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ON THE LEAST DEGREE OF POLYNOMIALS BOUNDING ABOVE THE DIFFERENCES BETWEEN MULTIPLICITIES AND LENGTH OF GENERALIZED FRACTIONS

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1. Introduction

Throughout this note, let A be a commutative local Noetherian ring with maximal ideal $\mathfrak{m}$ and $\dim A = d$. Denote by $\mathbf{N}$ the set of all positive integers.

The theory of modules of generalized fractions introduced by Sharp and Zakeri [S-Z1] has a wide range of application in Commutative Algebra; particularly, in the study of top cohomology modules and the Monomial Conjecture (see [O],[S-H],[S-Z1]-[S-Z4]). By [S-Z3, 2.1], the Monomial Conjecture holds for a system of parameters (abbr. s.o.p) $\mathbf{x} = (x_1, ..., x_d)$ of A if and only if $1/(x_1, ..., x_d, 1) \neq 0$ in $U(A)_{d+1}^{-d-1}A$. This fact suggests the study of the length $\ell(A(1/(x_1^{n_1}, ..., x_d^{n_d}, 1)))$. An interesting problem is to find conditions on $\mathbf{x}$ for this length to be a polynomial in $n_1, ..., n_d$ (see [S-H, Question 1.2]). In some concrete cases, the authors of [S-H] and [C-M] showed that there is certain intriguing link between polynomial properties of the functions $\ell(A(1/(x_1^{n_1}, ..., x_d^{n_d}, 1)))$ and $\ell(A/(x_1^{n_1}, ..., x_d^{n_d})A)$. It is worth noticing in all of these cases that

$$\ell(A/(x_1^{n_1}, ..., x_d^{n_d})A) = n_1...n_d.e(x_1, ..., x_d; A) + P(\mathbf{n}, \mathbf{x})$$

and

$$\ell(A(1/(x_1^{n_1}, ..., x_d^{n_d}, 1))) = n_1...n_d.e(x_1, ..., x_d; A) - Q(\mathbf{n}, \mathbf{x}),$$

where $P(\mathbf{n}, \mathbf{x})$ and $Q(\mathbf{n}, \mathbf{x})$ are polynomials (in $\mathbf{n}$) of degree at most $d - 1$.

On the other hand, it is known that the function

$$I_A(\mathbf{n}, \mathbf{x}) = \ell(A/(x_1^{n_1}, ..., x_d^{n_d})A) - n_1...n_d.e(x_1, ..., x_d; A)$$

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gives a lot of informations on the structure of A. Cuong showed in [C2] and [C3] that the least degree of all polynomials in $\mathbf{n}$ bounding above $I_A(\mathbf{n}, \mathbf{x})$ is independent of the choice of $\mathbf{x}$. This numerical invariant of A is called the *polynomial type* of A. Some applications of the polynomial type were given in [C2] - [C4] and [C-M].

Inspired by the study of Sharp, Zakeri and Cuong, we study in this paper the function:

$$J_A(\mathbf{n}, \mathbf{x}) = n_1 ... n_d . e(x_1, ..., x_d; A) - \ell(A(1/(x_1^{n_1}, ..., x_d^{n_d}, 1))).$$

We show that there are some similar properties between the functions $I_A(\mathbf{n}, \mathbf{x})$ and $J_A(\mathbf{n}, \mathbf{x})$. Our main result is the following theorem.

THEOREM 1.1. *Let $\mathbf{x} = (x_1, ..., .x_d)$be a s.o.p of A and $\mathbf{n} = (n_1, ..., n_d) \in \mathbf{N}^d$. Then the least degree of all polynomials in $\mathbf{n}$ bounding above $J_A(\mathbf{n}, \mathbf{x})$ is independent of the choice of $\mathbf{x}$.*

Before proving Theorem 1.1, we recall some standard facts on modules of generalized fractions in Section 2. Section 3 is devoted to the proof of Theorem 1.1. Section 4 gives some consequences and application. Especially, we get again a result of Hochster on the Monomial Conjecture.

I am grateful to my supervisor, Nguyen Tu Cuong, for drawing my attention to the subject and for hearty guidance during this research. I also would like to thank Le Tuan Hoa for many useful discussions and helpful comments.

2. Preliminaries

The reader is referred to [S-Z1] for details of the following brief summary of the theory on generalized fractions of Sharp and Zakeri.

Let k be a positive integer. We denote by $D_k(A)$ the set of all $k \times k$ lower triangular matrices with entries in A. For $H \in D_k(A)$, the determinant of H is denoted by $|H|$; and we use ^{T} to denote the matrix transpose.

A *triangular subset* of A^k is a non-empty subset U in A^k such that (i) whenever $(u_1, ..., u_k) \in U$, then $(u_1^{n_1}, ..., u_k^{n_k}) \in U$ for all choices of positive integers $n_1, ..., n_k$, and (ii) whenever $(u_1, ..., u_k)$ and $(v_1, ..., v_k) \in U$, then there exist $(w_1, ..., w_k) \in U$ and $H,\ K\ \in D_k(A)$ such that

$$H[u_1, ..., u_k]^T = [w_1, ..., w_k]^T = K[v_1, ..., v_k]^T.$$

Given such a U and an A-module M, R. Y. Sharp and H. Zakeri have constructed the module of generalized fractions $U^{-k}M$ of M with respect to U as follows.

Let $\alpha = ((u_1, ..., u_k), x)$ and $\beta = ((v_1, ..., v_k), y) \in U \times M$. Then we write $\alpha \sim \beta$ when there exist $(w_1, ..., w_k) \in U$ and $H,\ K\ \in D_k(A)$ such that $H[u_1, ..., u_k]^T = [w_1, ..., w_k]^T = K[v_1, ..., v_k]^T$ and $|H|x - |K|y \in (\sum_{i=1}^{k-1} Aw_i)M$. This relation is an equivalent relation on the set $U \times M$. For $x \in M$ and $(u_1, ..., u_k) \in U$, we denote by the formal symbol $x/(u_1, ..., u_k)$ the equivalence class of $((u_1, ..., u_k), x)$ and let $U^{-k}M$ denote the set of all these equivalence classes. Then $U^{-k}M$ is an A-module under the following operations.

Let $a \in A$; $x, y \in M$ and $(u_1, ..., u_k), (v_1, ..., v_k) \in U$. We define

$$x/(u_1, ..., u_k) + y/(v_1, ..., v_k) = (|H|x + |K|y)/(w_1, ..., w_k)$$

$$a(x/(u_1, ..., u_k)) = ax/(u_1, ..., u_k)$$

for any choice of $(w_1, ..., w_k) \in U$ and $H,\ K\ \in D_k(A)$ such that

$$H[u_1, ..., u_k]^T = [w_1, ..., w_k]^T = K[v_1, ..., v_k]^T.$$

An important example of triangular subsets is the following subset of A^{d+1} :
$U(A)_{d+1} = \{(x_1, ..., x_d, 1) \in A$: there exists j with $0 \leq j \leq d$ such that
$x_1, ..., x_j$ form a subset of a s.o.p of A and $x_{j+1} = ... = x_d = 1\}$.

We shall need the following basic properties of generalized fractions in the module $U(A)_{d+1}^{-d-1}A$:

REMARK 2.1. (see [S-H], [S-Z1], [S-Z4]) Let $a \in A$, and $(x_1, ..., x_d, 1) \in U(A)_{d+1}$. Then

i) $a/(x_1, ..., x_d, 1) = |H|a/(y_1, ..., y_d, 1)$ for any choice of $H \in D_{d+1}(A)$ and of $(y_1, ..., y_d, 1) \in U(A)_{d+1}$ such that $H[x_1, ..., x_d, 1]^T = [y_1, ..., y_d, 1]^T$.

ii) Let $n_1, ..., n_d \in \mathbf{N}$. Then $a/(x_1^{n_1}, ..., x_d^{n_d}, 1) = 0$ if and only if there exists $t \in \mathbf{N}$ such that $a.x_1^t...x_d^t \in \sum_{i=1}^{d} x_i^{n_i+t}A$.

iii) Let σ be a permutation of $\{1, ..., d\}$, then

$$a/(x_1, ..., x_d, 1) = sign(\sigma).a/(x_{\sigma(1)}, ..., x_{\sigma(d)}, 1).$$

LEMMA 2.2. *Let I be the annihilator of an $\mathfrak{m}$-primary ideal of A, let $\bar{A} = A/I$. Let $^- : A \to \bar{A}$ be the natural ring homomorphism. Let $x_1, ..., x_d \in \mathfrak{m}$. Then*

i) $x_1, ..., x_d$ is a s.o.p of $A \iff \bar{x}_1, ..., \bar{x}_d$ is a s.o.p of $\bar{A}$; when this is the case,

$$e(x_1, ..., x_d; A) = e(\bar{x}_1, ..., \bar{x}_d; \bar{A}).$$

ii) For all $(u_1, ..., u_d, 1) \in U(A)_{d+1}$,

$$\ell(A(1/(u_1, ..., u_d, 1))) = \ell(\bar{A}(1/(\bar{u}_1, ..., \bar{u}_d, \bar{1}))).$$

PROOF. It is straightforward from [S-H, 2.1].

LEMMA 2.3. (see [S-Z3, 2.4 and 2.7]) *Suppose that $d > 1$. Let $x_1 \in \mathfrak{m}$ be a parameter element of A. Put $\bar{A} = A/x_1A$ and let $^- : A \to \bar{A}$ denote the natural homomorphism. Then the map $\psi_{d+1} : U(\bar{A})_d^{-d}\bar{A} \longrightarrow U(A)_{d+1}^{-d-1}A$ defined by the formula*

$$\psi_{d+1}(\bar{y}/(\bar{u}_2, ..., \bar{u}_d, \bar{1}) = y/(x_1, u_2, ..., u_d, 1),$$

for all $\bar{y}/(\bar{u}_2, ..., \bar{u}_d, 1) \in U(\bar{A})_d^{-d}\bar{A}$, is a homomorphism of $A-$modules. Moreover, if x_1 is a non-zerodivisor on A, then $\ker \psi_{d+1} \cong H_{\mathfrak{m}}^{d-1}(A)/x_1H_{\mathfrak{m}}^{d-1}(A)$.

The following will be often used in the next sections.

LEMMA 2.4. *Let $\mathbf{x} = (x_1, ..., x_d)$ be a s.o.p of A and $\mathbf{n} = (n_1, ..., n_d) \in \mathbf{N}^d$. Assume that depth $A > 0$. Then there exists $a_1 \in (x_2^{n_2}, ..., x_d^{n_d})A$ such that $y_1 = x_1 + a_1$ satisfies the following properties:*

i) y_1 is a non-zerodivisor on A and pseudo-$H_{\mathfrak{m}}^{d-1}(A)$-coregular (in the terminology of [S-H]).

ii) For any $\mathbf{m} = (m_1, ..., m_d) \in \mathbf{N}^d$ such that $m_i \le n_i$, $i = 1, ..., d$,

$$(x_1^{m_1}, x_2^{m_2}, ..., x_d^{m_d})A = (y_1^{m_1}, x_2^{m_2}, ..., x_d^{m_d})A.$$

iii) For every $a \in A$ and $\mathbf{m}$ as in (ii),

$$a/(x_1^{m_1}, x_2^{m_2}, \ldots, x_d^{m_d}, 1) = a/(y_1^{m_1}, x_2^{m_2}, \ldots, x_d^{m_d}, 1)$$

in $U(A)_{d+1}^{-d-1}A$.

PROOF. Set $\mathfrak{V} = \operatorname{Ass}A \bigcup (\operatorname{Att} H_{\mathfrak{m}}^{d-1}(A) - \{\mathfrak{m}\})$. Then $(x_1, x_2^{n_2}, \ldots, x_d^{n_d})A \not\subset \bigcup_{\mathfrak{p} \in \mathfrak{V}} \mathfrak{p}$ since depth $A > 0$. It follows from [K, Theorem 124] that there exists $a_1 \in (x_2^{n_2}, \ldots, x_d^{n_d})A$ such that $y_1 = x_1 + a_1 \notin \bigcup_{\mathfrak{p} \in \mathfrak{V}} \mathfrak{p}$. Thus y_1 is a non-zerodivisor on A and pseudo-$H_{\mathfrak{m}}^{d-1}(A)$-coregular which gives (i). Note that, for each $t \in \mathbf{N}$, there is $b_t \in (x_2^{n_2}, \ldots, x_d^{n_d})A$ such that $y_1^t = x_1^t + b_t$. This proves (ii). It is clear that there exist elements $r_2, \ldots, r_d \in A$ such that $y_1^{m_1} = x_1^{m_1} + r_2 x_2^{m_2} + \cdots + r_d x_d^{m_d}$. Let

$$D_{d+1}(A) \ni H = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ r_2 & r_3 & \ldots & 1 & 0 \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix}.$$

Then $H[x_2^{m_2}, \ldots, x_d^{m_d}, x_1^{m_1}, 1]^T = [x_2^{m_2}, \ldots, x_d^{m_d}, y_1^{m_1}, 1]^T$. Hence (iii) follows from (i) of Remark 2.1.

3. Main result

Let $\mathbf{x} = (x_1, \ldots, x_d)$ be a s.o.p of A and $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbf{N}^d$. In this section, we study the function

$$J_A(\mathbf{n}, \mathbf{x}) = n_1 \ldots n_d . e(x_1, \ldots, x_d; A) - \ell(A(1/(x_1^{n_1}, \ldots, x_d^{n_d}, 1))).$$

First of all, we show that $J_A(\mathbf{n}, \mathbf{x})$ is non-negative.

LEMMA 3.1. *$\ell(A(1/(x_1, \ldots, x_d, 1))) \leq e(x_1, \ldots, x_d ; A)$.*

PROOF. We use induction on d.

In the case $d = 1$, we have $\ell(A(1/(x_1, 1))) = e(x_1; A)$ by [S-H, 3.1]. Assume that $d > 1$ and our assertion is proved for all rings of dimension smaller than

d. Lemmas 2.2 and 2.4 allow us to assume that depth $A > 0$ and x_1 is a non-zerodivisor on A. Let $\bar{A} = A/x_1A$ and let $^- : A \longrightarrow \bar{A}$ be the natural map. By Lemma 1.3, there is an epimorphism from $\bar{A}(\bar{1}/(\bar{x}_2, ..., \bar{x}_d, \bar{1}))$ onto $A(1/(x_1, ..., x_d, 1))$. Thus

$$\ell(A(1/(x_1, x_2, ..., x_d, 1))) \leq \ell(\bar{A}(\bar{1}/(\bar{x}_2, ..., \bar{x}_d, \bar{1}))) \leq e(\bar{x}_2, ..., \bar{x}_d;\ \bar{A})$$

by the induction hypothesis. Since x_1 is non-zerodivisor, $e(x_1, x_2, ..., x_d;\ A) = e(\bar{x}_2, ..., \bar{x}_d;\ \bar{A})$. Thus, $\ell(A(1/(x_1, ..., x_d, 1))) \leq e(x_1, ..., x_d;\ A)$ and the proof is completed.

LEMMA 3.2. *$J_A(\mathbf{n}, \mathbf{x}) \leq n_1...n_d J_A(\mathbf{1}, \mathbf{x})$, where $\mathbf{1} = (1, ..., 1) \in \mathbf{N}^d$.*

PROOF. By [S-Z3], $\ell(A(1/(x_1^{n_1}, ..., x_d^{n_d}, 1))) \geq n_1...n_d \ell(1/(x_1, ..., x_d, 1))$. Hence $J_A(\mathbf{n}, \mathbf{x}) \leq n_1...n_d J_A(\mathbf{1}, \mathbf{x})$.

This lemma gives an immediate consequence as follows.

COROLLARY 3.3. *If $J_A(\mathbf{n}, \mathbf{x})$ is a polynomial, then it is linear in each n_i, $i = 1, ..., d$.*

In the rest of this section, we shall consider $J_{A,\mathbf{x}}(t) := J_A((t, ..., t), \mathbf{x})$ as a function of one variable t.

LEMMA 3.4. *Let r be a positive integer satisfying $\mathfrak{m}^r \subseteq (x_1, ..., x_d)A$. Then, for every s.o.p $\mathbf{y} = (x_1, ..., x_{d-1}, y_d)$ and any $t \in \mathbf{N}$, we have*

$$J_{A,\mathbf{x}}(t) \leq (rd)^{d-1} J_{A,\mathbf{y}}(t).$$

PROOF. Note that, for all $t \in \mathbf{N}$, $\mathfrak{m}^{rdt} \subseteq (x_1, ..., x_d)^{dt}A \subseteq (x_1^t, ..., x_d^t)A$. We shall prove the following statement which is slightly stronger than Lemma 3.4.

Let $t_0 \in \mathbf{N}$ and $k \in \mathbf{N}$ such that $\mathfrak{m}^{kt} \subseteq (x_1^t, ..., x_d^t)A$ for all $t \geq t_0$. Then $J_{A,\mathbf{x}}(t) \leq k^{d-1} J_{A,\mathbf{y}}(t)$ for all $t \geq t_0$.

We do induction on d. In the case $d = 1$, by [S-H, 3.1], $J_{A,\mathbf{x}}(t) = J_{A,\mathbf{y}}(t) = 0$. Thus the statement is true in this case. Assume that $d > 1$ and our assertion is proved for all rings of smaller dimension. Lemma 2.2 enables us to assume that depth $A > 0$. Since $(x_1, x_2^t, ..., x_{d-1}^t, x_d^t y_d^t)A \not\subseteq \bigcup_{\mathfrak{p} \in \text{Ass } A} \mathfrak{p}$, there exists $a_1 \in$

$(x_2^t, ..., x_{d-1}^t, x_d^t y_d^t)A$ such that $y_1 = x_1 + a_1 \notin \bigcup_{\mathfrak{p} \in \text{Ass } A} \mathfrak{p}$. Thus, y_1 is a non-zero divisor on A. By the same method given in the proof of Lemma 2.4, we obtain

$$J_{A,\mathbf{x}}(t) = J_{A,\mathbf{z}}(t) \text{ and } J_{A,\mathbf{y}}(t) = J_{A,\mathbf{z}'}(t),$$

where $\mathbf{z} = (y_1, x_2, ..., x_{d-1}, x_d)$ and $\mathbf{z}' = (y_1, x_2, ..., x_{d-1}, y_d)$. Hence, without loss of generality, we can assume that x_1 is a non-zerodivisor on A. Set $\bar{A} = A/x_1^t A$ and use $^- : A \to \bar{A}$ to denote the natural ring homomorphism. By Lemma 2.3, there is an A-homomorphism $\Psi : U(\bar{A})_d^{-d}\bar{A} \to U(A)_{d+1}^{-d-1}A$ such that $\Psi(\bar{a}/(\bar{z}_2, ..., \bar{z}_d, \bar{1})) = a/(x_1^t, z_2, ..., z_d, 1)$ for all $\bar{a}/(\bar{z}_2, ..., \bar{z}_d, \bar{1})) \in U(\bar{A})_d^{-d}\bar{A}$. Note that

$$\Psi(\bar{A}(\bar{1}/(\bar{x}_2^n, ..., \bar{x}_d^n, \bar{1}))) = A(1/(x_1^t, x_2^n, ..., x_d^n, 1))$$

and

$$\Psi(\bar{A}(\bar{1}/(\bar{x}_2^n, ..., \bar{y}_d^{kn}, \bar{1}))) = A(1/(x_1^t, x_2^n, ..., y_d^{kn}, 1)).$$

It follows that

$$\ell(\bar{A}(\bar{1}/(\bar{x}_2^n, ..., \bar{x}_d^n, \bar{1}))) = \ell(\bar{A}(\bar{1}/(\bar{x}_2^n, ..., \bar{x}_d^n, \bar{1})) \cap \ker\Psi) + \ell(A(1/(x_1^t, x_2^n, ..., x_d^n, 1)))$$

and

$$\ell(\bar{A}(\bar{1}/(\bar{x}_2^n, ..., \bar{y}_d^{kn}, \bar{1}))) = \ell(\bar{A}(\bar{1}/(\bar{x}_2^n, ..., \bar{y}_d^{kn}, \bar{1})) \cap \ker\Psi) + \ell(A(1/(x_1^t, x_2^n, ..., y_d^{kn}, 1))).$$

Since $e(x_1^{n_1}, ..., x_d^{n_d}; A) = e(\bar{x}_2^{n_2}, ..., \bar{x}_d^{n_d}; \bar{A})$, we have

$$J_A((t, n, ..., n), (x_1, x_2, ..., x_d)) = J_{\bar{A},\bar{\mathbf{x}}'}(n) + \ell(\bar{A}(\bar{1}/(\bar{x}_2^n, ..., \bar{x}_d^n, \bar{1})) \cap \ker\Psi\ ) \tag{1}$$

and

$$J_A((t, n, ..., n), (x_1, x_2, ..., x_{d-1}, y_d^k)) = J_{\bar{A},\bar{\mathbf{y}}'}(n) + \ell(\bar{A}(\bar{1}/(\bar{x}_2^n, ..., \bar{y}_d^{kn}, \bar{1})) \cap \ker\Psi), \tag{2}$$

where $\bar{\mathbf{x}}' = (\bar{x}_2, ..., \bar{x}_d)$ and $\bar{\mathbf{y}}' = (\bar{x}_2, ..., \bar{x}_{d-1}, \bar{y}_d^k)$. Now let n be an arbitrary positive integer with $n \geq t$. Then $\overline{\mathfrak{m}}^{kn} \subseteq (\bar{x}_2^n, ..., \bar{x}_d^n)\bar{A}$ and hence there exist elements $\bar{r}_2, ..., \bar{r}_d \in \bar{A}$ such that $\bar{y}_d^{kn} = \bar{r}_2\bar{x}_2^n + \ldots + \bar{r}_d\bar{x}_d^n$. We choose

$$D_d(\bar{A}) \ni K = \begin{pmatrix} \bar{1} & \bar{0} & \ldots & \bar{0} & \bar{0} \\ \bar{0} & \bar{1} & \ldots & \bar{0} & \bar{0} \\ \vdots & \vdots & & \vdots & \vdots \\ \bar{r}_2 & \bar{r}_3 & \ldots & \bar{r}_d & \bar{0} \\ \bar{0} & \bar{0} & \ldots & \bar{0} & \bar{1} \end{pmatrix}.$$

Then $K[\bar{x}_2^n, ..., \bar{x}_d^n, \ \bar{1}]^T = [\bar{x}_2^n, ..., \bar{x}_{d-1}^n, \bar{y}_d^{kn}, \ \bar{1}]^T$. It follows from Remark 2.1 that

$$\bar{A}(\bar{1}/(\bar{x}_2^n, ..., \bar{x}_d^n, \bar{1})) \subseteq \bar{A}(\bar{1}/(\bar{x}_2^n, ..., \bar{x}_{d-1}^n, \bar{y}_d^{kn}, \bar{1})).$$

Hence $\ell(\bar{A}(\bar{1}/(\bar{x}_2^n, ..., \bar{x}_d^n, \bar{1})) \cap \ker\Psi) \leq \ell(\bar{A}(\bar{1}/(\bar{x}_2^n, ..., \bar{y}_d^{kn}, \bar{1})) \cap \ker\Psi)$. On the other hand, by the induction hypothesis,

$$J_{\bar{A},\bar{\mathbf{x}}'}(n) \leq k^{d-2}.J_{\bar{A},\bar{\mathbf{y}}'}(n). \tag{3}$$

Combining (1), (2) and (3), we get

$$J_A((t, n, ..., n), (x_1, x_2, ..., x_{d-1}, x_d)) \leq k^{d-2} J_A((t, n, ..., n), (x_1, x_2, ..., x_{d-1}, y_d^k))$$

$$\leq k^{d-1} J_A((t, n, ..., n), \ (x_1, x_2, ..., x_{d-1}, y_d))$$

for all $n \geq t$. In particular (for $n = t$), $J_{A,\mathbf{x}}(t) \leq k^{d-1}.J_{A,\mathbf{y}}(t)$ and the proof is completed.

PROPOSITION 3.5. *The least degree of all polynomials bounding above $J_{A,\mathbf{x}}(t)$ is independent of the choice of* $\mathbf{x}$.

PROOF. Let $\mathbf{y} = (y_1, ..., y_d)$ be an arbitrary s.o.p of A. Then we can connect $\mathbf{x}$ and $\mathbf{y}$ by a sequence of at most $(2d+1)$ s.o.p.'s of A with the property that two neighbour s.o.p.'s differ each from one by just one element. By repeated applications of Lemma 3.4, there exist two constants $k_1, k_2 \in \mathbf{N}$ such that $J_{A,\mathbf{x}}(t) \leq k_1.J_{A,\mathbf{y}}(t)$ and $J_{A,\mathbf{y}}(t) \leq k_2.J_{A,\mathbf{x}}(t)$ for all $t \geq 1$. The proof is then finished.

LEMMA 3.6. *Let $\mathbf{m} = (m_1, ..., m_d)$, $\mathbf{n} = (n_1, ..., n_d) \in \mathbf{N}^d$ with $m_i \leq n_i, i = 1, ..., d$. Then $J_A(\mathbf{m}, \mathbf{x}) \leq J_A(\mathbf{n}, \mathbf{x})$.*

PROOF. By Remark 2.1,(iii), we need only prove the lemma in the case where

$m_1 = n_1, \ldots, m_{d-1} = n_{d-1}$ and $m_d \le n_d$. We do induction on d. For $d = 1$, by [S-H, 3.1], $J_A(\mathbf{m}, \mathbf{x}) = J_A(\mathbf{n}, \mathbf{x}) = 0$. Assume that $d > 1$ and our assertion is true for all local ring of smaller dimension. Using Lemmas 2.2 and 2.4, we can assume that depth $A > 0$ and that x_1 is a non-zerodivisor on A. Let $\bar{A} = A/x_1^{n_1}A$ and let $^- : A \to \bar{A}$ be the natural homomorphism. By Lemma 2.3, there is an A- homomorphism $\Phi : U(\bar{A})_d^{-d}\bar{A} \longrightarrow U(A)_{d+1}^{-d-1}A$. By the same argument as in the proof of Lemma 3.4, we get

$$J_A(\mathbf{n}, \mathbf{x}) = J_{\bar{A}}(\mathbf{n}', \mathbf{x}') + \ell(\bar{A}(\bar{1}/(\bar{x}_2^{n_2}, \ldots, \bar{x}_d^{n_d}, \bar{1})) \cap \ker\Phi)$$

and

$$J_A(\mathbf{m}, \mathbf{x}) = J_{\bar{A}}(\mathbf{m}', \mathbf{x}') + \ell(\bar{A}(\bar{1}/(\bar{x}_2^{n_2}, \ldots, \bar{x}_{d-1}^{n_{d-1}}, \bar{x}_d^{m_d}, \bar{1})) \cap \ker\Phi),$$

where $\mathbf{n}' = (n_2, \ldots, n_d)$, $\mathbf{m}' = (n_2, \ldots, n_{d-1}, m_d)$, and $\mathbf{x}' = (\bar{x}_2, \ldots, \bar{x}_d)$. By induction hypothesis, $J_{\bar{A}}(\mathbf{m}', \mathbf{x}') \le J_{\bar{A}}(\mathbf{n}', \mathbf{x}')$. Since

$$\bar{A}(\bar{1}/(\bar{x}_2^{n_2}, \ldots, \bar{x}_{d-1}^{n_{d-1}}, \bar{x}_d^{m_d}, \bar{1})) \subseteq \bar{A}(\bar{1}/(\bar{x}_2^{n_2}, \ldots, \bar{x}_d^{n_d}, \bar{1})),$$

we get $J_A(\mathbf{m}, \mathbf{x}) \le J_A(\mathbf{n}, \mathbf{x})$, as required.

Now we are able to prove Theorem 1.1.

PROOF OF THEOREM 1.1. Let $t \in \mathbf{N}$. Then, by Proposition 3.5, the least degree of all polynomials of one variable t bounding above $J_{A,\mathbf{x}}(t)$ is independent of $\mathbf{x}$. Denote by $f(A)$ this invariant of A and by $pf(A, \mathbf{x})$ the least degree of all polynomials bounding above $J_A(\mathbf{n}, \mathbf{x})$ which is well-defined by Lemma 3.2. It is clear that $f(A) \le pf(A, \mathbf{x})$. But by Lemma 3.6, $J_{A,\mathbf{x}}(t) \ge J_A(\mathbf{n}, \mathbf{x})$ for all $t \ge \max\{n_1, \ldots, n_d\}$ which implies that $f(A) \ge pf(A, \mathbf{x})$. Thus $pf(A, \mathbf{x}) = f(A)$ is independent of the choice of $\mathbf{x}$. The theorem is proved.

4. Consequences and application

In this section, the numerical invariant of A given in Theorem 1.1 will be called the *polynomial type of fractions* of A and will be denoted by $pf(A)$. We stipulate that the degree of the zero-polynomial is equal to $-\infty$. We give some properties of polynomial types of fractions.

First, from Lemma 2.2 we get the following.

LEMMA 4.1. *Let I be the annihilator of an $\mathfrak{m}$-primary ideal of A. Then $pf(A) = pf(A/I)$.*

LEMMA 4.2. *Denote the $\mathfrak{m}$-adic completion of A by $\hat{A}$. Then $pf(A) = pf(\hat{A})$.*

PROOF. Let $\hat{}: A \longrightarrow \hat{A}$ be the canonical map. Let $\mathbf{x} = (x_1, ..., x_d)$ be a s.o.p of A and $\mathbf{n} = (n_1, ..., n_d) \in \mathbf{N}^d$. Then $\hat{\mathbf{x}} = (\hat{x}_1, ..., \hat{x}_d)$ is a s.o.p of $\hat{A}$ and

$$e(x_1, ..., x_d\ ;\ A) = e(\hat{x}_1, ..., \hat{x}_d\ ,\ \hat{A}).$$

It is not difficult to check that

$$\hat{A}(\hat{1}/(\hat{x}_1^{n_1}, ..., \hat{x}_d^{n_d},\ \hat{1})) \cong A(1/(x_1^{n_1}, ..., x_d^{n_d}, 1)) \otimes_A \hat{A}.$$

Thus

$$\begin{aligned}\ell_{\hat{A}}(\hat{A}(\hat{1}/(\hat{x}_1^{n_1}, \cdots, \hat{x}_d^{n_d},\ \hat{1}))) &= \ell_{\hat{A}}(\hat{A} \otimes A(1/(x_1^{n_1}, \cdots, x_d^{n_d}, 1))) \\ &= \ell_A(A(1/(x_1^{n_1}, ..., x_d^{n_d}, 1))).\end{aligned}$$

Hence, $J_A(\mathbf{n}, \mathbf{x}) = J_{\hat{A}}(\mathbf{n}, \hat{\mathbf{x}})$, and therefore $pf(A) = pf(\hat{A})$.

When we want to factor out by an ideal generated by a parameter element, we shall use the following fact.

PROPOSITION 4.3. *Let x be a parameter element of A with $\dim(0 : x)_A \leq d-2$. Then $pf(A/xA) \leq pf(A)$.*

PROOF. Let $\bar{A} = A/x_1A$ and let $\bar{}\ : A \to \bar{A}$ be the natural map. Choose a s.o.p $\mathbf{x} = (x_1, ..., x_d)$ of A such that $x_1 = x$ and let $\mathbf{n} = (n_1, ..., n_d) \in \mathbf{N}^d$. Then similar to the proof of Lemma 3.1, we obtain

$$\ell(A(1/(x_1^{n_1}, ..., x_d^{n_d}, 1))) \leq \ell(\bar{A}(\bar{1}/(\bar{x}_2^{n_2}, ..., \bar{x}_d^{n_d}, \bar{1}))).$$

As $\dim(0 : x)_A \leq d-2$, $e(x_1^{n_1}, ..., x_d^{n_d}, A) = e(\bar{x}_2^{n_2}, ..., \bar{x}_d^{n_d}, \bar{A})$. Thus, $J_{\bar{A}}(\mathbf{n}', \mathbf{x}') \leq J_A(\mathbf{n}, \mathbf{x})$, where $\mathbf{n}' = (n_2, ..., n_d)$ and $\mathbf{x}' = (x_2, ..., x_d)$. Hence, $pf(\bar{A}) \leq pf(A)$.

PROPOSITION 4.4. $pf(A) \leq d-2$.

PROOF. We do induction on d. From [S-H, 3.1 and 3.2], if $d=1$ then $pf(A) = -\infty$ and if $d=2$ then $pf(A) \leq 0$. Assume that $d > 2$ and the assertion is true for all rings of smaller dimension. By Lemma 4.1, we can assume that depth$A > 0$. Using Lemma 2.4, we can find a s.o.p $\mathbf{x} = (x_1, ..., x_d)$ such that x_1 is non-zerodivisor and pseudo-$H^{d-1}_{\mathfrak{m}}(A)$-coregular. Set $\bar{A} = A/x_1A$ and we use $^- : A \to \bar{A}$ to denote the natural homomorphism. By Lemma 2.3, there is an exact sequence of A-modules and A-homomorphisms

$$0 \longrightarrow H^{d-1}_{\mathfrak{m}}(A)/x_1H^{d-1}_{\mathfrak{m}}(A) \longrightarrow U(\bar{A})^{-d}_{d}\bar{A} \xrightarrow{\psi} U(A)^{-d-1}_{d+1}A.$$

Let $\mathbf{n} = (n_1, ..., n_d) \in \mathbf{N}^d$. Then similar to the proof of Lemma 3.4, we get

$$J_A((1, n_2, ..., n_d), \mathbf{x}) = J_{\bar{A}}((n_2, ..., n_d), (x_2, ..., x_d)) + \ell(\bar{A}(\bar{1}/(\bar{x}_2^{n_2}, ..., \bar{x}_d^{n_d}, \bar{1})) \cap \ker\psi).$$

Since x_1 is pseudo-$H^{d-1}_{\mathfrak{m}}(A)$-coregular,

$$R\ell(H^{d-1}_{\mathfrak{m}}(A)) \geq \ell(H^{d-1}_{\mathfrak{m}}(A)/x_1H^{d-1}_{\mathfrak{m}}(A)) = \ell(\ker\psi)$$

(where we use $R\ell(M)$ to denote the residual length (in the sense of [S-H]) of an Artinian-module M). Therefore,

$$\begin{aligned} J_A(\mathbf{n}, \mathbf{x}) \leq n_1 J_A((1, n_2, ..., n_d), \mathbf{x}) &= n_1(J_{\bar{A}}((n_2, ..., n_d), (x_2, ..., x_d)) \\ &\quad + \ell(\bar{A}(\bar{1}/(\bar{x}_2^{n_2}, ..., \bar{x}_d^{n_d}, \bar{1})) \cap \ker\psi) \\ &\leq n_1(J_{\bar{A}}((n_2, ..., n_d), (x_2, ..., x_d)) + R\ell(H^{d-1}_{\mathfrak{m}}(A))). \end{aligned}$$

Hence $pf(A) \leq \max\{pf(\bar{A}) + 1, 1\}$. Applying the induction hypothesis, we get $pf(A) \leq d-2$.

Recall that the polynomial type $p(A)$ of A, defined in [C2], is the least degree of all polynomials in $\mathbf{n}$ bounding above $I_A(\mathbf{n}, \mathbf{x})$. It was proved in [C-M] that if $p(A) \leq 1$, then $pf(A) \leq p(A)$. In fact, we can prove a more general statement. In the proof, we follow the terminology and notation of [S-H].

THEOREM 4.5. $pf(A) \leq p(A)$.

PROOF. We do induction on d. In the case $d = 1$, the proposition is proved by [S-H, 3.1]. Assume that $d > 1$ and our assertion is true for all rings of smaller dimension. The case A being a Cohen-Macaulay ring was proved in [C-M]. Now, suppose that $p(A) \geq 0$. Let $\hat{A}$ be the $\mathfrak{m}$-adic completion of A. Then we can assume that A is complete by the fact that $p(A) = p(\hat{A})$ and $pf(A) = pf(\hat{A})$. In that case there exists a system of parameters $x = \{x_1, ..., x_d\}$ of M so that the following conditions are satisfied:

$$(*) \begin{cases} x_d \in \mathfrak{a}(M); \\ x_i \in \mathfrak{a}(M/(x_{i+1}, ..., x_d)M), \quad i = 1, ..., d-1. \end{cases}$$

(see [C2]). Following [C4], a s.o.p $\mathbf{x} = (x_1, ..., x_d)$ of A is call a *p-standard system of parameter* (ps-s.o.p for short) if $\mathbf{x}$ satisfies the conditions (*).

Let $\mathbf{n} = (n_1, ..., n_d) \in \mathbf{N}^d$. Since depth $A > 0$, by virtue of Lemma 2.4, we can find $a_1 \in (x_2^{n_2}, ..., x_d^{n_d})A$ such that $y_1 = x_1 + a_1$ is a non-zerodisor on A and pseudo-$H_{\mathfrak{m}}^{d-1}(A)$-coregular. Note that $\mathbf{y} = (y_1, x_2, ..., x_d)$ is again a ps-s.o.p of A. Let $\bar{A} = A/y_1 A$ and let $^{-} : A \longrightarrow \bar{A}$ be the natural map. In the proof of Proposition 4.4 we have already shown that $pf(A) \leq \max\{pf(\bar{A}) + 1, 1\}$.

On the other hand, by [C4],

$$I_A(\mathbf{n}; \mathbf{y}) = \sum_{i=o}^{k} n_1 ... n_i . e_i,$$

where $e_i = e(y_1, x_2, ..., x_i; (x_{i+2}, ..., x_d)M : x_{i+1}/(x_{i+2}, ..., x_d)M)$ and $k = p(A)$. It is clear that $p(A) = p(\bar{A}) + 1$. Since $\bar{A}$ is not a Cohen-Macaulay ring, by induction hypothesis, we get

$$pf(A) \leq \max\{pf(\bar{A}) + 1, 1\} \leq p(\bar{A}) + 1 = p(A).$$

The proof is completed.

We conclude this note by an application to the Monomial Conjecture.

We say that the Monomial Conjecture holds for $x_1, ... x_d$ if for every integer $t \geq 0$,

$$x_1^t ... x_d^t \notin (x_1^{t+1}, ..., x_d^{t+1})A.$$

Proposition 4.4 provides a quick proof of a result of Hochster which supports the Monomial Conjecture.

COROLLARY 4.6. ([H, Proposition 2]) *Let* $\mathbf{x} = (x_1, ..., x_d)$ *be a s.o.p of* A. *Then there exists* $t \in \mathbf{N}$ *such that, whenever* $m_1 \geq t, ..., m_d \geq t$, *the s.o.p* $\mathbf{y} = (x_1^{m_1}, ..., x_d^{m_d})$ *of* A *satisfies the conclusion of the Monomial Conjecture.*

PROOF. Because $pf(A) \leq d-2$, there exists $t \in \mathbf{N}$ such that $t^d e(\mathbf{x}; A) > J_{A,\mathbf{x}}(t)$. Thus, for every $m_1 \geq t, ..., m_d \geq t$,

$$\ell(A(1/(x_1^{m_1}, ..., x_d^{m_d}, 1) \geq \ell(A(1/(x_1^t, ..., x_d^t, 1)) > 0.$$

Therefore, $1/(x_1^{m_1}, ..., x_d^{m_d}, 1) \neq 0$, $\forall m_1 \geq t, ..., m_d \geq t$. It follows from (ii) of Remark 2.1 that, for all $m_1 \geq t, ..., m_d \geq t$, the Monomial Conjecture holds for the s.o.p $\mathbf{y} = (x_1^{m_1}, ..., x_d^{m_d})$.

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