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# HYPERSURFACES IN (DFC)-SPACES

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**Abstract.** It is shown that every hypersurface H in a (DFM)-space E is of uniform type. This means that there exist a continuous semi-norm $\rho$ on E and a hypersurface $\hat{H}$ in $E_\rho$, the Banach space associated to $\rho$, such that $H = \omega_\rho^{-1}(\hat{H})$, where $\omega_\rho : E \longrightarrow E_\rho$ is the canonical map.

## Introduction

In [2] Dineen, Meise and Vogt have proved that every polar set in a (DFN)-space is of uniform type. This means that there exist a continuous seminorm $\rho$ on E and a polar set $A_\rho$ in $E_\rho$, the Banach space associated to $\rho$, such that $\omega_\rho(A) \subseteq A_\rho$, where $\omega_\rho : E \longrightarrow E_\rho$ is the canonical map. The aim of the present paper is to prove the above result for hypersurfaces in (DFM)-spaces.

To obtain the main result (Theorem 2.1) we study in Section 1 the extension of hypersurfaces in a Riemann domain over Banach spaces to its envelope of holomorphy. In the finite dimensional case the problem was solved by Dloussky in [3]. By extending the method of Dloussky to the infinite dimensional case we prove that for every hypersurface $H$ in a Riemann domain $D$ over a Banach space with a Schauder basic there exists an analytic set $\hat{H}$ in $\hat{D}$, the envelope of holomorphy of $D$, such that

$$(D \setminus H)^\wedge \cong \hat{D} \setminus \hat{H}.$$

Using the fact that every plurisubharmonic function on a separable (DFC)-space is of uniform type [8], we show in Section 2 that every hypersurface in either a (DFC)-space, which has the approximation property, or in a (DFM)-space is of uniform type.

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## 1. Extending hypersurfaces

We recall that a hypersurface in a Riemann domain $D$ over a locally convex space $E$ is an analytic set which locally is the zero-set of a holomorphic function.

In this section we prove the following.

THEOREM 1.1. *Let $H$ be a hypersurface in a Riemann domain $D$ over a Banach space $B$ with a Schauder basic. Then there exists an analytic set $\hat{H}$ in $\hat{D}$ which is either empty or a hypersurface in $\hat{D}$ such that*

$$(D \setminus H)^{\wedge} \cong \hat{D} \setminus \hat{H}.$$

First, as in [3] we give the following.

DEFINITION 1.2. We say that $(D, H)$, where $D$ is a Riemann domain over a locally convex space $E$ and $H$ is a hypersurface in $D$, is maximal if for every Riemann domain $D'$ containing $D$ as an open subset such that $D' \setminus D \subseteq H'$ we have $D' = D$ provided $H' \cap D = H$, where $H'$ is a hypersurface in $D'$.

PROPOSITION 1.3. *If $(D, H)$ is maximal and $D \setminus H$ is a domain of holomorphy, then $D$ is a domain of holomophy.*

For the proof of Proposition 1.3 we need the following lemma :

LEMMA 1.4. *Let $p : X \longrightarrow Y$ be a local homeomorphism between two connected topological spaces and $H \subseteq Y$ be a closed subset of $Y$ such that $H$ has an empty interior and $Y \setminus H$ is connected. If $p$ has a section $\sigma$ on $Y \setminus H$, then $p$ is injective.*

PROOF. Since $X \setminus p^{-1}(H)$ is dense in $X$, it suffices to show that $p$ is injective on $X \setminus p^{-1}(H)$. By hypothesis there exists a section $\sigma_1$ of $p$ on a connected neighbourhood $V$ in $Y$ such that $V \cap H \neq \emptyset$. We need to prove that $\sigma_1 = \sigma$ on $V$. Take an arbitrary point $y \in V$. Since $V \setminus p^{-1}(H)$ is connected, there exists a curve $\gamma : [0,1] \longrightarrow X \setminus p^{-1}(H)$ such that $\gamma(0) = \sigma(y)$ and $\gamma(1) = \sigma_1(y)$. For each $t \in [0,1]$ we have the equality $\gamma(t) = \sigma p(\gamma(t))$. Hence $\sigma_1(y) = \gamma(1) = \sigma p(\gamma(1)) = \sigma p(\sigma_1(y)) = \sigma(y)$. It follows that $\sigma_1(y) = \sigma(y)$, which means $\sigma_1|_V = \sigma|_V$. The lemma is proved.

Now we return to the proof of Proposition 1.3.

For each $t \in [0,1]$ we put

$$M_t = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| \leq 1 \text{ and } z_2 \in [0,t] \text{ or } |z_1| = 1 \text{ and } z_2 \in [t,1]\}.$$

Let $\varphi$ be an arbitrary holomorphic map from a neighbourhood $U$ of $M_0$ to $D$ such that $p\varphi : U \longrightarrow p\varphi(U)$ is a homeomorphism and $p\varphi(U)$ is contained in a subspace $B$ of $E$ of dimension 2. By [4] it suffices to show that $\varphi$ can be extended holomorphically to a neighbourhood of $M_1$, where $p : D \longrightarrow E$ is a locally homeomorphic map defining $D$ as a Riemann domain over $E$. By a result of Dloussky [3] we can find an analytic set $\tilde{H} \in \tilde{U}$ such that

$$(U \setminus \varphi^{-1}(H))^{\wedge} \cong \hat{U} \setminus \tilde{H}.$$

Since $\varphi(U \setminus \varphi^{-1}(H)) \subseteq D \setminus H$ and, by hypothesis, $\varphi$ can be extended to a holomorphic map $\hat{\varphi}$ on $\hat{U} \setminus \hat{H}$ with values in $D \setminus H$. Write $E = B \oplus B^{\perp}$. Replacing $U$ by a smaller neighbourhood of $M_0$ we can assume that there exists a neighbourhood $V$ of $0 \in B$ such that $p$ has a holomorphic section $\delta : p\varphi(U) \times V \longrightarrow D$. Put

$$\tilde{\varphi} = \delta \cdot (p\varphi \times id) : U \times V \longrightarrow D,$$

and assume that $Z$ is the domain of existence of $\tilde{\varphi}$ over $\hat{U} \times V$. Then $\tilde{\varphi}$ has a holomorphic extension $\hat{\tilde{\varphi}}$ on $Z$ with value in $D$ and

$$((\hat{U} \setminus \tilde{H}) \cup U) \times V \subseteq Z.$$

We have the following commutative diagram

$$\begin{array}{ccc} ((\hat{U} \setminus \tilde{H}) \cup U) \times V & \longrightarrow & Z \\ \downarrow & \swarrow_{\pi} & \\ \hat{U} \times V & & \end{array}$$

where $\pi : Z \longrightarrow \hat{U} \times V$ defines $Z$ as a Riemann domain over $\hat{U} \times V$. Since $\pi$ is injective (by Lemma 1.4), $Z$ is an open subset in $\hat{U} \times V$. Now on $(\hat{U} \times V) \sqcup D$ we can define an equivalent relation as follows. Let $x \in \hat{U} \times V$ and $y \in D$. We write $x \sim y$ if $\hat{\tilde{\varphi}} = y$. Put $'D = (\hat{U} \times V) \sqcup D/\sim$ and

$$'p = \begin{cases} p \text{ on } D, \\ \text{the holomorphic extension of } p\varphi \times id \text{ on } \hat{U} \times V. \end{cases}$$

It follows that $('D, 'p)$ is a Riemann domain over $E$ and $\hat{\tilde{\varphi}}$ can be extended holomorphically to a map from $\hat{U} \times V$ into $'D$. Moreover $'H = (\hat{H} \times V) \sqcup H/\sim$ is an analytic set in $'D$ with $D \cap {}'H = H$ and $'D \setminus D \subseteq' H$. By the maximality of $D$ we have $D = {}'D$ and hence $\hat{\tilde{\varphi}}$ is extended holomorphically on $\hat{U} \times V$. It implies that $Z = \hat{U} \times V$. Thus, $\varphi$ can be extended to a holomorphic map on a neighbourhood $\hat{U}$ of $M_1$. The proposition is proved.

As in [3] we give the following.

DEFINITION 1.5. Let $(D,p)$ be a Riemann domain over a Banach space $E$. A boundary point of $(D,p)$ is a basic of a filter $r$ consists of connected open set in $D$ such that

(i) $r$ has no limit point in $D$,

(ii) $p(r)$ converges to a point $x \in E$,

(iii) For every open connected neighbourhood $U(x)$ of $x$, $r$ contains one and only one connected component of $p^{-1}(U(x))$ and every element of $r$ has such a form.

Let $\check{D} = D \cup \partial D$, where $\partial D$ denotes the set of boundary points of $(D,p)$. If $q : (D_1, p_1) \longrightarrow (D_2, p_2)$ is a morphism between Riemann domains over $E$, then it can be extended to a continuous map $\check{q} : \check{D}_1 \longrightarrow \check{D}_2$. Now we assume that $r \in \partial D$. We say that $\partial D$ is a local hypersurface at $r$ if there exists a neighbourhood $U(r)$ of $r$ in $\check{D}$ such that $\check{p} : \check{U}(r) \longrightarrow \check{p}(U(r))$ is a homeomorphism, $\check{p}(U(r))$ is an open set in $E$ and $\check{p}(\partial D \cap U(r))$ is a hypersurface in $\check{p}(U(r))$.

PROPOSITION 1.6. *Let $(D,p)$ be a Riemann domain over a Banach space $E$ and $H$ a hypersurface in $D$ which is singular for a holomorphic function $f$ on $D \setminus H$. Then there exists a hypersurface $\hat{H}$ in $\hat{D}$ such that $\lambda^{-1}(\hat{H}) = H$ and*

$$(D \setminus H)^\wedge = \hat{D} \setminus \hat{H},$$

*where $\lambda : D \longrightarrow \hat{D}$ is the canonical map.*

PROOF. Let $\hat{\lambda} : (D\backslash H)^\wedge \longrightarrow \hat{D}$ be the holomorphic map such that the following diagram is commutative

$$\begin{array}{ccc} D \setminus H & \xrightarrow{\lambda_H} & (D \setminus H)^\wedge \\ \lambda \big\downarrow & \swarrow \hat{\lambda} & \\ \hat{D} & & \end{array}$$

Construct $({}'D, {}'\lambda)$, where ${}'D = (D \setminus H)^\wedge \cup Z$ and $Z$ denotes the set of boundary points of $(D \setminus H)^\wedge$ and as in [3], $Z$ is a local hypersurface. Put

$${}'\lambda = \overset{\vee}{\hat{\lambda}}, \quad {}'H = Z.$$

As in [3], ${}'H$ is a hypersurface of ${}'D$ such that ${}'H \cap D = H$ and $({}'D, {}'H)$ is maximal. By Proposition 1.3 it implies that ${}'D$ is a domain of holomophy. Hence ${}'D \cong \hat{D}$. Since ${}'D \setminus {}' = (D \setminus H)^\wedge$ we obtain $(D \setminus H)^\wedge = \hat{D} \setminus \hat{H}$, where $\hat{H} = {}'\lambda({}'H)$. The proposition is proved.

Now based on Proposition 1.6 and ideas of Dloussky [3] we prove Theorem 1.1. We can assume that $H$ is irreducible. Let ${}'H$ be the set of poins $h \in H$ such that for every holomorphic function $f$ on $D \setminus H$ there exists an open neighbourhood $V_f$ of $h$ to which $f$ can be extended holomorphically. Then $H \setminus {}'H$ is a hypersurface in $D$ which is singular for a holomorphic function on $D \setminus (H \setminus {}'H)$. Indeed, we have $(D \setminus (H \setminus {}'H))^\wedge$ is the domain of existence of a holomorphic function $f$. Let $x \in (H \setminus {}'H)$ be an arbitrary point. Assume that there exists a neighbourhood $U_x$ of $x$ such that $f$ is holomorphic on $U_x$. Then $U_x \subset (D \setminus (H \setminus {}'H))^\wedge$. By the definition of ${}'H$ it follows that if $g$ is holomorphic on $D \setminus H$, then $g$ is holomorphic on $(D \setminus (H \setminus {}'H))$. Hence, $g$ can be extended holomorphically to $(D \setminus (H \setminus {}'H))^\wedge$. Therefore, $g$ is holomorphic on $U_x$. This is impossible because for every neighbourhood $U_x, x \in (H \setminus {}'H)$, there always exists a holomorphic function $g$ on $D \setminus H$ such that $g$ can not be extended holomorphically to $U_x$. By Proposition 1.6 there exists a hypersurface $\hat{H}$ in $\hat{D}$ such that

$$(D \setminus (H \setminus {}'H))^\wedge \cong \hat{D} \setminus \hat{H}.$$

By the definition of $'H$ it follows that

$$(D\setminus(H\setminus 'H))^\wedge \cong (D\setminus H)^\wedge.$$

The theorem is completely proved.

## 2. Hypersurfaces in (DFC)-spaces

Using Theorem 1.1 and [8] we shall prove the following

THEOREM 2.1. *Let $E$ be a separable (DFC)-space and $H$ a hypersurface in $E$. Then $H$ is of uniform type if one of the following two conditions holds*

*(i) $E$ has the approximation property and $E$ has a fundamental system of continuous semi-norm$\{\rho\}$such that $E_\rho$ has the approximation property.*

*(ii) $E$ is a (DFM)-space.*

We need the following five lemmas.

LEMMA 2.2. *Let $S$ be a continuous linear map from a Banach space $A$ onto a Banach space $B$. Let $Z_0$ be a locally closed submanifold of codimension 1 in an open subset $D$ of $B$ such that $Cl[S^{-1}(Z_0)]_{S^{-1}}(D)$ is an analytic set of codimension 1 in $S^{-1}(D)$. Then $Cl[Z_0]_D$ is an analytic set of finite dimension in $D$.*

PROOF. (i) First we show that $S(W) = Cl[Z_0]_D$, where $W = Cl[S^{-1}(Z_0)]_{S^{-1}}(D)$. Note that $Cl[S(W)]_D = Cl[Z_0]_D$. It suffices to check that $S(W)$ is closed in $D$. Indeed, let $y_n \in S(W), y_n \longrightarrow y$. Since $S(W) \subseteq Cl[Z_0]_D$ we can assume that $y_n \in Z_0$ for $n \geq 1$. By hypothesis, $S$ is open, there exists a sequence $x_n \in S^{-1}(Z_0) \subseteq W$ such that $S(x_n) = y_n$ for $n \geq 1$ and $x_n \longrightarrow x \in W$. Since $S(x) = y$, it follows that $x \in S^{-1}(D)$ and hence $y \in S(W)$.

(ii) Given $y_0 \in Cl[Z_0]_D$. We may assume that $y_0 = 0$. Since $W$ is a hypersurface in $S^{-1}(D)$ we can find $e \in A, e \neq 0$ such that for a neighbourhood $U$ of $0 \in A$ the map $\sigma_1 : W \cap U \longrightarrow V := \sigma_1(U)$ is branched cover, where $\sigma_1 : A \longrightarrow A/\mathbb{C}e$ is the canonical projection. Without loss of generality we may assume that $B$ is a quotient space of $A$ and hence $B/\mathbb{C}\tilde{e}$ is also a quotient space of $B$, where $\tilde{e} = S(e)$. Moreover we can assume that $U$ is the open unit ball in $A$. Then $V, S(U)$ and $\tilde{S}(V)$ are open unit balles in $A/\mathbb{C}e, B$ and $B/\mathbb{C}\tilde{e}$

respectively, where $\tilde{S} : A/Ce \longrightarrow B/C\tilde{e}$ is the canonical map induced by $S$. Consider the commutative diagram

$$\begin{array}{ccc} U \cap V & \xrightarrow{S} & S(U \cap W) \\ \sigma_1 \downarrow & & \sigma_2 \downarrow \\ V & \xrightarrow{\tilde{S}} & \tilde{S}(V) \end{array}$$

with $\sigma_2 : B \longrightarrow B/C\tilde{e}$ is the canonical projection.

Assume that $\{y_n\} \subset \tilde{S}(V)$ and $y_n \longrightarrow y \in \tilde{S}(V)$. Choose $\delta > 0$ such that $\|y_n\| < 1 - \delta$ for $n \geq 1$. Let $\{y_{n_k}\} \subset \{y_n\}$ such that

$$\sum_{k \geq 1} \|y_{n_{k+1}} - y_{n_k}\| < \delta/2,$$

and $\epsilon_k \downarrow 0$ such that

$$\sum_{k \geq 1} (1 + \epsilon)\|y_{n_{k+1}} - y_{n_k}\| < \delta/2.$$

For each $k \geq 1$ take $z_k \in V$ such that $\tilde{S}(z_k) = y_{n_{k+1}} - y_{n_k}$ and $\|z_k\| < (1 + \epsilon_k)\|y_{n_{k+1}} - y_{n_k}\|$. Then

$$\sum_{k \geq 1} \|z_k\| < \delta/2.$$

Hence for

$$z = \sum_{k \geq 1} z_k,$$

we have $\tilde{S}(z) = y - y_{n_1}$.

Choose $z' \in V$ such that $\tilde{S}(z') = y_{n_1}$ and $\|z'\| < (1 + \delta/2)\|y_{n_1}\|$. Then $\tilde{S}(z + z') = y$ and $\|z + z'\| < \|z\| + \|z'\| < \delta/2 + 1 - \delta + \delta/2 = 1$. Thus the sequence $\{u_k\}$ with $u_k = z_1 + \cdots + z_k + z'$ is contained in $V$ and $u_k \longrightarrow u$ for which $\tilde{S}(u) = y$ and $\tilde{S}(u_k) = y_{n_{k+1}}$ for $k \geq 1$. By the properity of $\sigma_1$ this implies that $\sigma_2$ is also proper. Since $\sigma_1$ is finite and $S : U \cap W \longrightarrow S(U \cap W)$ is surjective, we get that

$$\infty > \sup\{\text{card } \sigma_1^{-1}(z) : z \in V\} = \sup\{\text{card } \sigma_2^{-1}(y) : y \in \tilde{S}(V)\}.$$

Hence $\sigma_2 : S(U \cap W) \longrightarrow \tilde{S}(V)$ is a branched covering map.

(iii) As in (ii) for every sequence $\{y_N\} \subset S(U \cap W)$, converging to $y \in S(U)$, there exists a subsequence $\{y_{n_k}\}$ such that $y_{n_k} = S(u_k), u_k \in U \cap W$ for $k \geq 1$, and $u_k \longrightarrow u \in U \cap W$. Then $\sigma_2(y_{n_k}) \longrightarrow \tilde{S}\sigma_1(u) \in \tilde{S}(V)$. Hence, by the properity of $\sigma_2$, we have $y \in S(U \cap W)$.

(iv) From (ii) and (iii) it implies that $S(U \cap W)$ is an analytic set in $S(U)$ of dimension 1. Now from the relation $U \cap S^{-1}(Z_0) \subseteq R(U \cap W)$, where $R(U \cap W)$ is the regular locus of $U \cap W$, we have

$$S(U) \cap Z_0 \subseteq S(R(U \cap W)).$$

This relation yields

$$S(U) \cap Cl(Z_0)_D = S(U \cap W).$$

The lemma is proved.

LEMMA 2.3. *Every separable Banach space is the image of the space $l^1$ under a continuous linear map.*

PROOF. Given $B$ a separable Banach space. Take a continuous map $S$ from a space $l^1(T)$ onto $B$, where $T$ is some index set. By the open mapping theorem there exists $C \geq 0$ such that for every $y \in B$ there exists $x \in l^1(T)$ for which $S(x) = y$ and $\|x\| < C\|y_n\|$.

Let $\{y_n\}$ be a dense sequence in $B$. For each $n \geq 1$ choose $x_n \in l^1(T)$ with $S(x_n) = y_n$ and $\|x_n\| < C\|y_n\|$. Put

$$T_0 = \bigcup_{n \geq 1} \{t \in T : x_{n,t} \neq 0\},$$

where $x_n = \{x_{n,t} : t \in T\}$ for $n \geq 1$. Then $T_0$ is countable and hence $l^1(T_0) \cong l^1$. Consider $S_0 = S|_{l^1(T_0)} : l^1(T_0) \longrightarrow B$. It remains to check that Im $S_0 = B$. Again by the open mapping theorem it suffices to show that $Cl\{S_0(U)\}$ is a neighbourhood of $0 \in B$, where $U = \{x \in l^1(T_0) : \|x\| < 1\}$. Given $y \in B$ with $\|y\| < 1/C$. For each $\delta > 0$ take $y_n$ with $\|y_n - y\| < \delta$. Then for $x_n$ we have $S(x_n) = y_n, \|x_n\| < 1$ and $\|S(x_n) - y\| < \delta$.

LEMMA 2.4. *Every separable Frechet space is a subspace of a Frechet space with the approximation property.*

PROOF. We write $E = \text{limproj } E_n$, where $E$ is a given separable Frechet space and $E_n$ are separable Banach spaces. For each $n \geq 1$ there exists a Banach space $F_n$ with a Schauder basis containing $E_n$ as a subspace. Then $F := \prod_{n\geq 1} F_n$ is a Frechet space having the approximation containing $E$ as a subspace.

LEMMA 2.5. *Let $G$ be an open subset in a normed space $E$ and $\sigma$ a holomorphic function on $G$. Then $\sigma$ can be extended to a holomorphic function $\hat{\sigma}$ on a neighbourhood $\hat{G}$ of $G$ in $\hat{E}$, the completion of $E$, such that $Z(\sigma)$ is dense in $Z(\hat{\sigma})$, where $Z(\sigma)$ and $Z(\hat{\sigma})$ are zero-sets of $\sigma$ and $\hat{\sigma}$, respectively.*

PROOF. Obviously $\sigma$ can be extended to a holomorphic function $\hat{\sigma}$ on a neighbourhood $\hat{G}$ of $G$ in $\hat{E}$. It remains to show that $Z(\sigma)$ is dense in $Z(\hat{\sigma})$. Since the regular locus $R(Z(\hat{\sigma}))$ is dense in $Z(\hat{\sigma})$ [7], it suffices to show that $Z(\sigma)$ is dense in $R(Z(\hat{\sigma}))$. Given $z_0 \in R(Z(\hat{\sigma}))$ with $x' = \hat{\sigma}'(z_0) \neq 0$. Choose $e \in E$ with $x'(e) = 1$ and write $E = \mathbb{C}e \oplus \text{Ker } x'_0$, with $x'_0 = x'|_E$. Then $\hat{E} = \mathbb{C}e \oplus \text{Ker } x'$. Define a biholomorphism $\theta$ from a neighbourhood $U$ of $z_0$ onto a neighbourhood $\Delta \times V$, where $\Delta$ is the open unit disc in $\mathbb{C}$, as follow :

$$\theta(te, u) = (\hat{\sigma}(te + u), u) \text{ for } u \in V; te + u \in U.$$

We may assume that $U \cap R(Z(\hat{\sigma})) = U \cap Z(\hat{\sigma})$. We have

$$\theta(Z(\hat{\sigma}) \cap U) = 0 \times V \text{and } \theta(Z(\sigma) \cap U) \subseteq 0 \times V_0,$$

where $V_0 = V \cap \text{Ker } x'_0$. Let $v_0 \in V_0$ and $(t_0 e, u_0) \in Z(\hat{\sigma}) \cap U$ such that $(0, v_0) = \theta(t_0 e, u_0) = (\hat{\sigma}(t_0 e, u_0), u_0) = (0, u_0)$. This means that $(t_0 e, u_0) \in Z(\sigma) \cap U$. Hence

$$\theta(Z(\sigma) \cap U) = 0 \times V_0.$$

Since $0 \times V_0$ is dense in $0 \times V$, it follows that $Z(\sigma) \cap U$ is dense in $Z(\hat{\sigma}) \cap U$. The lemma is proved.

LEMMA 2.6. *Let $H$ be a hypersurface in a normed space $E$. Then $H$ can be extended to a hypersurface in a neighbourhood of $E$ in $\hat{E}$.*

PROOF. Cover $E$ by open subset $\{U_i : i \in I\}$ such that on each $U_i$ we have a holomorphic function $h_i$ for which $H \cap U_i = Z(h_i)$. By Lemma 2.5 for each $i \in I$ there exist a neighbourhood $V_i$ of $U_i$ in $\hat{E}$ and a holomorphic function $\hat{h}_i$ on $V_i$ such that $\hat{h}_i|_{U_i} = h_i$ and $Z(h_i)$ is dense in $Z(\hat{h}_i)$. Let $D = \cup\{V_i : i \in I\}$ and let $\{W_i\}$ be a locally finite open cover of $D$ such that $Cl(W_i)_D \subseteq V_i$ for $i \in I$. Put

$$\tilde{H} = \cup\{Z(\hat{h}_i|_{W_i}) : i \in I\}.$$

We check that $\tilde{H}$ is a hypersurface in $D$. Given $z_0 \in D$. Choose a neighbourhood $W$ of $z_0$ such that $W \cap W_i = \emptyset$ for $i \notin I(z_0)$, where $I(z_0)$ is a finite subset of $I$. Then

$$\begin{aligned}\tilde{H} \cap W &= \cup\{Z(\hat{h}_i)|_{W_i}) \cap W\} = \cup\{Z(\hat{h}_i|_{W_i}) \cap W_j \cap W : i, j \in I\} \\ &= \cup\{Z(\hat{h}_i|_{W_i}) \cap W \cap W_j : i, j \in I(z_0)\} \\ &= \cup\{Z(\hat{h}_i|_{W_i}) : i \in I(z_0)\} \cap (\cup\{W_j : j \in I(z_0)\}) \cap W.\end{aligned}$$

Hence it suffices to show that $U\{Z(\hat{h}_i|_{W_i}) : i \in I(z_0)\}$ is closed in $\cup\{W_i : i \in I(z_0)\}$. For simplicity we assume that $I(z_0) = (i_1, i_2)$. Let $\{z_n\} \subset Z(\hat{h}_{i_1}|_{W_{i_1}}) \cup Z(\hat{h}_{i_2}|_{W_{i_2}})$ with $z_n \longrightarrow z \in W_{i_1} \cup W_{i_2}$. By Lemma 3.5 without loss of generality we may assume that $\{z_n\} \subset Z(\hat{h}_{i_1}|_{W_{i_1}})$. If $z \in W_{i_1}$, then $\hat{h}_{i_1}(z) = 0$, because $\hat{h}_1$ is holomorphic on $V_{i_1} \supseteq Cl(W_{i_1})_D$. Consider the case $z \in W_{i_2}$. Then with $n > N, z_n \in W_{i_1} \cap W_{i_2}$. Hence $\hat{h}_{i_2}(z) = 0$. The lemma is proved.

PROOF OF THEOREM 2.1. (i) Since $E$ has the approximation, $H$ is singular for a holomorphic function on $E \setminus H$ [5]. From the Lindelöfness of $E$ and $E \setminus H$ [5] there exist two open covers $\{z_j + U_j\}$ and $\{x_j + V_j\}$ of $E$ and $E \setminus H$ respectively such that

$$H \cap (z_j + U_j) = Z(h_i) \quad \text{for} \quad j \geq 1,$$

where $h_j$ are bounded holomorphic functions on $z_j + U_j$, $f(x_j + V_j)$ are bounded and $U_j, V_j$ are balanced convex neighbourhoods of $0 \in E$. Choose a sequence $\mu_j \to \infty$ such that

$$V = \bigcap_{j \geq 1} \mu_j (U_j \cap V_j)$$

is a neighbourhood of $0 \in E$. Then $\omega_V(z_j + U_j)$ and $\omega_V(x_j + V_j)$ are open in $E/\mathrm{Ker}\rho_V$, where we write $\omega_V = \omega_{\rho_V}$.

By Lemma 2.6 $\omega_V(H)$ can be extended to a hypersurface $\tilde{H}$ in a neighbourhood $D$ of $E/\mathrm{Ker}\ \rho_V$ in $E_V = E_{\rho_V}$. On the other hand, the function $f$ can be considered as a holomorphic function on $(E/\mathrm{Ker}\rho_V)\setminus\omega_V(H)$. Given $z_0 \in R(\tilde{H})$ with $x' = h'(z_0) \neq 0$, where $h$ is a holomorphic function on a neighbourhood $U(z_0)$ of $z_0$ in $E_V$ such that

$$U(z_0) \cap \tilde{H} = U(z_0) \cap Z(h) = U(z_0) \cap R(\tilde{H}).$$

Take $e \in E/\mathrm{Ker}\rho_V$ with $x'(e) = 1$ and write

$$E/\mathrm{Ker}\rho_V = \mathbb{C}e \oplus \mathrm{Ker}\ x'_0, \text{ with } x'_0 = x'|_{(E/\mathrm{Ker}\rho_V)}.$$

Then

$$E_V = \mathbb{C}e \oplus \mathrm{Ker}\ x'.$$

Define a biholomorphism $\theta_{z_0}$ from $U(z_0)$ onto a neighbourhood $\Delta \times W(z_0)$ of $0 \in E_V$, where $\Delta$ is the open disc in $\mathbb{C}$, by the formular

$$\theta_{z_0}(te, u) = (h(te+u), u) \text{ for } u \in W(z_0) \text{ and } te + u \in U(z_0),$$

Then as in the proof of Lemma 2.5, it follows that

$$\theta_{z_0}(Z(h) \cap U(z_0)) = 0 \times W(z_0)_0,$$

where $W(z_0)_0 = W(z_0) \cap E/\mathrm{Ker}\ \rho_V$. Since $f\theta_{z_0}^{-1}$ is holomorphic on $\Delta^* \times W(z_0)_0$, where $\Delta^* = \Delta \setminus 0$, it can be extended to a holomorphic function $g^{z_0}$ on $W_0^{z_0}$, where $W_0^{z_0}$ is a neighbourhood of $W(z_0)_0$ in Ker $x'$. This means also that $f|_{U(z_0)\cap(E/\mathrm{Ker}\rho_V\setminus Z(h))}$ can be extended to a holomorphic function $f^{z_0}$ on a neighbourhood $G^{z_0}$ of $U(z_0)\cap(E/\mathrm{Ker}\rho_V\setminus Z(h))$ in $E_V$. By the identity principle the family $\{f^{z_0} : z_0 \in R(\tilde{H})\}$ defines a holomorphic extension $g$ of $f$ to a neighbourhood $G_0$ of $E/\mathrm{Ker}\rho_V \setminus \tilde{H}$ in $E_V$. Put

$$G = \mathrm{Int}\ (G_0 \cup \tilde{H}).$$

Since

$$G \cap \theta_{z_0}^{-1}(\Delta e \times W_0^{z_0}) \subset G_0 \cap \tilde{H} \text{ for } z_0 \in R(\tilde{H}),$$

it follows that $G$ is a neighbourhood of $E/\mathrm{Ker}\rho_V \setminus S(\tilde{H})$, where $S(\tilde{H})$ is the singular locus of $\tilde{H}$. By hypothesis, we may assume that $E_V$ has the approximation property. Hence, by Theorem 1.1, there exists a hypersurface $\hat{\tilde{H}}$ in $\hat{G}$ such that

$$(G \setminus \tilde{H})^\wedge \cong \hat{G} \setminus \hat{\tilde{H}}.$$

Since codim $S(\tilde{H}) \geq 2$, we have $\hat{G} \supset E/\mathrm{Ker}\rho_V$. It is easy to see that $\tilde{H}$ is singular for $g$. This yields that

$$\hat{\tilde{H}} \cap E/\mathrm{Ker}\rho_V = \tilde{H}.$$

Consider the plurisubharmonic function $\varphi$ on $\hat{G}$ given by

$$\varphi(z) = -\log d(z, \partial\hat{G}).$$

By [8] we can find a plurisubharmonic function $\psi$ on $E_\rho$, with $\rho \geq \rho_V$, and $\rho$ is a continuous semi-norm on $E$, such that

$$\psi\omega_\rho = \psi\omega_V.$$

It remains to check that $\mathrm{Im}\,\omega_{\rho\rho_V} \subset \hat{G}$, where $\omega_{\rho\rho_V} : E_\rho \longrightarrow E_V$ is the canonical map. In the converse case there would exist $z \in E_\rho$ with $\omega_{\rho\rho_V}(z) \in \partial\hat{G}$. Choose a sequence $\{z_n\} \subset E/\mathrm{Ker}\rho$ which is convergent to $z$. Then

$$\infty = \lim \varphi_{\omega_{\rho\rho_V}}(z_n) = \lim \psi(z_n) \leq \psi(z) < \infty.$$

This is impossible. Hence $\mathrm{Im}\,\omega_{\rho\rho_V} \subset \hat{G}$. Then $\hat{H} = \omega_{\rho\rho_V}^{-1}(\hat{\tilde{H}})$ is a required hypersurface and (i) is proved.

(ii) Now assume that $E$ is a (DFM)-space.

(a) Let $F$ be a Frechet space given by Lemma 2.4 for $P = E'$, the dual space of $E$. Consider the restriction map

$$R : F'_c \longrightarrow P'_c = E.$$

Since $F'_c$ is $B$-complete and $E$ is a barrelled space, the open mapping theorem implies that $R$ is open. As in (i) we can finds a continuous semi-norm $\rho_1$ on $E$ and a hypersurface $H_1$ in a neighbourhood $G_1$ of $E/\mathrm{Ker}\rho_1$ in $E_{\rho_1}$ such that $H = \omega_{\rho_1}^{-1}(H_1)$.

(b) Let

$$\tilde{H} = R^{-1}(H).$$

Since $F'_c$ has the approximation property, we can find, as in (i), a continuous semi-norm $\beta \geq \rho_1 R$ on $F'_c$, a neighbourhood $\tilde{G}_1$ of $F'_c/\mathrm{Ker}\beta$, a hypersurface $\tilde{H}_1$ in $\tilde{G}_1$ and a holomorphic function $\tilde{g}_1$ on $\tilde{G}_1 \setminus \tilde{H}_1$ such that

$$\tilde{g}_1 \omega_\beta = f,$$

where $f$ is a holomorphic function on $F'_c \setminus \tilde{H}$ such that $\tilde{H}$ is singular for $f$.

(c) Choose a continuous linear map $S$ from $l^1$ onto $F'_{c\beta}$ and consider

$$M = S^{-1}(\tilde{H}_1),\ D = S^{-1}(\tilde{G}_1),\ g = \tilde{g}_1 S|_{(D\setminus M)}.$$

We check that $M$ is singular for $g$. Indeed, assume that there exists $z_0 \in M$ such that $g$ can be extended to a bounded holomorphic function $g_1$ on a neighbourhood $z_0 + U$ where $U$ is a balanced convex neighbourhood of $0 \in l^1$. Consider the Taylor expansion of $g_1$ on $z_0 + U/2$ at $z_0$

$$g_1(z_0 + z_1) = \sum_{k \geq 0} P_k g_1(z_0)(z),$$

where

$$P_k g_1(z_0)(z) = 1/2\pi i \int_{|t|=2} (g_1(z_0 + tz)/t^{k+1}) dt.$$

Let $\{z_n\} \subset D \setminus M$ with $z_n \to z_0$. Then

$$P_k g_1(z_n) \to P_k g_1(z_0) \text{ as } n \to \infty \text{ for } k \geq 1,$$

uniformly on every compact set in $l^1$. Since every compact set in $F'_{c\beta}$ is the image of a compact set in $l^1$, it follows that

$$P_k g_1(y_n) \to T_k \text{ as } n \to \infty \text{ for } k \geq 1,$$

uniformly on every compact set in $F'_{c\beta}$, where $y_n = S(z_n)$ for $n \geq 1$. We have

$$\sup\{|P_k g_1(z_n)(z)| : z \in U/2\} \leq C/2^k \text{ for } k \geq 0,$$

where

$$C = \sup\{|g_1(z_0 + z)| : z \in U/2\}.$$

Hence the series

$$\sum_{k\geq 0} T_k(y_0 + y)$$

is convergent uniformly on $y_0 + V/2$, where $V = S(U)$ is a neighbourhood of $0 \in F'_{c\beta}$, to a holomorphic extension of $\tilde{g}_1$ to $y_0 + V/2$. This is impossible, because $y_0 \in \tilde{H}_1$ is singular for $\tilde{g}_1$. By Theorem 1.1, $M$ can be extended to a hypersurface $\hat{M}$ in $\hat{D}$ such that

$$(D \setminus M)^\wedge = \hat{D} \setminus \hat{M}.$$

(d) Let $\alpha$ be a continuous seminorm on $E$ induced by $\beta$, and $\tilde{R} : F'_{c\beta} \longrightarrow E_\alpha$ be the map induced by $R$. Put $T = \tilde{R}S : l^1 \longrightarrow E_\alpha$. Note that $T$ is open. Moreover

$$T^{-1}(H_1) \subset M \text{ and } T^{-1}(G_1) \subset D.$$

Then as in (b) we have

$$T(\hat{D}) = \hat{G}_1.$$

Since $R(\tilde{H})$ is locally closed in $G_1$, it is so in $\hat{G}_1$. Using Lemma 2.2 to $T$, $\hat{G}_1$, and $R(\tilde{H})$, we conclude that $\hat{H} = Cl[R(\tilde{H})]_{\hat{G}_1}$ is a hypersurface in $\hat{G}_1$ for which

$$H = \omega_\alpha^{-1}(\tilde{H}).$$

(e) As in (i) we can find a continuous seminorm $\gamma \geq \alpha$ on $E$ such that $\operatorname{Im} \omega_{\gamma\alpha} \subset \hat{G}_1$. This completes the proof of (ii) and Theorem 2.1.

Now we have the following corollary on extending hypersurface from a subspace of a (DFC)-space $E$ to $E$.

COROLLARY 2.7. *Let $F$ be a subspace of a (DFC)-space $E$ satisfying the condition (i) or (ii) of Theorem 2.1. Assume that the topology of $E$ is defined by a system of Hibert-seminorms $\{\rho\}$ and $H$ is a hypersurface in $F$. Then $H$ can be extended to a hypersurface $\hat{H}$ in $E$.*

PROOF. By Theorem 2.1 there exists a semi-norm $\rho$ on $E$ and a hypersurface $\tilde{H} \subset E_\rho$ such that $\omega_\rho(H) = \tilde{H}$, where $\omega_\rho : E \longrightarrow E_\rho$ is the canonical map. By the hypothesis we can assume that $E_\rho$ is a Hilbert space and, hence, $F_\rho$ is a closed subspace of $E_\rho$. Consider the orthogonal projection $\pi_\rho : E_\rho \longrightarrow F_\rho$.

Then there exists a hypersurface $H' \subset E_\rho$ such that $\pi_\rho(H') = \tilde{H}$. Hence, $\hat{H} = \omega_\rho^{-1}(H')$ is a hypersurface in $E$ and it is extending the hypersurface $H$. The corollary is proved.

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