(x-X)^2+(y-Y)^2]^{3/2}}$$

is a linear combination of the regular integrals

$$\int_{R_{i,j}} \frac{x^m y^n dxdy}{[(x-X)^2+(y-Y)^2]^{3/2}}, \quad 0 \leq m,n \leq 3; \tag{13}$$

where $R^{i,j} \subseteq S \setminus S_{I,J}$. Let us evaluate analytically the integrals (13). With $t = x - X, s = y - Y$, we get

$$\int_{R_{i,j}} \frac{x^m y^n dxdy}{[(x-X)^2+(y-Y)^2]^{3/2}} = \int_{[y_j-Y, y_{j+1}-Y]} (s+Y)^n ds \int_{[x_i-X, x_{i+1}-X]} \frac{(t+X)^m dt}{(t^2+s^2)^{3/2}}, \tag{14}$$

and we are reduced to evaluate the indefinite integrals

$$\int s^n ds \int \frac{t^m dt}{(t^2+s^2)^{3/2}}, \quad 0 \leq m,n \leq 3.$$

From the tables of Gradshteyn-Ryzhiz [3], p. 83, we get the indefinite integrals

$$\begin{aligned} \int \frac{dt}{(t^2+s^2)^{3/2}} &= \frac{t}{s^2(t^2+s^2)^{1/2}}, \\ \int \frac{tdt}{(t^2+s^2)^{3/2}} &= -\frac{1}{(t^2+s^2)^{1/2}}, \\ \int \frac{t^2dt}{(t^2+s^2)^{3/2}} &= -\frac{t}{(t^2+s^2)^{1/2}} + arcsin(t/s), \\ \int \frac{t^3dt}{(t^2+s^2)^{3/2}} &= \frac{t^2+2s^2}{(t^2+s^2)^{1/2}} \end{aligned} \tag{15}$$

and also the indefinite integrals

$$\begin{aligned} \int \frac{dt}{(t^2+s^2)^{1/2}} &= \log(t+(t^2+s^2)^{1/2}) = arcsin(t/s), \\ \int \frac{tdt}{(t^2+s^2)^{1/2}} &= (t^2+s^2)^{1/2}, \\ m\int \frac{t^m dt}{(t^2+s^2)^{1/2}} &= t^{m-1}(t^2+s^2)^{1/2} - (m-1)s^2 \int \frac{t^{m-2}dt}{(t^2+s^2)^{1/2}}, \quad m \in Z \end{aligned}$$

$$\int \frac{dt}{t^2(t^2+s^2)^{1/2}} = -\frac{(t^2+s^2)^{1/2}}{s^2 t}. \tag{16}$$

From equations (15), (16) we obtain the indefinite integrals

$$\begin{aligned}
\int s^n ds \int \frac{dt}{(t^2+s^2)^{3/2}} &= t\int \frac{s^{n-2}ds}{(t^2+s^2)^{1/2}}, \\
\int s^n ds \int \frac{tdt}{(t^2+s^2)^{3/2}} &= -\int \frac{s^n ds}{(t^2+s^2)^{1/2}}, \\
\int s^n ds \int \frac{t^2dt}{(t^2+s^2)^{3/2}} &= -t\int \frac{s^n ds}{(t^2+s^2)^{1/2}} + \int arcsin(t/s)s^n ds.
\end{aligned} \tag{17}$$

By integrating by parts, with

$$u = arcsin(t/s),\ v' = s^n,\ u' = -\frac{t}{s(t^2+s^2)^{1/2}},\ v = \frac{s^{n+1}}{n+1},$$

we get

$$\begin{aligned}
\int arcsin(t/s)s^n ds &= arcsin(t/s)\frac{s^{n+1}}{n+1} + \left(\frac{t}{n+1}\right)\int \frac{s^n ds}{(t^2+s^2)^{1/2}}; \\
\int s^n ds \int \frac{t^2dt}{(t^2+s^2)^{3/2}} &= arcsin(t/s)\frac{s^{n+1}}{n+1} - \frac{tn}{n+1}\int \frac{s^n ds}{(t^2+s^2)^{1/2}}; \\
\int s^n ds \int \frac{t^3dt}{(t^2+s^2)^{3/2}} &= \int s^n ds \left[\frac{t^2}{(t^2+s^2)^{1/2}} + \frac{2s^2}{(t^2+s^2)^{1/2}}\right] \\
&= t^2 \int \frac{s^n ds}{(t^2+s^2)^{1/2}} + 2\int \frac{s^{n+2}ds}{(t^2+s^2)^{1/2}}.
\end{aligned} \tag{18}$$

for $0 \le n \le 3$. Let $U(x,y)$ be the infinite integral

$$U(x,y) = \int \frac{x^m y^n dxdy}{[(x-X)^2+(y-Y)^2]^{3/2}}$$

given by equations (15), (16), (17), (18). From (13) we get analytically the integrals

$$\int_{R_{i,j}} \frac{x^m y^n dxdy}{[(x-X)^2+(y-Y)^2]^{3/2}} =$$

$$U(x_{i+1}, y_{j+1}) - U(x_{i+1}, y_i) - U(x_i, y_{j+1}) + U(x_i, y_j)\ . \tag{19}$$

Consider the singular integrals of (10)

$$\oint_{S_{I,J}} \frac{x^m y^n dxdy}{[(x-X)^2+(y-Y)^2]^{3/2}} =$$

$$= \int_0^{2\pi} \oint_0^{R(\theta)} \frac{(X + r\cos(\theta))^m (Y + r\sin(\theta))^n dr d\theta}{r^2},$$

where $\theta \in [0, 2\pi] \longrightarrow R(\theta) > 0$ is the equation of the boundary of $S_{I,J}$. Then we are reduced to consider the finite-part integral $\oint_0^{R(\theta)} \frac{dr}{r^2}$ and the Cauchy singular integral $⨍_0^{R(\theta)} \frac{dr}{r}$. We get

$$\oint_0^{R(\theta)} \frac{dr}{r^2} = -\frac{1}{R(\theta)},\ ⨍_0^{R(\theta)} \frac{dr}{r} = \log(R(\theta)). \tag{20}$$

Now, let $U(x, y)$ be the infinite integral

$$\oint \frac{x^m y^n dx dy}{[(x - X)^2 + (y - Y)^2]^{3/2}}$$

given by equations (15), (16), (17), (18). Then we obtain

$$\oint_{S_{I,J}} \frac{x^m y^n dx dy}{[(x - X)^2 + (y - Y)^2]^{3/2}} = U(x_{I+1}, y_{J+1}) - U(x_{I+1}, y_{J-1})$$
$$-U(x_{I-1}, y_{J+1}) + U(x_{I-1}, y_{J-1}) \tag{21}$$

2. An error analysis.

Assume that $u \in C^{(4)}([a, b] \times [c, d])$, and let u_N be the bicubic spline interpolate to u, see Prenter [9], p. 132.

Setting

$$v = u - u_N,\ \|v\|_\infty = \sup\{|v(x, y)|; a \le x \le b, c \le y \le d\}, \tag{22}$$

we get the following error estimates:

$$\|v\|_\infty \le \beta_0 h^4$$

$$\left\|\frac{\partial v}{\partial v}\right\|_\infty \le \beta_1 h^3,\ \left\|\frac{\partial v}{\partial y}\right\|_\infty \le \beta_1 h^3,$$

$$\|\frac{\partial^2 v}{\partial x^2}\| \le \beta_2 h^2,\ \|\frac{\partial^2 v}{\partial x \partial y}\|_\infty \le \beta_2 h^2,\ \|\frac{\partial^2 v}{\partial y^2}\|_\infty \le \beta_2 h^2, \tag{23}$$

where β_0, β_1 and β_2 depend only on the norms of u, $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial x \partial y}$, $\frac{\partial^2 u}{\partial y^2}$, see Prenter [9], p. 132.

Consider the integral of (12)

$$\int_{S\setminus S_{I,J}} \frac{v(x,y)dxdy}{[(x-X)^2+(y-Y)^2]^{3/2}} . \tag{14}$$

By the following change of the variables

$$x = X + r\cos(\theta),\ y = Y + r\sin(\theta),\ r = [(x-X)^2+(y-Y)^2]^{1/2},\ \theta \in [0, 2\pi],$$

we get

$$\int_{S\setminus S_{I,J}} \frac{v(x,y)dxdy}{[(x-X)^2+(y-Y)^2]^{3/2}} = \int_0^{2\pi}\int_{R_1(\theta)}^{R_2(\theta)} \frac{V(r,\theta)drd\theta}{r^2} , \tag{25}$$

where $V(r,\theta) = v(X + r\cos(\theta), Y + r\sin(\theta)),\ \theta \in [0,2\pi],\ r \in [R_1(\theta), R_2(\theta)]$; and $\theta \longrightarrow (R_1(\theta), R_2(\theta))$ is the equation of the boundary of $S \setminus S_{I,J}$. From (8) it follows that

$$\frac{h}{2} \le R_1(\theta) < R_2(\theta)\ ,\ \forall\theta \in [0,2\pi]\ . \tag{26}$$

Then, (23), (25), (26) yield

$$|V(r,\theta)| \le \beta_0 h^4,$$

$$\begin{aligned}\left|\int_0^{2\pi}\int_{R_1(\theta)}^{R_2(\theta)} \frac{V(r,\theta)drd\theta}{r^2}\right| &\le 2\pi\beta_0 h^4 \int_{R_1(\theta)}^{R_2(\theta)} \frac{dr}{r^2} \\ &= 2\pi\beta_0 h^4 \left(\frac{1}{R_1(\theta)} - \frac{1}{R_2(\theta)}\right) \\ &\le \frac{2\pi\beta_0 h^4}{R_1(\theta)} \le 4\pi\beta_0 h^3\ ,\end{aligned}$$

$$\left|\int_{S\setminus S_{I,J}} \frac{v(x,y)dxdy}{[(x-X)^2+(y-Y)^2]^{3/2}}\right| \le 4\pi\beta_0 h^3\ . \tag{27}$$

Using the integral of (10) we get

$$\oint_{S_{I,J}} \frac{v(x,y)dxdy}{[(x-X)^2+(y-Y)^2]^{3/2}} =$$

$$\int_0^{2\pi}\oint_0^{R(\theta)} \frac{v(X + r\cos(\theta), Y + r\sin(\theta))drd\theta}{r^2} ; \tag{28}$$

$$x = X + r\cos(\theta)\ ,\ y = Y + r\sin(\theta)\ ;\ \ r = [(x-X)^2+(y-Y)^2]^{1/2}\ ,$$

where $\theta \in [0, 2\pi] \longrightarrow R(\theta) > 0$ is the equation of the boundary of $S_{I,J}$. Setting $R_1(\theta) = -R(\theta+\pi), R_2(\theta) = R(\theta), \forall\theta \in [0, \pi]$, equation (28) yields

$$\oint_{S_{I,J}} \frac{v(x,y)dxdy}{[(x-X)^2+(y-Y)^2]^{3/2}} = \int_0^\pi \oint_{R_1(\theta)}^{R_2(\theta)} \frac{V(r,\theta)drd\theta}{r^2}, \tag{29}$$

where

$$V(r,\theta) = v(X + r\cos(\theta), Y + r\sin(\theta)),$$

and equation (8) yields

$$\begin{aligned} \frac{h}{2} &\le -R_1(\theta) \le 2\sqrt{2}h, \\ \frac{h}{2} &\le R_2(\theta) \le 2\sqrt{2}h, \\ -R_1(\theta) + R_2(\theta) &\le 2\sqrt{2}h. \end{aligned} \tag{30}$$

Now, for each $\theta \in [0, \pi]$, we have

$$\begin{aligned} \oint_{R_1(\theta)}^{R_2(\theta)} \frac{V(r,\theta)dr}{r^2} = \int_{R_1(\theta)}^{R_2(\theta)} \frac{\left[V(r,\theta) - V(0,\theta) - r\left(\frac{\partial V}{\partial r}\right)(0,\theta)\right]dr}{r^2} \\ + \left(\frac{\partial V}{\partial r}\right)(0,\theta) ⨍_{R_1(\theta)}^{R_2(\theta)} \frac{dr}{r} + V(0,\theta) \oint_{R_1(\theta)}^{R_2(\theta)} \frac{dr}{r^2}; \end{aligned} \tag{31}$$

where

$$\int_{R_1(\theta)}^{R_2(\theta)} \frac{\left[V(r,\theta) - V(0,\theta) - r\left(\frac{\partial V}{\partial r}\right)(0,\theta)\right]dr}{r^2}$$

is a regular integral,

$$⨍_{R_1(\theta)}^{R_2(\theta)} \frac{dr}{r} = \log(R_2(\theta)) - \log(-R_1(\theta)) = \log\left(\left|\frac{R_2(\theta)}{R_1(\theta)}\right|\right)$$

is a Cauchy principal value integral, and

$$\oint_{R_1(\theta)}^{R_2(\theta)} \frac{dr}{r^2} = -\frac{1}{R_2(\theta)} + \frac{1}{R_1(\theta)}$$

is a Hadamard finite-part integral.

Equation (30) yields

$$0 < |R_2(\theta)| + |R_1(\theta)| \le 2\sqrt{2}h,$$

$$\frac{1}{4\sqrt{2}} \leq \left|\frac{R_2(\theta)}{R_1(\theta)}\right| \leq 4\sqrt{2},$$

$$\left|\log\left(\left|\frac{R_2(\theta)}{R_1(\theta)}\right|\right)\right| \leq \log(4\sqrt{2})$$

$$\left|-\frac{1}{R_2(\theta)} + \frac{1}{R_1(\theta)}\right| \leq \left|\frac{1}{R_2(\theta)}\right| + \left|\frac{1}{R_1(\theta)}\right| \leq \frac{4}{h}.$$

Therefore,

$$\left|\frac{V(r,\theta) - V(0,\theta) - r\left(\frac{\partial V}{\partial r}\right)(0,\theta)}{r^2}\right| \leq \frac{1}{2}\left\|\frac{\partial^2 V}{\partial r^2}\right\|_\infty, \tag{32}$$

$$\left|\int_{R_1(\theta)}^{R_2(\theta)} \frac{dr}{r}\right| = \left|\log\left(\left|\frac{R_2(\theta)}{R_1(\theta)}\right|\right)\right| \leq \log(4\sqrt{2}), \tag{33}$$

$$\left|\oint_{R_1(\theta)}^{R_2(\theta)} \frac{dr}{r^2}\right| = \left|-\frac{1}{R_2(\theta)} + \frac{1}{R_1(\theta)}\right| \leq \frac{4}{h}. \tag{34}$$

Now, for each $\theta \in [0,\pi]$, equations (30), (31), (32), (33), (33) yield

$$\left|\oint_{R_1(\theta)}^{R_2(\theta)} \frac{V(r,\theta)dr}{r^2}\right| \leq \left\|\frac{\partial^2 V}{\partial r^2}\right\|_\infty h\sqrt{2} + \left\|\frac{\partial V}{\partial r}\right\|_\infty \log(4\sqrt{2}) + \|V\|_\infty \frac{4}{h}. \tag{35}$$

From equation (23) we get

$$\|V\|_\infty \leq \beta h^4\ , \quad \left\|\frac{\partial V}{\partial r}\right\|_\infty \leq \beta h^3\ , \quad \left\|\frac{\partial^2 V}{\partial r^2}\right\|_\infty \leq \beta h^2\ , \tag{36}$$

where β depends only on the norms of $u\ , \ \frac{\partial u}{\partial x}\ , \ \frac{\partial u}{\partial y}\ , \ \frac{\partial^2 u}{\partial x^2}\ , \ \frac{\partial^2 u}{\partial x \partial y}\ , \ \frac{\partial^2 u}{\partial y^2}$.

Equations (35), (36), (29) yield

$$\left|\oint_{R_1(\theta)}^{R_2(\theta)} \frac{V(r,\theta)dr}{r^2}\right| \leq \beta h^3(\sqrt{2} + \log(4\sqrt{2}) + 4), \tag{37}$$

$$\left|\oint_{S_{I,J}} \frac{v(x,y)dxdy}{[(x-X)^2 + (y-Y)^2]^{3/2}}\right| \leq \pi\beta h^3(\sqrt{2} + \log(4\sqrt{2}) + 4). \tag{38}$$

Finally, from equations (27), (38) we get

$$\left|\int_S \frac{v(x,y)dxdy}{[(x-X)^2 + (y-Y)^2]^{3/2}}\right| \leq h^3(4\pi\beta_0 + \pi\beta(\sqrt{2} + \log(4\sqrt{2}) + 4)). \tag{39}$$

Hence our error estimate is of order 3.

3. An example.

Consider the following example:

$$I(X,Y) = \oint_S \frac{xdxdy}{[(x-X)^2+(y-Y)^2]^{3/2}} \tag{40}$$

where $S=[-1,1]\times[-1,1], u(x,y)=x$. For all $R_{i,j}\subseteq S\setminus S_{I,J}$, equation (14) yields

$$\int_{R_{i,j}} \frac{xdxdy}{[(x-X)^2+(y-Y)^2]^{3/2}} = \int_{[y_j-Y,y_{j+1}-Y]} ds \int_{[x_i-X,x_{i+1}-X]} \frac{(t+X)dt}{(t^2+s^2)^{3/2}}$$
$$= \int_{[y_j-Y,y_{j+1}-Y]} ds\Big(\Big[((x_i-X)^2+s^2)^{-1/2} - ((x_{i+1}-X)^2+s^2)^{-1/2}\Big]$$
$$+X\Big[(x_{i+1}-X)s^{-2}((x_{i+1}-X)^2+s^2)^{-1/2} - (x_i-X)s^{-2}((x_i-X)^2+s^2)^{-1/2}\Big]\Big) \tag{41}$$

We get

$$\int_{[y_j-Y,y_{j+1}-Y]} ((x_i-X)^2+s^2)^{-1/2}ds =$$
$$\log\left(\frac{y_{j+1}-Y+((x_i-X)^2+(y_{j+1}-Y)^2)^{1/2}}{y_j-Y+((x_i-X)^2+(y_j-Y)^2)^{1/2}}\right),$$
$$\int_{[y_j-Y,y_{j+1}-Y]} s^{-2}((x_i-X)^2+s^2)^{-1/2}ds =$$
$$(x_i-X)^{-2}\left(\frac{((x_i-X)^2+(y_i-Y)^2)^{1/2}}{y_j-Y} - \frac{((x_i-X)^2+(y_{j+1}-Y)^2)^{1/2}}{y_{j+1}-Y}\right).$$

Hence

$$\int_{R_{i,j}} \frac{xdxdy}{[(x-X)^2+(y-Y)^2]^{3/2}} =$$
$$\log\left(\frac{y_{j+1}-Y+((x_i-X)^2+(y_{j+1}-Y)^2)^{1/2}}{y_j-Y+((x_i-X)^2+(y_j-Y)^2)^{1/2}}\right)$$
$$-\log\left(\frac{y_{j+1}-Y+((x_{i+1}-X)^2+(y_{j+1}-Y)^2)^{1/2}}{y_j-Y+((x_{i+1}-X)^2+(y_j-Y)^2)^{1/2}}\right)$$
$$+\frac{X}{x_{i+1}-X}\Big(\frac{((x_{i+1}-X)^2+(y_j-Y)^2)^{1/2}}{y_j-Y} - \frac{((x_{i+1}-X)^2+(y_{j+1}-Y)^2)^{1/2}}{y_{j+1}-Y}\Big)$$
$$-\frac{X}{x_i-X}\Big(\frac{((x_i-X)^2+(y_j-Y)^2)^{1/2}}{y_j-Y} - \frac{((x_i-X)^2+(y_{j+1}-Y)^2)^{1/2}}{y_{j+1}-Y}\Big) \tag{42}$$

By a similar manner, from equation (21) we get

$$\begin{aligned}
\int_{S_{I,J}} \frac{x dx dy}{[(x-X)^2+(y-Y)^2]^{3/2}} =& \\
&\log\left(\frac{y_{J+1}-Y+((x_{I-1}-X)^2+(y_{J+1}-Y)^2)^{1/2}}{y_{J-1}-Y+((x_{I-1}-X)^2+(y_{J-1}-Y)^2)^{1/2}}\right) \\
&-\log\left(\frac{y_{J+1}-Y+((x_{I+1}-X)^2+(y_{J+1}-Y)^2)^{1/2}}{y_{J-1}-Y+((x_{I+1}-X)^2+(y_{J-1}-Y)^2)^{1/2}}\right) \\
&+\frac{X}{x_{I+1}-X}\left(\frac{((x_{I+1}-X)^2+(y_{J-1}-Y)^2)^{1/2}}{y_{J-1}-Y}\right. \\
&\qquad\left.-\frac{((x_{I+1}-X)^2+(y_{J+1}-Y)^2)^{1/2}}{y_{J+1}-Y}\right) \\
&-\frac{X}{x_{I-1}-X}\left(\frac{((x_{I-1}-X)^2+(y_{J-1}-Y)^2)^{1/2}}{y_{J-1}-Y}\right. \\
&\qquad\left.-\frac{((x_{I-1}-X)^2+(y_{J+1}-Y)^2)^{1/2}}{y_{J+1}-Y}\right)
\end{aligned} \tag{43}$$

We have the analytic solution

$$\begin{aligned}
I(X,Y) &= \oint_S \frac{x dx dy}{[(x-X)^2+(y-Y)^2]^{3/2}} \\
&= \log\left(\frac{1-Y+((1+X)^2+(1-Y)^2)^{1/2}}{-1-Y+((1+X)^2+(1+Y)^2)^{1/2}}\right) \\
&\quad -\log\left(\frac{1-Y+((1-X)^2+(1-Y)^2)^{1/2}}{-1-Y+((1-X)^2+(1+Y)^2)^{1/2}}\right) \\
&+\frac{X}{1-X}\left(\frac{((1-X)^2+(1+Y)^2)^{1/2}}{-1-Y}-\frac{((1-X)^2+(1-Y)^2)^{1/2}}{1-Y}\right) \\
&+\frac{X}{1+X}\left(\frac{((1+X)^2+(1+Y)^2)^{1/2}}{-1-Y}-\frac{((1+X)^2+(1-Y)^2)^{1/2}}{1-Y}\right)
\end{aligned} \tag{44}$$

Therefore,

$$\begin{aligned}
I(X,Y) =& \int_{S_{I,J}} \frac{x dx dy}{[(x-X)^2+(y-Y)^2]^{3/2}} \\
&+\sum\left\{\int_{R_{i,j}} \frac{u(x,y) dx dy}{[(x-X)^2+(y-Y)^2]^{3/2}}; R_{i,j} \subseteq S \setminus S_{I,J}\right\}.
\end{aligned}$$

REFERENCES

[1] Bui Doan Khanh, *A computation for tomographic reconstrution from projections*, Appl. Math. and Comput. **64** (1994), 79-91.

[2] B. G. Gabdulkhaev, *Cubature formulas for multidimensional singular integrals II*, Soviet Math. **19** (1975), 1-9.

[3] I. S. Gradshteyn and I. M. Ryzhiz, "Table of integrals, series, and products," Academic Press, 1980.

[4] T. S. Lee, S. H. Advani and J. K. Lee, *Indirect finite element evaluation of two-dimentional finite-part integral using Fourier transformation*, Int. J. Numer. Method Eng. **36** (1993), 2981-2996.

[5] K. Mayrkhofer, *Closed-form solutions for finite-part integrals occuring in shear-mode fracture mechanics*, Acta Mechanica **105** (1994), 85-100.

[6] K. Mayrkhofer and F. D. Fischer, *Derivation of a new analytical solution for a general two-dimensional finite-part integral applicable in fracture mechanics*, Int. J. Numer. Methods Eng. **33** (1992), 1027-1047.

[7] G. Monegato, *The numerical evaluation of a 2-D Cauchy principal value integral arising in boundary integral equation methods*, Math. Comput. **62** (1994), 765-777.

[8] G. Monegato, *Numerical evaluation of hypersingular integrals*, J. Comput. Appl. Math. **50** (1994), 9-31.

[9] P. M. Prenter, "Splines and variatinal methods," Wiley, 1975.

[10] P. S. Theocaris, *Modified Gauss-Legendre, Lobatto and Radau cubature formulas for the numerical evaluation of 2-D singular integrals*, Int. J. Math. Sc. **6** (1983), 567-587.

[11] P. S. Theocaris, N. I. Ioakimidis and J. G. Kazantzakis, *On the numerical evaluation of two-dimension principal value integrals*, Int. J. Numer, Methods Eng. **14** (1980), 629-634.

[12] F. Tricomi, *Equazioni integrali contenenti il valor principale di un integrale doppio*, Math. Zeit. **27** (1928), 368-377.

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