ACTA MATHEMATICA VIETNAMICA
Volume 20, Number 1, 1995, pp. 85-98

SPLINE COLLOCATION METHODS FOR FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS OF SECOND ORDER

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Abstract. A spline collocation method is used to solve approximately a general class of boundary value problem for integro-differential equations of second order. Theorem of existence and uniqueness of approximant via diagonally dominant matrix is shown. Theorem of convergence rate of second order for approximate solutions is obtained.

1. Existence and uniqueness theorem

We consider the following equation

$$Lx(t) = x''(t) + p(t)x'(t) + q(t)x(t) + \lambda \int_a^b K(t,s)x(s)ds = f(t), \tag{1a}$$

where $t \in [a,b]$, with the following boundary value conditions

$$\alpha_a x(a) + \beta_a x'(a) = \gamma_a, \tag{1b}$$

$$\alpha_b x(b) + \beta_b x'(b) = \gamma_b, \tag{1c}$$

where $\alpha_a, \alpha_b, \beta_a, \beta_b, \lambda \in R, {\alpha_a}^2 + {\beta_a}^2 > 0, {\alpha_b}^2 + {\beta_b}^2 > 0$ and $p(t), q(t), f(t)$ are continuous functions on $[a,b]$, and $K(t,s)$ is continuous in $\Omega \equiv [a,b] \times [a,b]$.

Let $G(t,s)$ be the Green function of the problem $x''(t) = 0$ with the following conditions

$$\begin{cases} \alpha_a x(a) + \beta_a x'(a) = 0, \\ \alpha_b x(b) + \beta_b x'(b) = 0. \end{cases} \tag{1d}$$

Then the solution of the equation $x''(t) = u(t)$ satisfying the condition $(1d)$ will be defined by

$$x(t) = \int_a^b G(t,s)u(s)ds$$

Received July 30, 1994; in revised form March 7, 1995.
This paper was supported in part by the National Basis Research Program in Natural Sciences, Vietnam.

(see [2], p. 132). It is clear that for the existence of the solution $x(t) \in C^4[a,b]$ of the equation (1a), (1d) it is sufficient to find conditions for $u(t) \in C^2[a,b]$ and satisfying the equation

$$u(t) + \int_a^b K_1(t,s)u(s)ds = f(t), \tag{1e}$$

where

$$K_1(t,s) = p(t)\frac{\partial G(t,s)}{\partial t} + q(t)G(t,s) + \lambda \int_a^b K(t,\zeta)G(\zeta,s)d\zeta.$$

Let T be an operator from $C[a,b]$ into $C[a,b]$ defined by

$$Tu = \int_a^b K_1(t,s)u(s)ds.$$

The equation (1e) may be written as follows

$$(I+T)u = f,$$

where I is the identity operator.

It is obviously that if there exists an inverse operator $(I+T)^{-1}$ which brings a function $f \in C^2[a,b]$ into $C^2[a,b]$ then the problem $(1a), 1(d)$ has solutions in $C^4[a,b]$.

Let π be an uniform partition of $[a,b]$:

$$\pi : a = t_0 < t_1 < \ldots < t_n = b,$$

where $t_i = a + ih, h = \frac{b-a}{n}$.

We introduce the spline space

$$S_3(\pi) = \left\{ v \in C^2[a,b] : v|_{[t_k,t_{k+1}]} \in P_3, k = 0,\ldots,n-1 \right\},$$

where P_3 is the class of cubic polynomials.

LEMMA 1. *With the above mentioned notation we have:*

i) $S_3(\pi)$ *is a real vector space with the basis* $\{B_i(t)\}_{i=-1}^{n+1}$, *where*

$$B_i(t) = \frac{1}{h^3}\begin{cases} (t - t_{i-2})^3 & \text{if } t \in [t_{i-2}, t_{i-1}], \\ h^3 + 3h^2(t - t_{i-1}) + 3h(t - t_{i-1})^2 - 3(t - t_{i-1})^3 & \text{if } t \in [t_{i-1}, t_i], \\ h^3 + 3h^2(t_{i+1} - t) + 3h(t_{i+1} - t)^2 - 3(t_{i+1} - t)^3 & \text{if } t \in [t_i, t_{i+1}], \\ (t_{i+2} - t)^3 & \text{if } t \in [t_{i+1}, t_{i+2}], \\ 0 & \text{otherwise}. \end{cases}$$

ii) If $\{\hat{B}_i(t)\}_{i=-1}^{n+1}$ is defined as follows

$$\begin{aligned} \hat{B}_{-1}(t) =& 3B_{-1}(t), \ \hat{B}_0(t) = B_0(t) - 2B_{-1}(t), \\ \hat{B}_i(t) =& B_i(t), \ i = 1, \dots, n-1, \\ \hat{B}_n(t) =& B_n(t) - 2B_{n+1}(t), \ \hat{B}_{n+1}(t) = 3B_{n+1}(t), \end{aligned}$$

then the system $\{\hat{B}_i(t)\}_{i=-1}^{n+1}$ is a basis of $S_3(\pi)$ too.

iii) For all $t \in [a, b]$,

$$\sum_{i=-1}^{n+1} |\hat{B}_i(t)| \le 12 \text{ and } \sum_{i=-1}^{n+1} |\hat{B}_i'(t)| \le \frac{22}{h}, \tag{2}$$

where $\hat{B}_i'(t)$ is the derivative of $\hat{B}_i(t)$.

PROOF. By the standart arguments (see [5], p.82), it is easy to prove i) and ii). Moreover we can calculate $B_j(t), B_j'(t), B_j''(t)$ at t_j as follows

$$\begin{array}{lll} B_j(t_{j\pm1}) = 1, & B_j(t_j) = 4, & B_j(t_i) = 0, \\ B_j'(t_{j-1}) = \dfrac{3}{h}, & B_j'(t_{j+1}) = \dfrac{-3}{h}, & B_j'(t_i) = 0, \\ B_j''(t_{j\pm1}) = \dfrac{6}{h^2}, & B_j''(t_j) = \dfrac{-12}{h^2}, & B_j''(t_i) = 0. \end{array} \tag{3}$$

For the proof of iii) it is only to notice that on $[t_i, t_{i+1}], i = 0, \dots, n-1$, in the system $\{B_i(t)\}_{i=-1}^{n+1}$ there are only four non-vanishing functions: $B_{i-1}(t)$, $B_i(t), B_{i+1}(t)$ and $B_{i+2}(t)$.

Now we define an element $x_N(t) \in S_3(\pi)$ as follows

$$x_N(t) = \sum_{j=-1}^{n+1} a_j \hat{B}_j(t) \tag{4}$$

satisfying

$$Lx_N(t_i) = f(t_i),\ i = 0, \dots, n,$$
$$\alpha_a x_N(t_0) + \beta_a x'_N(t_0) = \gamma_a,$$
$$\alpha_b x_N(t_n) + \beta_b x'_N(t_n) = \gamma_b.$$

From (3) it follows

$$\begin{cases} 3(\alpha_a h - 3\beta_a)a_{-1} + 2(\alpha_a h + 3\beta_a)a_0 + (\alpha_a h + 3\beta_a)a_1 = \gamma_a h, \\ (\alpha_b h - 3\beta_b)a_{n-1} + 2(\alpha_b h - 3\beta_b)a_n + 3(\alpha_b h + 3\beta_b)a_{n+1} = \gamma_b h, \\ \sum_{j=-1}^{n+1} a_j[\hat{B}_j"(t_i) + p_i\hat{B}'_j(t_i) + q_i\hat{B}_j(t_i)] + \lambda \sum_{j=-1}^{n+1} a_{ij}a_j = f_i,\ i = 0, \dots, n, \end{cases} \tag{5}$$

where $p_i = p(t_i), q_i = q(t_i), f_i = f(t_i)$,

$$a_{ij} = \int_{\alpha_j}^{\beta_j} K(t_i, s)\hat{B}_j(s)ds,\ i = 0, \dots, n;\ j = -1, \dots, n+1,$$

with

$$\alpha_j = \begin{cases} a, \text{ if } j = -1, 0, 1, 2, \\ a + (j-2)h, \text{ if } j = 3, \dots, n+1. \end{cases} \quad \beta_j = \begin{cases} b, \text{ if } j = n-2, n-1, n, n+1, \\ a + (j+2)h, \text{ if } j = -1, \dots, n-3, \end{cases}$$

and we have an equivalent system

$$\begin{cases} 3(\alpha_a h - 3\beta_a)a_{-1} + 2(\alpha_a h + 3\beta_a)a_0 + (\alpha_a h + 3\beta_a)a_1 = \gamma_a h, \\ 3(q_0h^2 - 3p_0h + 6)a_{-1} + 2(q_0h^2 + 3p_0h - 12)a_0 + \\ \qquad (c_0h^2 + 3p_0h + 6)a_1 + \lambda h^2 \sum_{j=-1}^{n+1} a_{0j}a_j = f_0h^2, \\ \qquad (q_ih^2 - 3p_ih + 6)a_{i-1} + 4(q_ih^2 - 3)a_i + \\ \qquad (q_ih^2 + 3p_ih + 6)a_{i+1} + \lambda h^2 \sum_{j=-1}^{n+1} a_{ij}a_j = f_ih^2, \\ \qquad i = 1, \dots, n-1 \\ (q_nh^2 - 3p_nh + 6)a_{n-1} + 2(q_nh^2 - 3p_nh - 12)a_n + \\ \qquad 3(q_nh^2 + 3p_nh + 6)a_{n+1} + \lambda h^2 \sum_{j=-1}^{n+1} a_{nj}a_j = f_nh^2, \\ (\alpha_b h - 3\beta_b)a_{n-1} + 2(\alpha_b h - 3\beta_b)a_n + 3(\alpha_b h + 3\beta_b)a_{n+1} = \gamma_b h. \end{cases} \tag{6}$$

LEMMA 2. *Given the following system of equations*

$$\begin{cases} b_{-1}x_{-1} + b_0x_0 + b_1x_1 = g_1, \\ c_{i,i-1}x_{i-1} + c_{i,i}x_i + c_{i,i+1}x_{i+1} + \sum\limits_{j=-1}^{n+1} d_{ij}x_j = g_i, i = 0,\dots,n, \\ b_{n-1}x_{n-1} + b_nx_n + b_{n+1}x_{n+1} = g_{n+1}. \end{cases} \tag{7}$$

If the coefficients in System (7) satisfy

$$\begin{cases} |b_0| + |b_1| < |b_{-1}|, \\ |c_{i,i-1}| + |c_{i,i+1}| + \sum\limits_{j=-1}^{n+1} |d_{ij}| < |c_{i,i}|, \; i = 0,\dots,n, \\ |b_{n-1}| + |b_n| < |b_{n+1}|, \end{cases} \tag{8}$$

then System (7) has a unique solution.

PROOF. System (7) can be rewritten as follows

$$\begin{cases} b_{-1}x_{-1} + b_0x_0 + b_1x_1 = g_1, \\ \sum\limits_{\substack{j=-1 \\ j\neq i-1,i,i+1}}^{n+1} d_{ij}x_j + (c_{i,i-1} + d_{i,i-1})x_{i-1} + \\ \quad (c_{i,i} + d_{i,i})x_i + (c_{i,i+1} + d_{i,i+1})x_{i+1} = g_i, \\ b_{n-1}x_{n-1} + b_nx_n + b_{n+1}x_{n+1} = g_{n+1}. \end{cases} \tag{9}$$

We have

$$\sum_{\substack{j=-1 \\ j\neq i-1,i,i+1}}^{n+1} |d_{ij}| + |c_{i,i-1} + d_{i,i-1}| + |c_{i,i+1} + d_{i,i+1}| \le \sum_{j=-1,\, j\neq i}^{n+1} |d_{ij}| + |c_{i,i-1}| + |c_{i,i+1}|.$$

On the other hand, by (8) we get

$$\sum_{\substack{j=-1 \\ j\neq i}}^{n+1} |d_{ij}| + |c_{i,i-1}| + |c_{i,i+1}| < |c_{i,i}| - |d_{ii}| \le |c_{i,i} + d_{i,i}|.$$

Therefore

$$\sum_{\substack{j=-1 \\ j\neq i-1,i,i+1}}^{n+1} |d_{ij}| + |c_{i,i-1} + d_{i,i-1}| + |c_{i,i+1} + d_{i,i+1}| < |c_{i,i} + d_{i,i}|.$$

By the first and last inequalities from (8) it is obvious that the coefficient matrix of System (9) is of diagonally dominant type. Consequently it has a unique solution. Lemma 2 is proved.

Now we find solvability conditions for System (6).

If the coefficient matrix of System (6) is of diagonally dominant type then the coefficients of the first and last equations of (6) must satisfy

$$|\alpha_a h - 3\beta_a| > |\alpha_a h + 3\beta_a|,$$

and

$$|\alpha_b h + 3\beta_b| > |\alpha_b h - 3\beta_b|.$$

Hence

$$\alpha_a\beta_a < 0 \text{ and } \alpha_b\beta_b > 0.$$

For the second equation of (6) we have

$$3|q_0h^2 - 3p_0h + 6| + |q_0h^2 + 3p_0h + 6| + |\lambda|h^2 \sum_{j=-1}^{n+1} |a_{oj}|$$

$$\leq 3|q_0h^2 - 3p_0h + 6| + |q_0h^2 + 3p_0h + 6| + |\lambda|h^2 \int_a^b |K(t_0, s)| \sum_{j=-1}^{n+1} |\hat{B}_j(s)|ds$$

$$\leq 3|q_0h^2 - 3p_0h + 6| + |q_0h^2 + 3p_0h + 6| + 12h^2|\lambda| \max_{0\leq i\leq n} \int_a^b |K(t_i, s)|ds.$$

For the coefficient matrix of (6) to be of diagonally dominant type we need

$$3|q_0h^2 - 3p_0h + 6| + |q_0h^2 + 3p_0h + 6| + 12\rho h^2 < 2|q_0h^2 + 3p_0h - 12|, \quad (10)$$

where $\rho = |\lambda| \max_{0\leq i\leq n} \int_a^b |K(t_i, s)|ds$. Let $q_0 < 0$. When $9p_0^2 + 48q_0 < 0$ or $\dfrac{-3p_0 + \sqrt{9p_0^2 + 48q_0}}{2q_0} < 0$ we take

$$\epsilon_0 = \min\left\{\frac{3p_0 - \sqrt{9p_0^2 - 24q_0}}{2q_0}, \frac{-3p_0 - \sqrt{9p_0^2 - 24q_0}}{2q_0}\right\}.$$

In the case when $\dfrac{-3p_0+\sqrt{9p_0^2+48q_0}}{2q_0}>0$ we take

$$\epsilon_0=\min\left\{\frac{3p_0-\sqrt{9p_0^2-24q_0}}{2q_0},\ \frac{-3p_0+\sqrt{9p_0^2+48q_0}}{2q_0}\right\}.$$

If $q_0<0$, $0<h<\epsilon_0$ and $0\le\rho<\dfrac{-q_0}{2}$ we have (10).

Analogously for the following equations from (6)

$$(q_ih^2-3p_ih+6)a_{i-1}+4(q_ih^2-3)a_i+(q_ih^2+3p_ih+6)a_{i+1}+\lambda h^2\sum_{j=-1}^{n+1}a_{ij}a_j=f_ih^2,$$

$i=1,\dots,n-1$, we get

$$\begin{aligned}&|q_ih^2-3p_ih+6|+|q_ih^2+3p_ih+6|+|\lambda|h^2\sum_{j=-1}^{n+1}|a_{ij}|\\&\le|q_ih^2-3p_ih+6|+|q_ih^2+3p_ih+6|+12\rho h^2.\end{aligned}$$

So we need to show

$$|q_ih^2-3p_ih+6|+|q_ih^2+3p_ih+6|+12\rho h^2<|4q_ih^2-12|, i=1,\dots,n-1. \quad (11)$$

Under the conditions : $q_i<0,\ i=1,\dots,\ n-1,$

$$0<h<\min\left\{\epsilon_i=\frac{\pm3p_i-\sqrt{9p_i^2-24q_i}}{2q_i}, i=1,\dots,n-1\right\},$$

and $0\le\rho<\dfrac{-q_i}{2}$, we obtain (11). We get $\pm3p_i\le\sqrt{9p_i^2}<\sqrt{9p_i^2-24q_i}$, which yields $\epsilon_i>0$.

Finally for the following equation from (6)

$$\begin{aligned}&(q_nh^2-3p_nh+6)a_{n-1}+2(q_nh^2-3p_nh-12)a_n+\\&3(q_nh^2+3p_nh+6)a_{n+1}+\lambda h^2\sum_{j=-1}^{n+1}a_{nj}a_j=f_nh^2,\end{aligned}$$

we have

$$|q_nh^2-3p_nh+6|+3|q_nh^2-3p_nh+6|+|\lambda|h^2\sum_{j=-1}^{n+1}|a_{nj}|$$

$$\leq \; |q_n h^2 - 3p_n h + 6| + 3|q_n h^2 - 3p_n h + 6| + 12\rho h^2.$$

Therefore we need to show

$$|q_n h^2 - 3p_n h + 6| + 3|q_n h^2 + 3p_n h + 6| + 12\rho h^2 < 2|q_n h^2 - 3p_n h - 12|. \quad (12)$$

When $q_n < 0$ and either $9p_n^2 + 48q_n < 0$ or $\dfrac{3p_n + \sqrt{9p_n^2 + 48q_n}}{2q_n} < 0$ we take

$$\epsilon_n = \min\left\{ \frac{-3p_n - \sqrt{9p_n^2 - 24q_n}}{2q_n}, \frac{3p_n - \sqrt{9p_n^2 - 24q_n}}{2q_n} \right\}.$$

In the case when $\dfrac{3p_n + \sqrt{9p_n^2 + 48q_n}}{2q_n} > 0$ we take

$$\epsilon_n = \min\left\{ \frac{-3p_n - \sqrt{9p_n^2 - 24q_n}}{2q_n}, \frac{3p_n + \sqrt{9p_n^2 + 48q_n}}{2q_n} \right\}.$$

So if $q_n < 0$, $0 < h < \epsilon_n$ and $0 \leq \rho < \dfrac{-q_n}{2}$. we get (12)

Therefore we obtain the following

THEOREM 3. *Assume that $p(t), q(t), f(t)$ are continuous on $[a,b]$, $K(t,s)$ is continuous on $\in \Omega$, $q(t) < 0, \forall t \in [a,b]$ and $\alpha_a, \alpha_b, \beta_a, \beta_b$, satisfy the conditions*

$$\alpha_a \beta_a < 0, \;\; \alpha_b \beta_b > 0.$$

Set

$$l = \min_{0 \leq i \leq n} \frac{-q_i}{2}, \;\; \epsilon = \min\{\epsilon_0, \dots, \epsilon_n\}.$$

Then, for $0 \leq \rho < l$ and $0 < h < \epsilon$ the collocation solution (4) of the boundary value problem (1a–1b–1c) exists uniquely and will be defined by solving System (6).

2. Estimate of convergence rate

Let $y(t) \in S_3(\pi)$ be an interpolation spline of the solution of (1). Then

$$y(t_i) = x(t_i), i = 0, \dots, n; \;\; y'(t_0) = x'(t_0), y'(t_n) = x'(t_n).$$

If $x(t) \in C^4[a,b]$ then

$$\|D^j(x-y)\|_\infty \le \gamma_j h^{4-j}, j=0,1,2,3, \tag{13}$$

where

$$\|x\|_\infty = \max_{1\le i\le n} \sup_{t_{i-1}\le t\le t_i} |x(t)|$$

and D^j is the j-th derivative (see [5], p. 112).

Let

$$y(t) = \sum_{j=-1}^{n+1} b_j \hat{B}_j(t).$$

Setting $Ly(t_i) = \hat{f}_i,\ i=0,\dots,n$, with $\hat{f}_i \in R$, we have

$$Lx_N(t_i) - Ly(t_i) = \sum_{j=-1}^{n+1} (a_j - b_j)[\hat{B}_j"(t_i) + p_i \hat{B}_j'(t_i) + q_i \hat{B}_j(t_i)] +$$

$$\lambda \sum_{j=-1}^{n+1} a_{ij}(a_j - b_j) = f_i - \hat{f}_i, i=0,\dots,n.$$

Denoting $\delta_j = a_j - b_j$ we get the following system

$$\begin{cases} 3(\alpha_a h - 3\beta_a)\delta_{-1} + 2(\alpha_a h + 3\beta_a)\delta_0 + (\alpha_a h + 3\beta_a)\delta_1 = 0, \\ 3(q_0 h^2 - 3p_0 h + 6)\delta_{-1} + 2(q_0 h^2 + 3p_0 h - 12)\delta_0 + \\ \qquad (q_0 h^2 + 3p_0 h + 6)\delta_1 + \lambda h^2 \sum_{j=-1}^{n+1} a_{oj}\delta_j = \tau_0 h^2; \\ (q_i h^2 - 3p_i h + 6)\delta_{i-1} + 4(q_i h^2 - 3)\delta_i + \\ \qquad (q_i h^2 + 3p_i h + 6)\delta_{i+1} + \lambda h^2 \sum_{j=-1}^{n+1} a_{ij}\delta_j = \tau_i h^2, i=1,\dots,n-1 \\ (q_n h^2 - 3p_n h + 6)\delta_{n-1} + 2(q_n h^2 - 3p_n h - 12)\delta_n + \\ \qquad 3(q_n h^2 + 3p_n h + 6)\delta_{n+1} + \lambda h^2 \sum_{j=-1}^{n+1} a_{nj}\delta_j = \tau_n h^2, \\ (\alpha_b h - 3\beta_b)\delta_{n-1} + 2(\alpha_b h - 3\beta_b)\delta_n + 3(\alpha_b h + 3\beta_b)\delta_{n+1} = 0, \end{cases} \tag{14a}$$

where $\tau_i = f(t_i) - \hat{f}(t_i),\ i=0,\dots,n$ and $f(t_i) = f_i, \hat{f}(t_i) = \hat{f}_i$.

We have

$$|\tau_i| = |Lx(t_i) - Ly(t_i)| \leq |x"(t_i) - y"(t_i)| + |p_i||x'(t_i) - y'(t_i)| +$$
$$|q_i||x(t_i) - y(t_i)| + |\lambda|| \int_a^b K(t_i, s)[x(s) - y(s)]ds|$$
$$\leq \|x" - y"\|_\infty + \|p\|_\infty \|x' - y'\|_\infty + \|q\|_\infty \|x - y\|_\infty + \rho \|x - y\|_\infty.$$

Using (13) we obtain

$$|\tau_i| \leq \gamma_2 h^2 + \|p\|_\infty \gamma_1 h^3 + (\|q\|_\infty + \rho) h^4 \gamma_0$$
$$= [\gamma_2 + \|p\|_\infty \gamma_1 h + (\|q\|_\infty + \rho) h^2 \gamma_0] h^2.$$

Consequently

$$|\tau_i| \leq \beta h^2, \ i = 0, \dots, n \text{ with } \beta = \gamma_2 + \|p\|_\infty \gamma_1 h + (\|q\|_\infty + \rho) h^2 \gamma_0.$$

Now by the hypotheses of Theorem 3 and by the first and last equations of System (14a) we see that if $e_{-1} > 0, e_{n+1} > 0$ then $e_{-1} < e_0$ or $e_{-1} < e_1$ and $e_{n+1} < e_n$ or $e_{n+1} < e_{n-1}$, where $e_i = |\delta_i|, i = -1, \dots, n+1$. Indeed from

$$|\alpha_a h - 3\beta_a| > |\alpha_a h + 3\beta_a|,$$

it follows that

$$3|\alpha_a h - 3\beta_a| e_{-1} > 2|\alpha_a h + 3\beta_a| e_{-1} + |\alpha_a h + 3\beta_a| e_{-1}.$$

If $e_{-1} \geq e_0$ and $e_{-1} \geq e_1$ then

$$3|\alpha_a h - 3\beta_a| e_{-1} > 2|\alpha_a h + 3\beta_a| e_0 + |\alpha_a h + 3\beta_a| e_1,$$

which contradicts to the first equation of System $(14a)$.

Similarly it is proved that

$$e_{n+1} < e_n \quad \text{or} \quad e_{n+1} < e_{n-1}.$$

Let us denote

$$e = \max\{e_0, e_1, \dots, e_n\} = \max\{e_{-1}, e_0, \dots, e_{n+1}\}. \tag{14b}$$

If $e_{-1} = e_{n+1} = 0$ then the equality (14b) remains valid. From (14a) we have

$$(4q_ih^2 - 12)\delta_i = -\lambda h^2 \sum_{j=-1}^{n+1} a_{ij}\delta_j + h^2\tau_i - (q_ih^2 - 3p_ih + 6)\delta_{i-1} - (q_ih^2 + 3p_ih + 6)\delta_{i+1}, i = 1, \ldots, n-1$$

It follows that

$$\begin{aligned}|4q_ih^2 - 12|e_i \le & h^2|\lambda| \sum_{j=-1}^{n+1} |a_{ij}|e_j + h^2|\tau_i| + \\ & |q_ih^2 - 3p_ih + 6|e_{i-1} + |q_ih^2 + 3p_ih + 6|e_{i+1} \\ & \le 12\rho h^2 e + \beta h^4 + (2q_ih^2 + 12)e,\end{aligned}$$

with $i = 1, \ldots, n-1$, and for $0 < h < \epsilon$. Hence

$$4(3 - q_ih^2)e_i \le 2(q_ih^2 + 6)e + 12\rho h^2 e + \beta h^4, i = 1, \ldots, n-1.$$

Since

$$q_ih^2 e \le q_ih^2 e_i,$$

we have

$$6(2 - q_ih^2)e_i \le 12e + 12\rho h^2 e + \beta h^4, \; i = 1, \ldots, n-1. \tag{15}$$

Let $0 < h < \epsilon$. Now we prove that (15) holds also for $i = 0$ and $i = n$.

If $e_0 = 0$ then we also get (15). If $e_0 > 0$ then either $e_0 \le e_1$, hence (15) holds immediately for $i = 0$, or $e_0 > e_1$ then from the second equation of (14a) we obtain

$$2(q_0h^2 + 3p_0h - 12)\delta_0 = \tau_0 h^2 - \lambda h^2 \sum_{j=-1}^{n+1} a_{0j}\delta_j - 3(q_0h^2 - 3p_0h + 6)\delta_{-1} - (q_0h^2 + 3p_0h + 6)\delta_1.$$

It follows that

$$-6q_0h^2e_0 \le 12\rho h^2 e + \beta h^4. \tag{16}$$

Taking into account the following inequality

$$e_0 \le e\,,$$

from (16) we can assert that (15) holds for $i = 0$.

Similarly (15) is proved for $i = n$. Therefore

$$6(2 - q_i h^2)e \leq 12e + 12\rho h^2 e + \beta h^4, \; i = 0, \dots, n,$$

and

$$e \leq \beta_1 h^2 \quad \text{with} \quad \beta_1 = \beta/12(l - \rho).$$

Consequently

$$e = \max_{-1 \leq i \leq n+1} |a_i - b_i| \leq \beta_1 h^2. \tag{17}$$

Now let us estimate $\|x(t) - x_N(t)\|_\infty$. We have

$$|y(t) - x_N(t)| \leq \max |a_i - b_i| \sum_{i=-1}^{n+1} |\hat{B}_i|. \tag{18}$$

By (2), (17), (18) we obtain

$$\|y(t) - x_N(t)\|_\infty \leq 12\beta_1 h^2.$$

Because of

$$\|x(t) - x_N(t)\|_\infty \leq \|x(t) - y(t)\|_\infty + \|y(t) - x_N(t)\|_\infty,$$

and of (13) we get

$$\|x(t) - x_N(t)\|_\infty \leq \eta h^2,$$

where $\eta = \gamma_0 h^2 + 12\beta_1$. Similarly, we have

$$|y'(t) - x'_N(t)| \leq \max |a_i - b_i| \sum_{i=-1}^{n+1} |\hat{B}'_i(t)|. \tag{19}$$

By (2), (17), (19) we get

$$\|y'(t) - x'_N(t)\|_\infty \leq 22\beta_1 h.$$

Next we estimate $\|x'(t) - x'_N(t)\|_\infty$ as follows

$$\begin{aligned} \|x'(t) - x'_N(t)\|_\infty &\leq \|x'(t) - y'(t)\|_\infty + \|y'(t) - x'_N(t)\|_\infty \\ &\leq \gamma_1 h^3 + 22\beta_1 h \\ &= \theta h, \end{aligned}$$

where $\theta = \gamma_1 h^2 + 22\beta_1$. So we get the following

THEOREM 4. *Let $x(t)$ be a solution of (1a), (1b), (1c) and $x(t) \in C^4[a,b]$. Assume that all conditions in Theorem 3 are satisfied. Then we have the following estimates*

$$\|x(t) - x_N(t)\|_\infty = 0(h^2),$$

and

$$\|x'(t) - x'_N(t)\|_\infty = 0(h),$$

Now we show a simple example as an application of Theorem 3 and 4.

Let us consider the following equation

$$x"(t) - x(t) + 1/20 \int_0^1 s^{39} x(s) ds = -t^2 - 2t + 2521/688800$$

satisfying the conditions

$$\begin{cases} x(0) - x'(0) = 0, \\ x(1) + x'(1) = 9. \end{cases}$$

The exact solution is $x(t) = t^2 + 2t + 2$. If $\epsilon = \sqrt{6},\ 0 < h < \sqrt{6}$ then the collocation solution exists uniquely.

Choosing $h = 1/15$ we can calculate the approximate solution and the error given in the following list.

$a_{-1} = 0.32286839262$	$a_8 = 0.55560452377$
$a_0 = 0.32996216531$	$a_9 = 0.59057752118$
$a_1 = 0.35270835532$	$a_{10} = 0.62702354870$
$a_2 = 0.37691845573$	$a_{11} = 0.66494327750$
$a_3 = 0.40278417732$	$a_{12} = 0.70433733518$
$a_4 = 0.43040903237$	$a_{13} = 0.74520632522$
$a_5 = 0.45950287959$	$a_{14} = 0.78755081966$
$a_6 = 0.49006663398$	$a_{15} = 0.83137136666$
$a_7 = 0.52210115174$	$a_{16} = 0.84647040819$

t_i	$\lvert x(t_i) - x_{15}(t_i)\rvert$	$\lvert x'(t_i) - x'_{15}(t_i)\rvert$
00	0.0187621362	0.0187632353
01	0.0200637354	0.0203002644
02	0.0212780889	0.0132546767
03	0.0215334191	0.0070759488
04	0.0205212581	0.0190082688
05	0.0192905931	0.0179254058
06	0.0181294328	0.0169222468
07	0.0170353464	0.0158717072
08	0.0160143431	0.0147535034
09	0.0150618428	0.0138561219
10	0.0141627843	0.0131257011
11	0.0133104506	0.0124537250
12	0.0125010566	0.0118371474
13	0.0117309887	0.0112734683
14	0.0109968073	0.0107601981
15	0.0102952225	0.0102952210

References

[1] V. P. Ilin, *On spline solutions of ordinary differential equations*, Vysch. Math. i Math. Fiz. 18 (1978), 620-627.

[2] M. L. Crasnov, A. W. Kisilev and G. I. Makarenko, "Integral Equations," Moscow, Nauka (in Russian), 1980.

[3] Yu. S. Zavialov, B. I. Kvacov, B. V. Miroshnishenko, "Spline methods," Moscow, Nauka (in Russian), 1980.

[3] Nguyen Minh Chuong, Ya. B. Mamedov and Khuat Van Ninh, "Approximate Solutions of Operator Equations," Sci. and Techn. Pubbl. House, Hanoi (in Vietnamese), 1993.

[5] P. M. Prenter, "Spline and variational methods," Wiley-Iterscience, New-York, 1975.

[6] S. B. Steshkin, Yu. N. Subbotin, "Splines in computational mathematics," Moscow, Nauka (in Russian), 1976.

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