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P-STANDARD SYSTEMS OF PARAMETERS AND P-STANDARD IDEALS IN LOCAL RINGS

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1. Introduction

Let $(A, \mathfrak{m})$ be a commutative Noetherian local ring with the maximal ideal m and M a finitely generated A-module with $\dim M = d$. Let $\mathfrak{q}$ be a parameter ideal of M. It is well-known that the difference between the length and the multiplicity of $\mathfrak{q}$

$$I(\mathfrak{q}; M) = \ell(M/\mathfrak{q}M) - e(\mathfrak{q}; M)$$

gives a lot of informations on the structure of the module M. For example, modules, which satisfy the condition that $I(\mathfrak{q}; M)$ is constant for all parameter ideals of M, are called Buchsbaum modules and their structure is well-known [S-V]. Furthermore, we set $I(M) = \sup\limits_{\mathfrak{q}} I(\mathfrak{q}; M)$, where $\mathfrak{q}$ runs through all parameter ideals of M. Then $I(M) < +\infty$ if and only if $\ell(H^i_{\mathfrak{m}}(M)) < +\infty$ for $i = 0, ..., d-1$, thereby $H^i_{\mathfrak{m}}(M))$ is the i-th local cohomology module of M with respect to $\mathfrak{m}$. Note that the notion of modules M with $I(M) < +\infty$ is a generalization of that of Cohen-Macaulay module in a natural way. This module is called a generalized Cohen-Macaulay and has been studied first in [C-S-T]. In the theory of generalized Cohen-Macaulay modules, a so-called standard system of parameters plays a central roll. Recall that a standard system of parameters $x = \{x_1, ..., x_d\}$ of M is characterized by the equality $I(M) = I(\mathfrak{q}; M)$, where $\mathfrak{q} = (x_1, ..., x_d)A$. Then M is a generalized Cohen-Macaulay module if and only if M admits a standard system of parameters. Now, we will extend the above idea to the following situation:

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Let $x = \{x_1, ..., x_d\}$ be a system of parameters of M and $n = (n_1, ..., n_d)$ a d-tuple of positive integers. We consider the difference

$$I(n; x) = \ell(M/(x_1^{n_1}, ..., x_d^{n_d})) - n_1 ... n_d\, e(x; M)$$

as a function in n. Then we prove in [C2] that the least degree of all polynomials in n bounding above $I(n; x)$ is independent of the choice of x. This invariant is called the polynomial type of M and denoted by $p(M)$. Therefore M is not generalized Cohen-Macaulay if and only if $p(M) > 0$. The purpose of this paper is to define a new kind of system of parameters called *p-standard system of parameters*, which is closely related to the invariant $p(M)$ and plays a roll in M with $p(M) > 0$ like that of standard systems of parameters in the theory of generalized Cohen-Macaulay modules.

Let us give a summary of this paper. In Section 2, we will give several basic properties of a p-standard system of parameters. In particular, we prove that if x is a p-standard system of parameters then the function $I(n; x)$ is a polynomial having very simple form; moreover, we can also show in this case that x is a strong d-sequence in the sense of Huneke [Hu]. We define in this section one more notion called *p-standard ideal* and examine the relation between p-standard systems of parameters and p-standard ideals. All definitions in Section 2 will be globalized in Section 3. We show in this section a geometric meaning of the polynomial type $p(M)$ which has been proved in [C2] for the local case. Also the existence of global p-standard ideals will be proved based on a result of Faltings on the annihilators of local cohomology. In the last section, we define the blowing up rings with respect to a p-standard ideal and give some relations between their polynomial types.

2. P-standard systems of parameters

Let $(A, \mathfrak{m})$ be a local ring and M a finitely generated A-module with $\dim M = d$. Let $x = \{x_1, ..., x_d\}$ be a system of parameters of M and $n = (n_1, ..., n_d)$ a d-tuple of positive integers. We consider the difference

$$I_M(n; x) = \ell(M/(x_1^{n_1}, ..., x_d^{n_d})M) - n_1 ... n_d e(x; M)$$

as a function in n, where $\ell(M)$ is the length of the A-module M and $e(x; M)$ is the multiplicity of M with respect to the system of parameters x. In [C$_2$], Theorem 2.3, we have shown that the least degree of all polynomials in n bounding above $I_M(n; x)$ is independent of the choice of x. Therefore we have the following definition.

DEFINITION 2.1. The least degree of all polynomials in n bounding above $I_M(n; x)$ is called *polynomial type* of M and is denoted by $p(M)$.

Many properties of polynomial types of modules were studied in [C$_1$], [C$_2$]. Here we recall some of them which we need for the further investigation in this paper. First of all, it is easy to see that $p(M) \leq \dim M - 1$. We stipulate that the degree of the zero polynomial is equal to $-\infty$. Then the module M is Cohen-Macaulay if and only if $p(M) = -\infty$. Next, we denote by $\mathfrak{a}_i(M)$ the annihilator of the i-th local cohomology module $H^i_{\mathfrak{m}}(M)$ of M with respect to the maximal ideal $\mathfrak{m}$ and put $\mathfrak{a}(M) = \mathfrak{a}_o(M)...\mathfrak{a}_{d-1}(M)$. We also denote by $NC(M)$ the non-Cohen-Macaulay locus of M, i.e. $NC(M) = \{\mathfrak{p} \in \operatorname{Supp}(M)$, $M_{\mathfrak{p}}$ is not a Cohen-Macaulay module$\}$. An A-module M is called equidimensional if $\dim M = \dim(A/\mathfrak{p})$ for all minimal associated prime ideals $\mathfrak{p}$ of M. Then the first meaning of the polynomial type is given in the following theorem.

THEOREM 2.2([C$_1$], Theorem 1.2). *Suppose that A admits a dualizing complex. Then:*

(i) $p(M) = \dim(A/\mathfrak{a}(M))$.

(ii) If M is equidimensional then $p(M) = \dim NC(M)$.

Note that if A admits a dualizing complex then, by [Sh], the formal fiber of A are Gorenstein, and hence also Cohen-Macaulay. Therefore by the same argument as in the proof of Theorem 4.1 of [C$_2$] we get the following lemma.

LEMMA 2.3. *Let k be an integer. Suppose that A admits a dualizing complex. Then the following conditions are equivalent:*

(i) $p(M) \leq k$;

(ii) for any $\mathfrak{p} \in \operatorname{Supp} M$ with $\dim(A/\mathfrak{p}) > k$, $M_{\mathfrak{p}}$ is Cohen-Macaulay and $\dim M_{\mathfrak{p}} + \dim(A/\mathfrak{p}) = d$.

It is known that if A has a dualizing complex then $\dim A/\mathfrak{a}(M) < d$ (see [Sc₁], Korollar 2.2.4). Moreover, since any homomorphic image of A also has a dualizing complex, there always exists in that case a system of parameters $x = \{x_1, ..., x_d\}$ of M so that the following conditions are satisfied:

$$(*) \quad \begin{cases} x_d \in \mathfrak{a}(M) \ ; \\ \overline{x}_i \in \mathfrak{a}(M/(x_{i+1}, ..., x_d)M), \ \ i = 1, ..., d-1, \end{cases}$$

where $\overline{x}_i$ is the image of x_i in $A/(x_{i+1}, ..., x_d)A$.

DEFINITION 2.4. Let A be a local ring and and M a finitely generated A-module with $\dim M = d$. Then, a system of parameters $x = \{x_1, ..., x_d\}$ of M is called *p-standard system of parameters* if x satisfies the above condition (*).

DEFINITION 2.5. Given A and M as above. Let $p(M) = k \geq 0$.

(i) An ideal $I \subset A$ is called *p-standard ideal* of M if the following conditions are satisfied:

1) $I \subseteq \mathfrak{a}(M)$;

2) There exists a p-standard system of parameters $x = \{x_1, ..., x_d\}$ of M such that $I = (x_{k+1}, ..., x_d)A$.

In this case, $\{x_{k+1}, ..., x_d\}$ will be called *p-standard basis* of I on M.

(ii) An ideal $I \subset A$ is called *quasi-p-standard ideal* of M if it is generated by a subsystem of a p-standard basis of M.

It should be mentioned that p-standard systems of parameters of local rings have been examined by Brodmann [B₃] and Schenzel [Sc₂] in the case $p(M) \leq 0$ and by Faltings [F₁] and Brodmann [B₁] , [B₂] in the case $p(M) \leq 1$.

Recall that a system $\{x_1, ..., x_t\}$ in A is said to be a d-sequence of M if

$$(x_1, ..., x_{i-1})M : x_j = (x_1, ..., x_{i-1})M : x_i x_j$$

for all $i = 1, ..., t$ and $j \geq i$. The notion of d-sequence was introduced by Huneke [Hu] and has become a useful tool in different topics of commutative algebra.

In the sequel, we will show that the p-standard system of parameters has many nice properties.

THEOREM 2.6. *Let $x = \{x_1, ..., x_d\}$ be a p-standard system of parameters of M and $n = (n_1, ..., n_d)$ a d-tupel of positive integers. Put $p(M) = k$. Then:*

(i) $\{x_1^{n_1}, ..., x_d^{n_d}\}$ *is a d-sequence of* M .

(ii) $I_M(n;x) = \sum_{i=o}^{k} n_1...n_i e_i$,

where $e_i = e(x_1, ..., x_i; (x_{i+2}, ..., x_d)M : x_{i+1}/(x_{i+2}, ..., x_d)M)$, for $i \neq 0$ and $e_0 = \ell((x_2, ..., x_d)M : x_1/(x_2, ..., x_d)M)$.

PROOF. (i) By [C₁], Theorem 1.1, $I_M(n;x)$ is a polynomial and for all $n_1, ..., n_d > 0$ we have

$$(x_1^{n_1}, ..., x_i^{n_i})M : x_{i+1}^{n_{i+1}} = (x_1^{n_1}, ..., x_i^{n_i})M : x_{i+1},\ i = 0, ..., d-1.$$

On the other hand, we get by Lemma 3.4 of [C₁]

$$(x_1^{n_1}, ..., x_i^{n_i})M : x_{i+1} \subseteq \bigcup_{k=0}^{\infty}((x_1^{n_i}, ..., x_i^{n_i})M : \mathfrak{m}^k).$$

Therefore

$$(x_1^{n_1}, ..., x_i^{n_i})M : x_{i+1}^{n_{i+1}} = \bigcup_{k=0}^{\infty}((x_1^{n_1}, ..., x_i^{n_i})M : \mathfrak{m}^k).$$

It follows that $\{x_1^{n_1}, ..., x_d^{n_d}\}$ is a d-sequence by [T], Theorem 1.1.

(ii) We prove our statement by induction on d. It is obviously true for $d = 1$. Put $M' = M/x_dM$, $n' = \{n_1, ..., n_{d-1}\}$ and $x' = \{x_1, ..., x_{d-1}\}$. Then x' is a p-standard system of parameters of M'. Let $p(M) = k$. Since $\{x_1^{n_1}, ..., x_d^{n_d}\}$ is a d-sequence of M (by (i)), it is also a reducing sequence in the sense of Auslander and Buchsbaum in [A-B]. Thus by Corollary 4.8 of [A-B] we have

$$\begin{aligned} I_M(n;x) &= \ell((x_1^{n_1}, ..., x_{d-1}^{n_{d-1}})M : x_d^{n_d}/(x_1^{n_1}, ..., x_{d-1}^{n_{d-1}})M) \\ &= \ell((x_1^{n_1}, ..., x_{d-1}^{n_{d-1}})M : x_d/(x_1^{n_1}, ..., x_{d-1}^{n_{d-1}})M) \\ &= I_M(n, 1; x). \end{aligned}$$

On the other hand, since

$$e(x; M) = e(x'; M') - e(x_1, ..., x_{d-1}; 0_M : x_d) = e(x'; M') - e_{d-1},$$

it follows that

$$I_M(n;x) = I(n', 1; x) = I_{M'}(n'; x') + n_1...n_{d-1}e_{d-1}.$$

Thus, if $k = p(M) < d-1$ then $e_{d-1} = 0$ and therefore $p(M) = p(M')$. So by induction on d we get

$$\begin{aligned} I_M(n;x) &= I_{M'}(n';x') \\ &= \sum_{i=0}^{k} n_1...n_i e(x_1,...,x_i;(x_{i+2},...,x_{d-1})M' : x_{i+1}/(x_{i+2},...,x_{d-1})M') \\ &= \sum_{i=0}^{k} n_1...n_i e_i. \end{aligned}$$

Now, suppose that $k = d-1$ and $k' = p(M') \leq d-2$. Write briefly $I_M(x)$ instead of $I((1,\dots,1);x)$. Using Corollary 4.3 of [A-B] to the system x written in following order $x_d,\dots,x_1$ we have $I_M(x) = \sum_{i=0}^{d-1} e_i$. By the induction hypothesis we get $I_{M'}(x') = \sum_{i=0}^{k'} e_i$. Note that $e(x;M) = e(x';M') - e_{d-1}$. Therefore

$$0 = I_M(x) - I_{M'}(x') - e_{d-1} = \sum_{i=k'+1}^{d-2} e_i.$$

Hence $e_{k'+1} = \dots = e_{d-2} = 0$ since $e_i \geq 0$. It follows that $I_M(n;x) = \sum_{i=0}^{d-1} n_1...n_i e_i$, as required.

LEMMA 2.7. *Let $I \subseteq A$ be a p-standard ideal of M. Then for every prime ideal $\mathfrak{p} \not\supseteq I$, $M_{\mathfrak{p}}$ is a Cohen-Macaulay module and $\dim M_{\mathfrak{p}} + \dim(A/\mathfrak{p}) = d$.*

PROOF. Let $\mathfrak{p}$ be a prime ideal, $I \not\subseteq \mathfrak{p}$. Then $\mathfrak{a}(M) \not\subseteq \mathfrak{p}$. Choose a subsystem of parameters $\{x_1,...,x_j\}$ of M contained in $\mathfrak{p}$ such that j is maximal. Thus $\mathfrak{p} \in \mathrm{Ass}(M/(x_1,...,x_j)M)$. If $\dim(A/\mathfrak{p}) < d-j$ then there exists an $x_{j+1} \in \mathfrak{p} \setminus (\bigcup \mathfrak{q})$, where $\mathfrak{q}$ runs through all prime ideals of $\mathrm{Ass}(M/(x_1,...,x_j)M)$ with $\dim(A/\mathfrak{q}) = d-j$. Therefore $\{x_1,...,x_j,x_{j+1}\}$ is a subsystem of parameters of M contained in $\mathfrak{p}$. This conflicts with the maximality of j. Hence $\dim(A/\mathfrak{p}) = d-j$ and $\mathfrak{p}$ is a minimal prime ideal in $\mathrm{Ass}(M/(x_1,...,x_j)M)$. By [Sc$_1$], Satz 2.4.2, we have

$$\begin{aligned} (x_1,...,x_i)M_{\mathfrak{p}} : x_{i+1}A_{\mathfrak{p}} &= ((x_1,...,x_i)M : x_{i+1}) \otimes A_{\mathfrak{p}} \\ &\subseteq ((x_1,...,x_i)M : \mathfrak{a}(M)) \otimes A_{\mathfrak{p}} \\ &= (x_1,...,x_i)M_{\mathfrak{p}}, \end{aligned}$$

for $i = 0, ..., j-1$. Therefore $\{x_1, ..., x_j\}$ is a $M_{\mathfrak{p}}$-regular sequence. Since $j \geq \dim M_{\mathfrak{p}}$ it follows that $\operatorname{depth} M_{\mathfrak{p}} = \dim M_{\mathfrak{p}} = j$. Thus $M_{\mathfrak{p}}$ is a Cohen-Macaulay module and $\dim M_{\mathfrak{p}} + \dim(A/\mathfrak{p}) = d$ as required.

The following result shows a relation between p-standard ideals and p-standard systems of parameters.

THEOREM 2.8. *Let $x = \{x_1, ..., x_d\}$ be a p-standard system of parameters of M and $p(M) = k$. Then $I = (x_{k+1}^{n_{k+1}}, ..., x_d^{n_d})A$ is a p-standard ideal of M, for all $n_{k+1}, ..., n_d \geq d$.*

To prove Theorem 2.8 we need the following lemma.

LEMMA 2.9. *Let $x = \{x_1, ..., x_d\}$ be a p-standard system of parameters of M. Then*

$$x_j H^i_{\mathfrak{m}}(M/(x_1^{n_1}, ..., x_h^{n_h})M) = 0$$

for all $j = 1, \ldots, d$, $h + i < j$ and $n_1, ..., n_d > 0$.

PROOF. Since $x_1^{n_1}, ..., x_d^{n_d}$ is a d-sequence by Theorem 2.6, (i) , the lemma is proved if we can show the following statement:

Let $\{x_1^{n_1}, ..., x_d^{n_d}\}$ be a d-sequence for all $n_1, ..., n_d > 0$. Then

$$x_j H^i_{\mathfrak{m}}(M/(x_1^{n_1}, ..., x_h^{n_h})M) = 0$$

for all $j = 1, ..., d$ and $h + i < j$.

In fact, we have

$$x_j((x_1^{n_1}, ..., x_h^{n_h})M : \mathfrak{m}^n/(x_1^{n_1}, ..., x_h^{n_h})M) = 0,$$

for all $j > h$ and $n > 0$. Therefore

$$x_j H^0_{\mathfrak{m}}(M/(x_1^{n_1}, ..., x_h^{n_h})M) = 0,$$

for $j > h$. Suppose that

$$x_j H^i_{\mathfrak{m}}(M/(x_1^{n_1}, ..., x_h^{n_h})M) = 0,$$

for a fixed i and all $h < j - i$. We only need to prove that

$$x_j H^{i+1}_{\mathfrak{m}}(M/(x_1^{n_1}, ..., x_{h-1}^{n_{h-1}})M) = 0.$$

Set

$$F = M/(x_1^{n_1}, ..., x_h^{n_h})M \ , \ E = M/(x_1^{n_1}, ..., x_{h-1}^{n_{h-1}})M.$$

Since $\{x_h^{n_h}, ..., x_d^{n_d}\}$ is again a d-sequence on E (see [Hu]), $0_E : x_h^{n_h} = 0_E : x_h = \bigcup_{n>0}(0_E : \mathfrak{m}^n)$. So $H^i_{\mathfrak{m}}(E) \cong H^i_{\mathfrak{m}}(E/0_E : x_n)$ for all $i > 0$. From the exact sequence

$$0 \longrightarrow E/0_E : x_h \xrightarrow{x_h^{n_h}} E \longrightarrow F \longrightarrow 0,$$

we get the following long exact cohomology sequence

$$\cdots \longrightarrow H^i_{\mathfrak{m}}(E) \longrightarrow H^i_{\mathfrak{m}}(F) \longrightarrow H^{i+1}_{\mathfrak{m}}(E) \xrightarrow{x_h^{n_h}} H^{i+1}_{\mathfrak{m}}(E) \longrightarrow \cdots$$

This sequence yields epimorphisms

$$H^i_{\mathfrak{m}}(F) \longrightarrow 0_{H^{i+1}_{\mathfrak{m}}(E)} : x_h^{n_h} \ .$$

From the induction hypothesis we have $x_j H^i_{\mathfrak{m}}(F) = 0$. It follows that

$$x_j(0_{H^{i+1}_{\mathfrak{m}}(E)} : x_h^{n_h}) = 0$$

for all $n_h > 0$. Thus

$$x_j H^{i+1}_{\mathfrak{m}}(E) = x_j(\bigcup_{n_h>0}(0_{H^{i+1}_{\mathfrak{m}}(E)} : x_h^{n_h})) = 0.$$

The lemma is proved.

Proof of Theorem 2.8. Put $J = (x_{k+1}, ..., x_d)A$. It is enough to show that $JH^i_m(M) = 0$ for $i = 0, ..., d-1$. The case $k = -\infty$ is obvious. So, let $k \geq 0$. First, we prove that $JH^i_{\mathfrak{m}}(M) = 0$ for $i = k, ..., d-1$ by induction on d. It is clear for $d \leq 1$. Let $d > 1$. If $d = k+1$, the statement follows from the definition of p-standard system of parameters. Suppose now that $d-1 > k$. Put $M' = M/x_dM$. Then $p(M') = p(M)$ by the proof of Theorem 2.6, (ii). Therefore $JH^i_{\mathfrak{m}}(M') = 0$ for $i = k, ..., d-2$ by the induction hypothesis. Consider the exact sequences

$$0 \longrightarrow 0_M : x_d \longrightarrow M \longrightarrow M/0_M : x_d \longrightarrow 0 \ ,$$

$$0 \longrightarrow M/0_M : x_d \longrightarrow M \longrightarrow M' \longrightarrow 0 \ .$$

As $\mathfrak{a}(M)(0_M : x_d) = 0$ and $\dim A/\mathfrak{a}(M) = k$ by Theorem 2.2, (i) we deduce that $\mathfrak{a}(M)(0_M : x_d) = 0$; therefore $H^j_{\mathfrak{m}}(0_M : x_d) = 0$ for all $j > k$. By our choice of x we moreover have $x_d H^i_{\mathfrak{m}}(M) = 0$ for $i = 1, ..., d-1$. So, applying cohomology to the above exact sequences, we get the following exact sequence

$$0 \longrightarrow H^i_{\mathfrak{m}}(M) \longrightarrow H^i_{\mathfrak{m}}(M') \longrightarrow H^{i+1}_{\mathfrak{m}}(M) \longrightarrow 0$$

for $i = k, ..., d-2$. Therefore $JH^i_{\mathfrak{m}}(M) = 0$ for $i = k, ..., d-1$. On the other hand, using Lemma 2.9 with $h = 0$ and $i = 0, ..., k-1$ we thus get $JH^i_{\mathfrak{m}}(M) = 0$ for $i = 0, ..., d-1$ and the theorem is proved.

LEMMA 2.10. *Suppose that $x = \{x_1, ..., x_t\}$ is a subsystem of parameters of M satisfying the following conditions:*

(i) $(x_1, ..., x_t)A \subseteq \mathfrak{a}(M)$;

(ii) $t \leq d - p(M)$.

Then $\{x_1^{n_1}, ..., x_t^{n_t}\}$ is a subsystem of a p-standard basis on M for all $n_1, ..., n_t \geq d$.

PROOF. Since there always exists a p-standard system of parameters in $M/(x_1^{n_1}, \cdots, x_t^{n_t})M$, we have only to show that the system $\{x_1^{n_1}, ..., x_t^{n_t}\}$ satisfies the condition (*) for fixed integers $n_1, ..., n_t \geq d$, i.e. $x_1^{n_1} \in \mathfrak{a}(M)$ and $\overline{x}_i^{n_i} \in \mathfrak{a}(M/(x_1^{n_1}, ..., x_i^{n_i}))M$ for $i = 1, ..., t-1$. We denote by $r(M)$ the intersection of annihilators of A-modules $(y_1, ..., y_i)M : y_{i+1}/(y_1, ..., y_i)M$ for $i = 1, ..., d-1$ and for all systems of parameters $\{y_1, ..., y_d\}$ of M. Then by [Sc1], Satz 2.4.5 we have $\mathfrak{a}(M) \subseteq r(M)$ and $(r(M))^d \subseteq \mathfrak{a}(M)$. We set $M_i = M/(x_1^{n_1}, ..., x_i^{n_i})M$, $A_i = A/(x_1^{n_1}, ..., x_i^{n_i})A$ for $i = 1, ..., t$ and denote by $\overline{a}$ the image of $a \in A$ in A_i. Then, from the definition of $r(M)$ we deduce that $\overline{r(M)} \subseteq r(M_i)$. Therefore

$$\overline{x}_{i+1} \in \overline{\mathfrak{a}(M)} \subseteq \overline{r(M)} \subseteq r(M_i).$$

Since $(r(M_i))^d \subseteq \mathfrak{a}(M_i)$ we obtain that $\overline{x}_{i+1}^{n_{i+1}} \in \mathfrak{a}(M_i)$. As $x_1^{n_1} \in \mathfrak{a}(M)$ our claim is proved.

COROLLARY 2.11. *Let $I = (x_1, ..., x_t)A$ be a quasi-p-standard ideal of M. Then*

$$I^{n+m}M \bigcap (x_1^n, ..., x_t^n)M = I^m(x_1^n, ..., x_t^n)M$$

. *for all positive integers n, m.*

PROOF. Let $\{y, x\} = \{y_1, ..., y_k, x_1, ..., x_t\}$ be a p-standard system of parameters of M . By Krull's Intersection Theorem and Theorem 2.6, (i) we get

$$\begin{aligned}(x_1^{n_1}, ..., x_{i-1}^{n_{i-1}})M : x_i^{n_i}x_j^{n_j} &= \bigcap_{n\geq 1}((y_1^n, ..., y_k^n, x_1^{n_1}, ..., x_{i-1}^{n_{i-1}})M : x_i^{n_i}x_j^{n_j}) \\ &= \bigcap_{n\geq 1}((y_1^n, ..., y_k^n, x_1^{n_1}, ..., x_{i-1}^{n_{i-1}})M : x_j^{n_j}) \\ &= (\bigcap_{n\geq 1}(y_1^n, ..., y_k^n, x_1^{n_1}, ..., x_{i-1}^{n_{i-1}})M) : x_j^{n_j} \\ &= (x_1^{n_1}, ..., x_{i-1}^{n_{i-1}})M : x_j^{n_j} ,\end{aligned}$$

for $i = 1, ..., t$ and $t \geq j \geq i$. Thus $\{x_1^{n_1}, ..., x_t^{n_t}\}$ is a d-sequence for all $n_1, ..., n_t > 0$. Since $I \subseteq \mathfrak{a}(M)$,

$$\begin{aligned}(x_1^{n_1}, ..., \widehat{x_i^{n_i}}, ..., x_t^{n_t})M : x_i^{n_i} &\subseteq (x_1^{n_1}, ..., \widehat{x_i^{n_i}}, ..., x_t^{n_t})M : \mathfrak{a}(M) \\ &\subseteq (x_1^{n_1}, ..., \widehat{x_i^{n_i}}, ..., x_t^{n_t})M : x_i ,\end{aligned}$$

for $i = 1, ..., t$, by [Sc$_1$], Satz 2.4.2. Using the Krull's Intersection Theorem as above, we easily show that for every permutation $\alpha = (\alpha_1, ..., \alpha_d)$ of $\{1, ..., d\}$

$$(x_{\alpha_1}^{n_1}, ..., x_{\alpha_{i-1}}^{n_{i-1}})M : x_{\alpha_i}^{n_i} = (x_{\alpha_1}^{n_1}, ..., x_{\alpha_{i-1}}^{n_{i-1}})M : \mathfrak{a}(M)$$

for $i = 1, ..., t-1$. Therefore by [T], Theorem 1.1 it follows that $\{x_1, ..., x_t\}$ is an unconditioned strong d-sequence in the sense of Goto and Yamagishi [G-Y]. Thus the corollary follows by Theorem 2.6 of [G-Y].

3. Global p-standard ideals

In this section we shall extend some results of the previous section to the case that A is not local. Let M be an A-module of finite dimension. We denote by $\Omega(M)$ the set of all maximal ideals of A contained in Supp M.

DEFINITION 3.1. Let M be an A-module as above. Then:

(i) The *polynomial type* of M is defined by

$$p(M) = \max_{\mathfrak{m}\in\Omega(M)} p(M_{\mathfrak{m}}).$$

(ii) Let I be an ideal of A. I is called a *p-standard ideal* of M if the following conditions are satisfied:

1) $\dim(A/I) = p(M)$;

2) There exists a system of generators $x_1, ..., x_t$ of I with $t = d - p(M)$ such that $\{x_1/1, ..., x_t/1\}$ is a subsystem of a p-standard basis on $A_{\mathfrak{m}}$ for all $\mathfrak{m} \in \Omega(M) \bigcap NC(M)$, where $x_i/1$ is the image of x_i in $A_{\mathfrak{m}}$.

Without difficulty we can generalize Theorem 2.2 to the following.

LEMMA 3.2. *Suppose that M is equidimensional and that A admits a dualizing complex. Then*

$$p(M) = \dim NC(M).$$

PROOF. First, since A admits a dualizing complex, $NC(M)$ is a closed set in Supp M (see [Sh]) and $\dim A < \infty$ ([Ha$_1$], Ch. V, Cor. 7.2). Let $\mathfrak{p} \in NC(M)$ such that $\dim(A/\mathfrak{p}) = \dim NC(M)$ and let $\mathfrak{m} \in \Omega(M)$ such that $\dim(A/\mathfrak{p}) = \dim(A_{\mathfrak{m}}/\mathfrak{p}A_{\mathfrak{m}})$. Since $M_{\mathfrak{m}}$ admits a dualizing complex over $A_{\mathfrak{m}}$, $NC(M_{\mathfrak{m}})$ is again closed. As $\mathfrak{p}A_{\mathfrak{m}} \in NC(M_{\mathfrak{m}})$, we have $\dim NC(M_{\mathfrak{m}}) \geq \dim NC(M)$. The converse inequality $\dim NC(M_{\mathfrak{n}}) \leq \dim NC(M)$ is easily shown to hold true for any $n \in \Omega(M)$. So we get $\dim NC(M_{\mathfrak{m}}) = \dim NC(M)$. Therefore by Theorem 2.2,

$$p(M) \geq p(M_{\mathfrak{m}}) = \dim NC(M_{\mathfrak{m}}) = \dim NC(M).$$

Now let $p(M) = p(M_{\mathfrak{n}})$ for an $\mathfrak{n} \in \Omega(M)$. Then, again by Theorem 2.2 we get

$$p(M) = p(M_{\mathfrak{n}}) = \dim NC(M_{\mathfrak{n}}) \leq \dim NC(M).$$

Thus $p(M) = \dim NC(M)$ as required.

LEMMA 3.3. *Suppose that A admits a dualizing complex and $\dim M_{\mathfrak{m}} = \dim M$ for every $\mathfrak{m} \in \Omega(M)$. Let k be an integer. Then, $p(M) \leq k$ if and only if $M_{\mathfrak{p}}$ is a Cohen-Macaulay module and $\dim M_{\mathfrak{p}} + \dim(A/\mathfrak{p}) = \dim M$ for all $\mathfrak{p} \in$ Supp M, with $\dim(A/\mathfrak{p}) > k$.*

PROOF. The sufficient condition is immediate by Lemma 2.3. For the converse, let $\mathfrak{m}$ be a maximal ideal in Supp M where $p(M) = p(M_{\mathfrak{m}})$. By Lemma 2.3 we only have to show that whenever $\dim(A/\mathfrak{p})_{\mathfrak{m}} > k$ with $\mathfrak{p} \in$ Supp $M_{\mathfrak{m}}$, $M_{\mathfrak{p}}$ is

Cohen-Macaulay and $\dim M_{\mathfrak{p}} + \dim(A/\mathfrak{p})_{\mathfrak{m}} = \dim M_{\mathfrak{m}}$. But, $\dim(A/\mathfrak{p})_{\mathfrak{m}} > k$ implies $\dim(A/\mathfrak{p}) > k$. Therefore, by the assumption, $M_{\mathfrak{p}}$ is Cohen-Macaulay and $\dim M_{\mathfrak{p}} + \dim(A/\mathfrak{p}) = \dim M = \dim M_{\mathfrak{m}}$.

On the other hand, since $\mathfrak{p} \in \operatorname{Supp} M$, from the assumption we deduce that

$$\dim(A/\mathfrak{p}) = \dim(M/\mathfrak{p}M) = \dim(M/\mathfrak{p}M)_{\mathfrak{m}} = \dim(A/\mathfrak{p})_{\mathfrak{m}}.$$

Hence the lemma follows by Lemma 2.3.

COROLLARY 3.4. *Given A and M as in Lemma 3.3. Let $\mathfrak{p}$ be a prime ideal contained in $NC(M)$. Then*

$$p(M_{\mathfrak{p}}) \leq p(M) - \dim(A/\mathfrak{p}).$$

PROOF. Since A admits a dualizing complex, A is catenary. Hence the lemma follows immediately by Lemma 3.3.

The following result on the existence of a p-standard ideal is important for the further examination.

THEOREM 3.5. *Suppose that A admits a dualizing complex and that M is equidimensional with $\dim M = \dim M_{\mathfrak{m}}$ for all $\mathfrak{m} \in \Omega(M)$. Then there always exists a p-standard ideal of M.*

PROOF. We write $Y = NC(M)$ and $Z = \Omega(M) \bigcap Y$. Then, since A is catenary and equidimensional for all $\mathfrak{p} \notin Y$ and $\mathfrak{m} \in V(\mathfrak{p}) \cap Z$, $M_{\mathfrak{p}}$ is a Cohen-Macaulay module and $\dim M_{\mathfrak{p}} + \dim(A/\mathfrak{p})_{\mathfrak{m}} = \dim M_{\mathfrak{m}} = \dim M$. It follows by Satz 1 of [F2] that there exists an ideal $\mathfrak{a} \subseteq A$ such that $V(\mathfrak{a}) \subseteq NC(M)$ and $\mathfrak{a}H^i_{\mathfrak{m}}(M_{\mathfrak{m}}) = 0$ for all $\mathfrak{m} \in \Omega(M)$ and for $i = 0, ..., d-1$. Thus we can find an ideal $J \subseteq \mathfrak{a}$ such that $V(J) = NC(M)$. Then, by Lemma 3.2 $\dim(A/J) = p(M)$. Put $t = d - p(M)$. We can choose a sequence of elements $x_1, ..., x_t$ of A such that $I = (x_1, ..., x_t)A \subseteq J$ and $\dim(A/I + \operatorname{Ann}(M)) = p(M)$. Taking powers (of order at most d) of $x_1, ..., x_t$, if necessary, we can assume that $IA_{\mathfrak{m}} \subseteq \mathfrak{a}(M_{\mathfrak{m}})$ for all $\mathfrak{m} \in Z$. As $\dim M_{\mathfrak{m}} - p(M_{\mathfrak{m}}) \geq \dim M_{\mathfrak{m}} - p(M) = t$ it follows from Lemma 2.10 that $\{x_1^d, ..., x_t^d\}$ can be extended to a p-standard basis on $M_{\mathfrak{m}}$. So $(x_1^d, ..., x_t^d)A$ is a p-standard ideal of M as required.

4. The polynomial types of p-standard blowing up rings

In this section we shall examine the relationship between the polynomial types of an A-module M on the one hand and of its associated graded module and of its Rees module with respect to a p-standard ideal on the other hand.

First, we need some auxiliary results when M is a graded module. Let $A = \oplus_{n\geq 0} A_n$ be a Noetherian graded ring defined over A_0 which admits a dualizing complex. Let M be a Noetherian graded A-module of finite dimension. In the sequel, we shall show that in order to compute the polynomial types in the graded case we can only work with the homogeneous prime ideals.

LEMMA 4.1. *Let A be a graded ring and M a graded A-module as above. Suppose that $\dim M = \dim M_{\mathfrak{m}}$ for all maximal homogeneous ideals $\mathfrak{m} \in \Omega(M)$. Then $p(M) \leq k$ for an integer k if and only if $M_{\mathfrak{p}}$ is a Cohen-Macaulay module and $\dim M_{\mathfrak{p}} + \dim(A/\mathfrak{p}) = \dim M$, for each homogeneous prime ideal $\mathfrak{p} \in \operatorname{Supp} M$ with $\dim(A/\mathfrak{p}) > k$.*

PROOF. By Lemma 3.3 we only have to show that the condition in the lemma imposed on the homogeneous prime ideals implies the same condition for all prime ideals. So, let $\mathfrak{p}$ be an inhomogeneous prime ideal in $\operatorname{Supp} M$ with $\dim(A/\mathfrak{p}) > k$. Let S be the set of all homogeneous elements of $A \setminus \mathfrak{p}$ and denote by $H(\mathfrak{p})$ the biggest homogeneous prime ideal contained in $\mathfrak{p}$. As A is a finitely generated A_0-algebra, A also admits a dualizing complex (see [Sh], Theorem 3.9). Thus A is catenary; therefore $\dim(A/H(\mathfrak{p})) = \dim(A/\mathfrak{p})+1 > k$ by [H-I-O], 9.1. So we get

$$\dim M_{\mathfrak{p}} + \dim(A/\mathfrak{p}) = (\dim M_{H(\mathfrak{p})} + 1) + (\dim(A/H(\mathfrak{p})) - 1) = \dim M.$$

Moreover, $H(\mathfrak{p})A_S$ is the unique maximal homogeneous ideal of A_S and $M_{H(\mathfrak{p})}$ is a Cohen-Macaulay module by the hypothesis. Therefore $M_{\mathfrak{p}}$ is a Cohen-Macaulay module by virtue of [M-R], Theorem. This proves our claim.

COROLLARY 4.2. *Let A be a graded ring and M a graded A-module as above. Then*

$$p(M) = \max_{\mathfrak{m}} p(M_{\mathfrak{m}}),$$

where $\mathfrak{m}$ runs through all maximal homogeneous ideals in $\Omega(M)$.

COROLLARY 4.3. *Given A, M as above. Then $p(M_{\mathfrak{p}}) = p(M_{(\mathfrak{p})})$ for each homogeneous prime ideal $\mathfrak{p} \subseteq A$, where $M_{(\mathfrak{p})}$ is the module of elements of degree zero in the localized module $M_{\mathfrak{p}}$.*

PROOF. As the canonical homomorphism $A_{(\mathfrak{p})} \longrightarrow A_{\mathfrak{p}}$ is local flat by [H-I-O], 12.18 and $\dim(M_{\mathfrak{p}}/(\mathfrak{p})M_{\mathfrak{p}}) = 0$, where $(\mathfrak{p})A_{(\mathfrak{p})}$ is the unique maximal ideal of $A_{(\mathfrak{p})}$, by virtue of Lemma 4.1 and Corollary 4.2 we can generalize Theorem 5.1 of [C$_2$] to modules and our claim follows similarly as Corollary 5.5 of [C$_2$].

Next, we recall some notations.

Let A be a Noetherian ring and M a finitely generated A-module. Let I be an ideal of A. The associated graded module of M with respect to the ideal I is defined by

$$G_M(I) = \sum_{i=0}^{\infty} I^i M / I^{i+1} M.$$

The Rees module of M with respect to the ideal I is defined by

$$R_M(I) = \sum_{i=0}^{\infty} I^i M.$$

If the ideal I is fixed we also write G_M, R_M instead of $G_M(I)$, $R_M(I)$ respectively and G, R instead of $G_A(I)$, $R_A(I)$ respectively. We always consider G_M and R_M as R-modules and sometime we identify R_M with the submodule $M[IT]$ of $M[T]$, where T is an indeterminate. Now, we are ready to prove the main result of this section.

THEOREM 4.4. *Let A be a local ring admitting a dualizing complex and M a finitely generated A-module of dimension d. Let I be a quasi-p-standard ideal of M. Suppose that $p(G_M) \leq \dim(A/I)$. Then $p(R_M) \leq \dim(A/I)$.*

PROOF. Put $k = \dim(A/I)$. If $0_M : I \neq 0$ we set $\overline{M} = M/(0_M : I)$, $\overline{R}_M = R_{\overline{M}}(I)$ and $\overline{G}_M = G_{\overline{M}}(I)$. Note that I is generated by a part of a system of parameters which is a d-sequence of $\overline{M}$. Then, we have

$$\overline{R}_M = \bigoplus_{n \geq 0} ((I^n M + 0_M : I)/(0_M : I))$$

$$\cong \bigoplus_{n\geq 0}(I^nM/(I^nM \bigcap (0_M : I)))$$
$$= M/(0_M : I) \bigoplus_{n>0}(I^nM).$$

Therefore we get the following exact sequence

$$0 \longrightarrow 0_M : I \longrightarrow R_M \longrightarrow \overline{R}_M \longrightarrow 0,$$

where $0_M : I$ is considered as a graded R-module concentrated in degree 0. From this exact sequence and the inequality $\dim(0_M : I) \leq k$, we obtain isomorphisms $H^i_{\mathfrak{M}}(R_M) \cong H^i_{\mathfrak{M}}(\overline{R}_M)$ for $i = k+1, ..., d$ and a surjection $H^k_{\mathfrak{M}}(R_M) \longrightarrow H^k_{\mathfrak{M}}(\overline{R}_M)$, where $\mathfrak{M}$ is the unique maximal homogeneous ideal of R. Therefore, applying Theorem 2.2, (i), Corollary 4.2 and Korollar 2.2.4 of [Sc$_1$] we can easily verify that $p(R_M) \leq k$ when $p(\overline{R}_M) \leq k$. Note that in this case I is a quasi-p-standard ideal of $\overline{M}$ and $p(M) \leq p(\overline{M}) \leq k$. As above we can also show that $p(G_M) \leq p(\overline{G}_M) \leq k$. Hence, without loss of generality, we can assume that $0_M : I = 0$. Now, following Lemma 4.1 we have only to show that $(R_M)_P$ is a Cohen-Macaulay module and that $\dim(R/P) + \dim(R_M)_P = d+1$ for every homogeneous prime ideal P of R with $\dim(R/P) > k$. To prove this we set $\mathfrak{p} = P \cap A$.

Case 1: $I \subseteq \mathfrak{p}$. Identify R with the subring $A[IT]$ of $A[T]$, where T is an indeterminate. If $IT \subseteq P$ then $\dim(R/P) \leq \dim(A/I) = k$. This contradicts the assumption. Thus, there exists an element $x \in I$ such that $xT \notin P$. Therefore

$$(R_M/xR_M)_P = (R_M)_P/x(R_M)_P = (R_M/IR_M)_P \cong (G_M)_P.$$

Thus $(R_M)_P/x(R_M)_P$ is a Cohen-Macaulay module by Lemmas 2.7 and 4.1. So $(R_M)_P$ is a Cohen-Macaulay module since x is $(R_M)_P$-regular. As $I \subseteq \mathfrak{p}$, PG is obviously a prime ideal of G with $\dim(G/P) > k$. Moreover, according to Lemma 4.1 we get

$$\dim(R_M)_P + \dim(R/P) = (\dim(G_M)_P + 1) + \dim(G/PG) = \dim G_M + 1 = d+1.$$

Case 2: $I \not\subseteq \mathfrak{p}$. Then we have $R_{\mathfrak{p}} = A_{\mathfrak{p}}[T]$ and $(R_M)_{\mathfrak{p}} = M_{\mathfrak{p}}[T]$. Thus $(R_M)_{\mathfrak{p}}$ is a Cohen-Macaulay module by Lemma 2.7 and so is $(R_M)_P$ as a localization of $(R_M)_{\mathfrak{p}}$. Furthermore, since $I \not\subseteq \mathfrak{p}$ we can verify that either $P = (\mathfrak{p}, IT)R$ or $P = \tilde{\mathfrak{p}} = \mathfrak{p} \oplus_{i\geq 1} (\mathfrak{p} \cap I^i)T^i$. If $P = (\mathfrak{p}, IT)R$ then $\dim(R/P) = \dim(A/\mathfrak{p})$ and $(R_M)_P \cong (M_{\mathfrak{p}}[T])_Q$, where $Q = (\mathfrak{p}, T)A_{\mathfrak{p}}[T]$. It follows that $\dim(R_M)_P = \dim M_{\mathfrak{p}} + 1$. Thus

$$\dim(R_M)_P + \dim(R/P) = \dim(A/\mathfrak{p}) + \dim M_{\mathfrak{p}} + 1 = d + 1.$$

If $P = \tilde{\mathfrak{p}}$ then $(R_M)_P \cong (M_{\mathfrak{p}}[T])_Q$, where $Q = \mathfrak{p}A_{\mathfrak{p}}[T]$. Therefore $\dim(R_M)_P = \dim M_{\mathfrak{p}}$. On the other hand, we have $R/P \cong R_{A/\mathfrak{p}}(I)$. Hence $\dim(R/P) = \dim(A/\mathfrak{p}) + 1$. It follows that

$$\dim(R_M)_P + \dim(R/P) = \dim M_{\mathfrak{p}} + \dim(A/\mathfrak{p}) + 1 = d + 1.$$

The proof of our theorem is now complete.

Theorem 4.4 leads to the following corollary.

COROLLARY 4.5. *Given A, M as in Theorem 4.4. Let I be a p-standard ideal on M. Suppose that $p(G_M) \leq p(M)$. Then $p(R_M(I)) \leq p(M)$.*

REMARKS 4.6. (i) Theorem 3.5 guarantees the existence of a quasi-p-standard ideal I in Theorem 4.4. However, Theorem 4.4 is still true without the assumption that A admits a dualizing complex provided there exists a quasi-p-standard ideal I on M.

(ii) At the time we are writing this paper, we still do not know whether the assumption about the polynomial type $p(G_M) \leq \dim(A/I)$ in Theorem 4.4 is really needed, at least for the case that A satisfies Serre's condition (S_2). Therefore, we close this paper with the following question.

Open question. Let I be a quasi-p-standard ideal of M. Suppose that A satisfies the condition (S_2). Is it true that $p(R_M) \leq \dim(A/I)$?

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