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PROPERTY $\overline{\overline{\Omega}}$ AND HOLOMORPHIC FUNCTIONS WITH VALUES IN A PSEUDOCONVEX SPACE HAVING A STEIN MORPHISM INTO A COMPLEX LIE GROUP.

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Abstract. It is shown that a Fréchet-Schwartz space E with an absolute basis has the property $\overline{\overline{\Omega}}$ if and only if every holomorphic function on $D \times E$ with values in a pseudoconvex space having a Stein morphism into a complex Lie group, where D is a Stein space, is of uniform type. In the scalar case, where E is a nuclear Fréchet space, the result was established by Meise and Vogt [5].

Introduction

Let E be a Fréchet space with a fundamental system of semi-norms $\|.\|_1 \leq \|.\|_2 \leq \dots$. For each $k \in \mathbb{N}$ define a norm

$$\|.\|_k^* : E^* \longrightarrow [0, \infty],$$

by

$$\|x^*\|_k^* = \sup\{|x^*(x)| : \|x\|_k \leq 1\},$$

where E^* denotes the dual space of E. We say that E has the property $\overline{\overline{\Omega}}$ if $\forall\, p, \exists\, q$ such that $\forall\, k, d > 0\ \exists\, C > 0$:

$$\|x^*\|_q^{*1+d} \leq C \|x^*\|_k^* \|x^*\|_p^{*d}$$

for all $x^* \in E^*$. The property $\overline{\overline{\Omega}}$ and another properties were introduced and investigated by Vogt (see, for example, [11], [12], [13]). Meise and Vogt [5] gave an important characterization of nuclear Fréchet spaces having the property $\overline{\overline{\Omega}}$. They proved that a nuclear Fréchet space E has property $\overline{\overline{\Omega}}$ if and only if every holomorphic function on $D \times E$, where D is a bounded balanced convex domain

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in $\mathbf{C}^{\mathbf{N}}$, is of uniform type. This means that there exists p and a holomorphic function g on $D \times E_p$, such that

$$f(z,x) = g(z, \omega_p(x)),$$

for every $(z,x) \in D \times E$, where E_p is the Banach space associated to $\|.\|_p$ and $\omega_p : E \to E_p$ is the canonical map.

In the present note, we extend the result of Meise and Vogt to the case of holomorphic maps with values in a pseudoconvex space having a Stein morphism [1] into a complex Lie group. To get the main result (Theorem 2.1), in Section 1 we will investigate an approximation of continuous plurisubharmonic functions on open polynomially convex sets in metric topological spaces. Using this result and the method of Meise and Vogt we shall prove the main result in Section 2.

1. Approximation of continuous plurisubharmonic functions in metric topological vector spaces.

Given E a topological vector space and G an open set in E. An upper-semi continuous function $\varphi : G \to [-\infty, +\infty)$ is called plurisubharmonic on G if it is subharmonic on the interestion of G with every complex line in E. In [7] Noverraz proved that if G is a polynomially convex domain in a Banach space E with a Schauder basis then every continuous plurisubharmonic function φ on G can be written in the form

$$\varphi(z) = \lim_{n \to +\infty} \max \{c_j^n \log |f_j^n(z)| : 1 \leq j \leq m_n\}, \tag{BN}$$

where f_j^n is holomorphic function on E and $0 \leq c_j^n \leq 1$. Moreover, the convergence is uniformly on compact sets in G. In particular, it contains the result of Bremermann in the finite dimensional case. We prove in this section the following

THEOREM 1.1. *Let E be a metric topological vector space with a Schauder basis. Then every continuous plurisubharmonic function on a polynomially convex set in E can be written in the form (BN) if and only if E has a continuous norm.*

PROOF. First we prove the necessity. Let us show that there exists a sequence $\{\lambda_j\} \subset \mathbf{C}$ such that $\lambda_{j_k} e_{j_k} \nrightarrow 0$, for every subsequence $\{\lambda_{j_k}\} \subset \{\lambda_j\}$, where $\{e_j\}$ is a Schauder basis in E. Let D be an open polynomially convex set in $\mathbf{C}$ consisting of infinitely many connected components $D = \bigsqcup_{j \geq 1} D_j$. We can assume that $0 \in D_1$. Put

$$G = \bigsqcup_{j \geq 1} D_j e_j + M,$$

where $M = Cl(\text{ span } \{e_j\}_{j \geq 2})$. Obviously, G is polynomially convex in E. On G, define a continuous plurisubharmonic function φ given by

$$\varphi(z) = |e_j^*(z)| \text{ for } z \in D_j e_1 + M,$$

where $\{e_j^*\}$ is the dual system of $\{e_j\}$. By the hypothesis, there exist constants $0 < c_k^n < 1$ and holomorphic functions f_k^n, $1 \leq k \leq m_n$, $n \geq 1$ on E such that the sequence of plurisubharmonic functions $\{\psi_n\}$ converges uniformly to φ on every compact set in G, where

$$\psi_n(z) = \max \{c_k^n \log |f_k^n(z)| : 1 \leq k \leq m_n\}.$$

For each $j \geq 1$ consider on $D_j e_1 + \mathbf{C} e_j$ the functions, which converge uniformly on compact sets to the function

$$|e_j^*(z)| = |z_j|, \text{ with } z = z_1 e_1 + z_j e_j.$$

This implies that there exists $z_1^j \in \mathbf{C}$ with $|z_1^j| < 1/j$ such that $\psi_{n_j}(z_1^j, z_j)$ depends on z_j. Then, there exists $\lambda_j \in \mathbf{C}$ such that

$$|\psi_{n_j}(z_1^j, \lambda_j)| > j \text{ for every } j \geq 1.$$

We claim that $\{\lambda_j\}$ is the desired sequence. Indeed, assume that there exists a subsequence $\{\lambda_{j_p}\} \subset \{\lambda_j\}$ such that $\lambda_{j_p} e_{j_p} \to 0$. Consider the compact set in E given by

$$K = \{z_1^{j_p} e_1 + \lambda_{j_p} e_j, 0 : j \geq 1\}.$$

Since $0 \in D_1 e_1 + M$ for p sufficiently large, we have

$$z_1^{j_p} e_1 + \lambda_{j_p} e_{j_p} \in D_1 e_1 + M.$$

Hence,

$$\varphi(z_1^{j_p} e_1 + \lambda_{j_p} e_{j_p}) = |e^*(z_1^{j_p} e_1 + \lambda_{j_p} e_{j_p})| = |z_1^{j_p}| \to 0,$$

as $p \to \infty$ and

$$\begin{aligned}\|\varphi - \psi_{n_j}\|_K &\geq |\varphi(z_1^{j_p} e_1 + \lambda_{j_p} e_{j_p}) - \psi_{n_{j_p}}(z_1^{j_p} e_1 + \lambda_{j_p} e_{j_p})| \\ &\geq |\psi_{n_{j_p}}(z_1^{j_p} e_1 + \lambda_{j_p} e_{j_p})| - |\varphi(z_1^{j_p} e_1 + \lambda_{j_p} e_{j_p})| \\ &> j_p - |z_1^{j_p}| \to \infty,\end{aligned}$$

as $p \to \infty$, what is impossible.

Since $\{e_j\}$ is a Schauder basis, we have

$$0 = \lim_{j\to\infty} e_j^*(z) e_j = \lim_{j\to\infty} (e_j^*(z)/\lambda_j)\lambda_j e_j,$$

for every $z \in E$. Put

$$\rho(z) = \sup\{|e_j^*(z)/\lambda_j| : j \geq 1\}.$$

By the Banach-Steinhaus theorem ρ is a continuous norm on E.

We turn now to the proof of the sufficiency. Assume that G is an open polynomially convex set in E and $\|.\|$ is a continuous norm on E. Let φ be a continuous plurisubharmonic function on G. Write

$$G = \bigcup_{m\geq 1} F_m = \bigcup_{m\geq 1} \text{Int } F_m,$$

where F_m are closed sets in E, $\forall m \geq 1$. For each $j \geq 1$, put

$$Q_j = \{z \in G : \|z\| < j\} \text{ and } K_j = Cl(E_j \cap Q_j \cap A_j(E)),$$

where

$$A_j(z) = \sum_{1\leq k\leq j} e_k^*(z) e_k \text{ for every } z \in E.$$

Then

$$K_j \subseteq F_j \cap A_j(E) \subseteq G \cap A_j(E) \text{ for } j \geq 1.$$

Since the topology of $A_j(E)$ is defined by $\|.\||_{A_j}$, it follows that K_j is compact in $G \cap A_j(E)$ for every $j \geq 1$. Thus by the polynomial convexity of $G \cap A_j(E)$, according to the Bremermann theorem, there exist polynomials P_k^j on $A_j(E)$ and constants $0 < c_k^j < 1, 1 \le k \le m_j$ such that

$$\|\varphi - \psi_j\|_{K_j} < \frac{1}{j},$$

where $\psi_j(z) = \max \{c_k^j \log |P_k^j(z)| : 1 \le k \le m_j\}$. Obviously, $\psi_j \circ A_j$ is a plurisubharmonic function on E. We prove that $\{\psi_j \circ A_j\}_{j \ge 1}$ converges uniformly on compact sets in G to φ. Given K a compact set in G. Take m_0 such that

$$K + V \subset K + Cl(V) \subset \text{ Int } F_{m_0}, \tag{1}$$

for some neighbourhood V of zero in E. Since $A_j(z) \to z$ uniformly on compact set K in E, we get

$$A_j(K) \subset K + V \text{ for } j \ge j_0. \tag{2}$$

From (1) and (2) we have

$$A_j(K) \subset F_{m_0} \subset F_j \text{ for } j > j_1 = \max (j_0, m_0). \tag{3}$$

On the other hand, since $\cup_{j>1} A_j(K)$ is relatively compact in E and $\|.\|$ is continuous on E, it follows that

$$\bigcup_{j \ge j_1} A_j(K) \subset Q_{j_2} \text{ for some } j_2 > j_1. \tag{4}$$

From (3) and (4) we have

$$A_j(K) \subset Q_j \cap F_j \cap A_j(E) \subset K_j \text{ for } j > j_2.$$

Hence,

$$\begin{aligned} \|\psi_j A_j - \varphi\|_K &\le \|\psi_j A_j - \varphi A_j\|_K + \|\varphi A_j - \varphi\|_K \\ &\le \|\psi_j - \varphi\|_{A_j(K)} + \|\varphi A_j - \varphi\|_K \\ &\le \|\psi_j - \varphi\|_{K_j} + \|\varphi A_j - \varphi\|_K \\ &\le \frac{1}{j} + \|\varphi A_j - \varphi\|_K, \end{aligned}$$

for $j > j_2$. This implies that $\|\psi_j A_j - \varphi\|_K \to 0$, as $j \to \infty$. The theorem is proved.

2. Holomorphic functions on a Fréchet-Schwartz space with values in a pseudoconvex space having a Stein morphism into a complex Lie group.

First we recall that a holomorphic map θ from a complex space X to a complex space Y is called a Stein morphism if for every $y \in Y$ there exists a neighbourhood V of y such that $\theta^{-1}(V)$ is a Stein space. (see [1])

In this section we shall prove the converse of a result by Meise and Vogt in [5]:

THEOREM 2.1. *Let E be a Fréchet-Schwartz space with an absolute basis. Then E has the property $\overline{\overline{\Omega}}$ if and only if every holomorphic function on $D \times E$, where D is a Stein space, with values in a pseudoconvex space X having a Stein morphism into a complex Lie group, is of uniform type.*

The sufficiency was proved in [5]. In orded to prove the necessity, we need some auxiliary results.

LEMMA 2.2. *$H(X)$ has the property $(\underline{DN})$ for every irreducible complex space X.*

PROOF. In [14] Vogt proved the following result:

Let X be a real analytic manifold and $\mathcal{S}$ be a subsheaf of the sheaf of germs of real analytic functions. Then $H^0(X, \mathcal{S})$ equipped with the compact-open topology has the property $(\underline{DN})$.

Consider the Hironaka singular resolution $\theta : Z \to X$ of X. Since X is irreducible we may assume that Z is connected. By the above mentioned Vogt's result, it follows that $H(Z)$ and hence $H(X)$ have the property $(\underline{DN})$.

LEMMA 2.3. *Let E be a Fréchet-Schwartz space with an absolute basis $\{e_j\}_{j\geq 1}$ and F a Fréchet space. Assume that $E \in (\overline{\overline{\Omega}})$ and $F \in (\underline{DN})$. Then there exists a compact balanced convex set B such that*

(i) $\{e_j\}_{j\geq 1} \subset E_B$.

(ii) $H_{ub}((E_B, \tau_E), F) = H((E_B, \tau_E), F)$, where E_B is the Banach space spanned by B and τ_E is the topology of E_B induced by the topology of E.

PROOF. Suppose that $\{e_j\}_{j\in\mathbb{N}}$ is an absolute basis in E and $\{e_j^*\}_{j\in\mathbb{N}}$ is the dual basis in E^*, i.e. the strongly dual space of E. Given $f \in H(E, F)$.

(i) First assume that F is a Banach space. Then there exists a continuous semi-norm $\|.\|_p$ on E such that

$$\sup_{\|x\|_p \le 1} \{\|f(x)\|\} = M < +\infty.$$

Since $E \in (\overline{\overline{\Omega}})$, there exists a continuous semi-norm q on E such that for every compact set B' in E, $d > 0$ there exists $C > 0$, with

$$\|x^*\|_q^{*1+d} \le C\|x^*\|_{B'}^*\|x^*\|_p^{*d} \text{ for all } x^* \in E^*.$$

Since E is a Fréchet space, we can assume that $\{e_j\}_{j\ge 1} \subset E_{B'}$. Put

$$B" = \{\sup_{x\in B'} \{|e_j^*(x)|e_j\} : j \ge 1\}.$$

Then $B"$ is bounded in E. Indeed, take p_1 a continuous semi-norm on E. Since B' is bounded, there exists $\alpha > 0$ such that

$$\|x\|_{p_1} \le \alpha \text{ for } x \in B'.$$

This implies that $|e_j^*(x)|\|e_j\|_{p_1} \le \alpha$, $\forall\, x \in B'$, $\forall\, j \ge 1$. Thus $B" \subset \alpha U_{p_1}$. This means that $B"$ is bounded. On the other hand , since E is a Schwartz space then $B = \overline{conv}(B' \cup B")$ is a compact set and we have

$$\|x^*\|_q^{*1+d} \le C\|x^*\|_B^*\|x^*\|_p^{*d},$$

for all $x^* \in E^*$. Thus

$$\|e_j^*\|_q^{*1+d} \le C\|e_j^*\|_B^*\|e_j^*\|_p^{*d} \text{ for } j \ge 1.$$

Note that

$$\|e_j^*\|_q^* = \frac{1}{\|e_j\|_q} \quad \text{for } j \ge 1,$$

$$\|e_j^*\|_p^* = \frac{1}{\|e_j\|_p} \quad \text{for } j \ge 1.$$

Obviously, $\|e_j^*\|_B^* \neq 0$ for $j \geq 1$. By the construction of B we have $\|e_j^*\|_B^* e_j \in B$ for $j \geq 1$, i.e. $e_j \in \frac{1}{\|e_j^*\|_B^*} B$ $(j \geq 1)$. This means that

$$\|e_j^*\|_B^* \leq \frac{1}{\|e_j\|_B} \quad \text{for } j \geq 1.$$

We have therefore,

$$\frac{1}{\|e_j\|_q^{1+d}} \leq \frac{C}{\|e_j\|_B \|e_j\|_p^d} \quad \text{for } j \geq 1.$$

By the compactness of B, for every $r > 0$,

$$\sup_{\|x\|_B \leq r} \{\|f(x)\|\} = N(r) < +\infty.$$

For each $x = \Sigma_{j\geq 1} e_j^*(x) e_j \in E_B$, with

$$\|x\|_q = \sum_{j\geq 1} |e_j^*(x)| \|e_j\|_q \leq r,$$

we have

$$\begin{aligned}
\|f(x)\| &\leq \sum_{n\geq 0} \|P_n f(x)\| = \sum_{n\geq 0} \|P_n f(\sum_{j\geq 1} e_j^*(x) e_j)\| \\
&\leq \sum_{n\geq 0} \sum_{j_1,\dots,j_n} |e_{j_1}^*(x)| \dots |e_{j_n}^*(x)| \|\hat{P}_n f(e_{j_1}, \dots, e_{j_n})\| \\
&= \sum_{n\geq 0} \sum_{j_1,\dots,j_n} |e_{j_1}^*(x)| \dots |e_{j_n}^*(x)| \|e_{j_1}\|_q \dots \|e_{j_n}\|_q \frac{\|\hat{P}_n f(e_{j_1}, \dots, e_{j_n})\|}{\|e_{j_1}\|_q \dots \|e_{j_n}\|_q} \\
&\leq \sum_{n\geq 0} \left(C^{1/1+d}\right)^n \sum_{j_1,\dots,j_n} |e_{j_1}^*(x)| \dots |e_{j_n}^*(x)| \|e_{j_1}\|_q \dots \|e_{j_n}\|_q \\
&\qquad \times \left[\frac{\|\hat{P}_n f(e_{j_1}, \dots, e_{j_n})\|}{\|e_{j_1}\|_B \dots \|e_{j_n}\|_B}\right]^{1/1+d} \left[\frac{\|\hat{P}_n f(e_{j_1}, \dots, e_{j_n})\|}{\|e_{j_1}\|_p \dots \|e_{j_n}\|_p}\right]^{d/1+d} \\
&\leq \sum_{n\geq 0} \frac{1}{\left(\rho^{1/1+d}\right)^n} \sum_{j_1,\dots,j_n} |e_{j_1}^*(x)| \dots |e_{j_n}^*(x)| \|e_{j_1}\|_q \dots \|e_{j_n}\|_q \\
&\qquad \times \left[\frac{\|\hat{P}_n f(C\rho e_{j_1}, \dots, C\rho e_{j_n})\|}{\|e_{j_1}\|_B \dots \|e_{j_n}\|_B}\right]^{1/1+d} \left[\frac{\|\hat{P}_n f(e_{j_1}, \dots, e_{j_n})\|}{\|e_{j_1}\|_p \dots \|e_{j_n}\|_p}\right]^{d/1+d} \\
&\leq \sum_{n\geq 0} \frac{1}{\left(\rho^{1/1+d}\right)^n} \sum_{j_1,\dots,j_n} |e_{j_1}^*(x)| \dots |e_{j_n}^*(x)| \|e_{j_1}\|_q \dots \|e_{j_n}\|_q
\end{aligned}$$

$$
\begin{aligned}
&\quad \times \left[\frac{1}{1+d}\frac{\|\hat{P}_n f(C\rho e_{j_1},\dots,C\rho e_{j_n})\|}{\|e_{j_1}\|_B\dots\|e_{j_n}\|_B} + \frac{d}{1+d}\frac{\|\hat{P}_n f(e_{j_1},\dots,e_{j_n})\|}{\|e_{j_1}\|_p\dots\|e_{j_n}\|_p}\right] \\
&\le \sum_{n\ge 0}\frac{1}{\left(\rho^{1/1+d}\right)^n}\sum_{j_1,\dots,j_n}|e^*_{j_1}(x)|\dots|e^*_{j_n}(x)|\|e_{j_1}\|_q\dots\|e_{j_n}\|_q \\
&\quad \times \left[\frac{N(C\rho)}{1+d}\frac{n^n}{n!(C\rho)^n} + \frac{dM}{1+d}\frac{n^n}{n!}\right] \\
&= \frac{N(C\rho)}{1+d}\sum_{n\ge 0}\frac{n^n}{n!\left(C\rho^{(2+d)/(1+d)}\right)^n}\left(\sum_{j\ge 1}|e^*_j(x)|\|e_j\|_q\right)^n \\
&\quad + \frac{dM}{1+d}\sum_{n\ge 0}\frac{n^n}{n!\left(\rho^{1/1+d}\right)^n}\left(\sum_{j\ge 1}|e^*_j(x)|\|e_j\|_q\right)^n \\
&\le \frac{N(C\rho)}{1+d}\sum_{n\ge 0}\frac{n^n r^n}{n!\left(C\rho^{(2+d)/(1+d)}\right)^n} + \frac{dM}{1+d}\sum_{n\ge 0}\frac{n^n r^n}{n!\left(\rho^{1/1+d}\right)^n} < +\infty,
\end{aligned}
$$

for ρ sufficiently large.

(ii) By (i) for each continuous semi-norm p there exists a continuous semi-norm $q = q(p)$ such that

$$M(p,r) = \sup\{\|f(x)\|_p : x \in E_B, \|x\|_q \le r\} < +\infty,$$

for all $r > 0$. Take p_1 such that: $\forall\, p\ \exists\, p_2, d > 0$ and $C > 0$:

$$\|.\|_p^{1+d} \le C\|.\|_{p_2}\|.\|_{p_1}^d.$$

Put $q_1 = q(p_1)$ and $q_2 = q(p_2)$. Since $E \in (\overline{\overline{\Omega}})$ we can find $q_0, D > 0$ such that

$$\|e_j\|_{q_2}\|e_j\|_{q_1} \le D\|e_j\|_{q_0},$$

for all j. Consider the Taylor expansion of f at $0 \in E$,

$$f(z) = \sum_{n\ge 0} P_n f(z),$$

where

$$P_n f(z) = \frac{1}{2\pi i}\int_{|t|=r}\frac{f(tz)}{t^{n+1}}dt,$$

for all $n \ge 0$ and all $r > 0$.

For $z = \Sigma_{j\geq 1} e_j^*(z) e_j$ where $\{e_j^*\}$ is the sequence of coefficient functionals, we have

$$\|f(z)\|_p \leq \sum_{n\geq 0} \|P_n f(z)\|_p$$

$$\leq \sum_{n\geq 0} \sum_{j_1,\dots,j_n} |e_{j_1}^*(z)| \dots |e_{j_n}^*(z)| \|e_{j_1}\|_{q_0} \dots \|e_{j_n}\|_{q_0} \frac{\|P_n f(e_{j_1},\dots,e_{j_n})\|_p}{\|e_{j_1}\|_{q_0} \dots \|e_{j_n}\|_{q_0}}$$

$$\leq \sum_{n\geq 0} \sum_{j_1,\dots,j_n} \left(C^{1/1+d}/D^{n/1+d}\right) |e_{j_1}^*(z)| \dots |e_{j_n}^*(z)| \|e_{j_1}\|_{q_0} \dots \|e_{j_n}\|_{q_0}$$

$$\times \left[\frac{\|P_n f(e_{j_1},\dots,e_{j_n})\|_{p_1}}{\|e_{j_1}\|_{q_1} \dots \|e_{j_n}\|_{q_1}}\right]^{d/1+d} \left[\frac{\|P_n f(e_{j_1},\dots,e_{j_n})\|_{p_2}}{\|e_{j_1}\|_{q_2} \dots \|e_{j_n}\|_{q_2}}\right]^{1/1+d}$$

$$\leq C^{1/1+d} \sum_{n\geq 0} \left(\frac{1}{D}\right)^{(n/1+d)} \sum_{j_1,\dots,j_n} |e_{j_1}^*(z)| \dots |e_{j_n}^*(z)| \|e_{j_1}\|_{q_0} \dots \|e_{j_n}\|_{q_0}$$

$$\times \left[\frac{\|P_n f(e_{j_1},\dots,e_{j_n})\|_{p_1}}{\|e_{j_1}\|_{q_1} \dots \|e_{j_n}\|_{q_1}} + \frac{\|P_n f(e_{j_1},\dots,e_{j_n})\|_{p_2}}{\|e_{j_1}\|_{q_2} \dots \|e_{j_n}\|_{q_2}}\right]$$

$$\leq [M(p_1,r) + M(p_2,r)] C^{1/1+d} \sum_{n\geq 0} \frac{1}{n!} \frac{n^n}{r^n D^{n/1+d}} \Big(\sum_{j\geq 1} |e_j^*(z)| \|e_j\|_{q_0}\Big)^n$$

$$= [M(p_1,r) + M(p_2,r)] C^{1/1+d} \sum_{n\geq 0} \frac{1}{n!} n^n \|z\|_{q_0}^n \frac{1}{D^{n/1+d}}$$

$$\leq [M(p_1,r) + M(p_2,r)] C^{1/1+d} \sum_{n\geq 0} \frac{n^n}{n!(e+1)^n} < +\infty,$$

for all $z \in E$ with $\|z\|_{q_0} \leq \frac{rD^{1/1+d}}{e+1}$. We may here assume that $\|z\|_q = \sum_{j\geq 1} |e_j^*(z)| \|e_j\|_q$, for every $z \in E$ and every q.

Consequently, f is of uniformly bounded type and hence Lemma 2.3 is proved.

PROPOSITION 2.4. *Let E be a Fréchet-Schwartz space with an absolute basis and let $E \in (\overline{\overline{\Omega}})$. Assume that F is a Fréchet space with $F \in (\underline{DN})$. Then for every family $\mathcal{F}$ of holomorphic functions from E to F which is locally bounded, there exists $q \in \mathbb{N}$ such that every $f \in \mathcal{F}$ can be factorized holomorphically through the canonical map $\omega_q : E \to E_q$.*

PROOF. It suffices to prove that there exists a compact balanced convex set B such that $\|e_j\|_{j\geq 1} \subset E_B$ and for every family $\mathcal{F}$ of holomorphic functions

$f \in H((E_B, \tau_E), F)$, there exists $\bar{f} \in H_{ub}(E, F)$ such that $\bar{f}|_{E_B} = f$, for each $f \in H((E_B, \tau_E), F)$.

First we construct a subset B as in the proof of Lemma 2.3. Let $f \in \mathcal{F}$. By Lemma 2.3, $f \in H_{ub}((E_B, \tau_E), F)$. Thus there exists a continuous semi-norm $\|.\|_{q_0}$ on E such that for all $r > 0$, $f(r(E_B \cap U_{q_0}))$ is bounded in F, where $U_{q_0} = \{x \in E : \|x\|_{q_0} \leq 1\}$. By the Cauchy inequality, the completness of F and by standard arguments, f can be extended on nU_{q_0} for all $n \in \mathbb{N}$. It follows that f has an extension $\bar{f} \in H(E, F)$. For each $r > 0$, we have

$$\bar{f}(rU_{q_0}) = \bar{f}(\overline{r(E_B \cap U_{q_0})}) \subset \overline{\bar{f}(r(E_B \cap U_{q_0}))} = \overline{f(r(E_B \cap U_{q_0}))}.$$

Thus $\bar{f}(rU_{q_0})$ is bounded in F for all $r > 0$. Hence $\bar{f} \in H_{ub}(E, F)$. The proposition is proved.

PROOF OF NECESSITY OF THEOREM 2.1.

1. Assume first that the necessity of the theorem holds for the case, where X a is complex Lie group.

Given $f : D \times E \to G$ a holomorphic function where D is a Stein space and G an arbitrary pseudoconvex space having a Stein morphism into a complex Lie group. Let φ be a continuous plurisubharmonic exhaustion function on G and let $\theta : G \to S$ be a Stein morphism, where S is a complex Lie group. Since the necessity of the theorem holds for S, there exists p and a holomorphic function $g : D \times E_p \to S$ such that

$$\theta f = g(id \times \omega_p),$$

where $\omega_p : E \to E_p$ is the canonical map.

a) First consider the case where E has a continuous norm.

(i) Using Theorem 1.1 to the continuous plurisubharmonic function $\psi = \varphi \circ f$ on $D \times E$, we can write

$$\psi(z, x) = \lim_{n \to \infty} \max \{c_j^n \log |f_j^n| : 1 \leq j \leq m_n\},$$

where $0 \leq c_j^n < 1$ and f_j^n are holomorphic functions on $D \times E$, for every $1 \leq j \leq m_n$ and $n \geq 1$. Moreover, the convergence is uniform on compact sets in $D \times E$. This implies that the sequence $\{\hat{f}_j^n\}$ is locally bounded in $H(E, H(D))$, where $\hat{f}_j^n(x)(z) = f_j^n(z, x)$.

Since E has the property $(\overline{\overline{\Omega}})$, by Proposition 2.4 we can find $q \geq q_1$ independent of $1 \leq j \leq m_n$, $n \geq 1$ and holomorphic functions g_j^n on $D \times E_q$ such that

$$f_j^n = g_j^n(id \times \omega_q)$$

for $1 \leq j \leq m_n$ and $n \geq 1$. Moreover, we also have

$$\sup\{|g_j^n(z,x)| : z \in K, \|x\|_q \leq r\} \leq \exp[\gamma N(K,r) + \beta M]/c_j^n < +\infty,$$

for every compact set K in D and every $r > 0$, $1 \leq j \leq m_n$ and $n \geq 1$. These inequalities imply that

$$\sup\{\psi(z,x) : z \in K, \|x\|_q \leq r\} \leq 2[\gamma N(K,r) + \beta M],$$

for every compact set K in D and every $r > 0$. Thus $f(z, x+ \text{ Ker } \|.\|_q)$ is relatively compact in the Stein manifold $\theta^{-1}g(z, \omega_q(x))$ for every $(z,x) \in D \times E$. Hence, by the Liouville theorem, one has

$$f(z,x) = f(z, x + \text{ Ker } \|.\|_q),$$

for every $(z,x) \in D \times E$ and hence the form h, given by

$$h(z, x + \text{ Ker } \|.\|_q) = f(z,x),$$

for every $(z,x) \in D \times E$, defines a function h on $D \times E/\text{ Ker } \|.\|_q$ with values in G, where $E/\text{ Ker } \|.\|_q$ is the image of E under the canonical projection $\omega_q : E \to E_q$. Given $(z_0, \omega_q(x_0)) \in D \times E/\text{ Ker } \|.\|_q$. By the hypothesis there exists a neighbourhood V of $g(z_0, \omega_q(x_0))$ such that $\theta^{-1}(V)$ is a Stein space. Take $\delta > 0$ small enough and W a neighbourhood of z_0 in D such that

$$g(W \times \omega_q(x_0 + \delta U_q)) \subseteq V,$$

where U_q is the open ball defined by the semi-norm $\|.\|_q$. Since

$$\sup\{\varphi f(z, x_0 + \delta x) : z \in W, \|x\|_q \leq 1\} < +\infty,$$

it follows that

$$h(W \times \omega_q(x_0 + \delta U_q)) = f(W \times (x_0 + \delta U_q))$$

is relatively compact in $\theta^{-1}(V)$. Since $\theta^{-1}(V)$ is a Stein and h is Gateaux holomorphic on $D \times E/\operatorname{Ker} \|.\|_q$ we conclude that h is holomorphic on $D \times E/\operatorname{Ker} \|.\|_q$.

(ii) Extend h to a holomorphic function $\hat{h}$ on a neighbourhood Z of $D \times E/\operatorname{Ker} \|.\|_q$ in $D \times E_q$. Consider the domain of existence Z_h of $\hat{h}$ over $D \times E_q$. Since $E/\operatorname{Ker} \|.\|_q$ is dense in E_q, it is easy to see that Z_h is an open set of $D \times E_q$. We prove that Z_h is pseudoconvex. By [8] it suffices to show that Z_h satisfies the weak disc condition. This means that if a sequence $\{\sigma_n\} \subset H(\Delta, Z_h)$, converges to $\sigma \in H(\Delta^*, Z_h)$ in $H(\Delta^*, Z_h)$, the σ can be holomorphically extended to Δ and $\{\sigma_n\}$ converges to σ in $H(\Delta, Z_h)$. Here $H(\Delta, Z_h)$ and $H(\Delta^*, Z_h)$ denote the space of holomorphic maps from the open unit disc Δ in $\mathbf{C}$ (resp., $\Delta^* = \Delta\setminus\{0\}$) into Z_h equipped with the compact-open topology.

First observe that the complex Lie group S satisfies the weak disc condition. This follows from the fact that S is a holomorphic bundle over a commutative Lie group whose fiber are Stein manifolds and that Stein manifolds satisfy the weak disc condition. Since G has a Stein morphism into a complex Lie group, it is easy to see that G satisfies the weak disc condition. Hence the sequence $\{\hat{h}\sigma_n\}$ converges to $\hat{h}\sigma$ in $H(\Delta, G)$. Take a Stein neighbourhood V which can be considered as a closed submanifold of $\mathbf{C}^m$ for some $m \geq 1$ of $\hat{h}\sigma(0)$, and $\varepsilon > 0, N > 0$ such that $\hat{h}\sigma_n(\varepsilon\Delta) \subseteq V$ for every $n > N_0$.

For each $n > N_0$, define a holomorphic function

$$\hat{\sigma}_n : \varepsilon\Delta \longrightarrow \varinjlim_{k\geq 1} H^\infty(W_k, \mathbf{C}^m),$$

by

$$\hat{\sigma}_n(t)(x) = \hat{h}(\sigma_n(t) + x),$$

where $\{W_k\}_{k\geq 1}$ is a basis of neighbourhoods of $0 \in \mathbf{C}^N \times E_q$. It follows that the sequence $\{\hat{\sigma}_n\}$ converges to $\hat{\sigma}$ in $H(\varepsilon\Delta^*, H^\infty(W_k, \mathbf{C}^m))$. Indeed, given K a compact set in $\varphi\Delta^*$ and hence, $\sigma(K)$ is a compact set in Z_h. Then there exists $V \subset Z$ such that $\hat{h}$ is uniform continuous on $\sigma(K) + V$, i. e. for every $\delta > 0$ there exists $V(\delta) \subset V$ such that for $x, y \in \sigma(K) + V$, $x - y \in V(\sigma)$ we have

$$\|h(x) - h(y)\| < \delta.$$

For each $k \geq 1$ and $r > 0$, put

$$U_{kr} = \{f \in H^\infty(W_k, \mathbf{C}^m) : \|f\|_{W_k} \leq \frac{1}{r}\}$$

and consider $\{U_l\}$ with $l : \mathbf{N} \to \mathbf{N}$, defined by

$$U_l = \overline{conv}(\bigcup_{k\geq 1} j_k(U_{k,l(k)})),$$

where $j_k : H^\infty(W_k, \mathbf{C}^m) \to \varinjlim_{k\geq 1} H^\infty(W_k, \mathbf{C}^m)$ is the canonical embedding. It is easy to see that $\{U_l\}$ is a basis of neighbourhoods of 0 in $\varinjlim_{k\geq 1} H^\infty(W_k, \mathbf{C}^m)$. Given a U_l in $\varinjlim_{k\geq 1} H^\infty(W_k, \mathbf{C}^m)$. Take k_0 such that $W_{k_0} \subset V$ and N_0 sufficiently large such that

$$\sigma_n(t) - \sigma(t) \subset W_{k_0},$$
$$\sigma_n(t) - \sigma(t) \subset V(\frac{1}{l(k_0)}),$$

for every $n > N_0$ and all $t \in K$. Thus, for all $n > N_0$ we get $\sigma(t) \in H^\infty(W_{k_0}, \mathbf{C}^m)$ for all $t \in K$ and

$$\sup_{t\in K} \sup_{x\in W_{k_0}} \|h(\sigma_n(t) + x) - h(\sigma(t) + x)\| < \frac{1}{l(k_0)},$$

i.e.

$$\sup_{t\in K} \sup_{x\in W_{k_0}} \|\hat{h}\sigma_n(t)(x) - \hat{\sigma}(t)(x)\| < \frac{1}{l(k_0)}.$$

Then, $\hat{\sigma}_n(t) - \hat{\sigma}(t) \subset U_{k_0,l(k_0)}$ for all $t \in K$. Thus we infer that $\{\hat{\sigma}_n\}$ converges to $\hat{\sigma}$ in $H(\varepsilon\Delta^*, \varinjlim_{k\geq 1} H^\infty(W_k, \mathbf{C}^m))$ and hence $\hat{\sigma}$ can be extended holomorphically to $\varepsilon\Delta$ and $\{\hat{\sigma}_n\}$ converges to $\hat{\sigma}$ in $H(\varepsilon\Delta, \varinjlim_{k\geq 1} H^\infty(W_k, \mathbf{C}^m))$.

Since $\{\hat{\sigma}_n(\varepsilon\Delta/2)\}$ is bounded in $\varinjlim_{k\geq 1} H^\infty(W_k, \mathbf{C}^m)$ and the inductive limit is regular [8] there exists k_1 such that $\hat{\sigma}(t) \in H^\infty(W_{k_1}, \mathbf{C}^m)$, for every $|t| \leq \varepsilon/2$ and every $n > N_0$ Observe that σ can be extended holomorphically to $\varepsilon\Delta$ and $\sigma_n \to \sigma$ in $H(\Delta, D \times E_q)$. It remains to check that $\sigma(0) \in Z_h$. We have $\hat{\sigma}_n(0)(x) = \hat{h}(\sigma_n(0) + x)$ for every $x \in W_{k_1}$ and $n > N_0$. This yields that $\sigma(0) \in Z_h$.

(iii) Since the topology of E is defined by Hilbert semi-norms, without loss of generality we may assume that E_q is a Hilbert space. Choose $q > p$ such that the canonical map $\omega_{qp} : E_q \to E_p$ is compact. Let τ denote the linear metric topology on $H(Z_h)$ generated by the uniform convergence on sets

$$K_r = \{\omega_{qp}(z) : \|z\| \leq r, (x, \omega_{qp}(z)) \in Z_h, \text{ dist } ((x, \omega_{qp}(z)), \partial Z_h) \geq 1/r, x \in D\}.$$

Since the canonical map $[H(Z_h), z] \to H(E)$ is continuous and

$$H(E)_{bor} \cong \varinjlim_k H_b(E_k)$$

(see [4]), where $H(E)_{bor}$ denotes the bornological space associated to $H(E)$ and for every $k \geq 1$, $H_b(E_k)$ denotes the Fréchet space of holomorphic functions on E_k, which are bounded on every bounded set in E_k, we can find $k > q$ such that $H(Z_h) \subseteq H_b(E_k)$. It remains to check that $\text{Im } (id \times \omega_{kp}) \subset Z_h$. In the converse case, there exists $z \in E_k$ such that $(x, \omega_{kp}(z)) \in \partial Z_h$. Choose a sequence $\{z_n\} \subset E/ \text{ Ker } \|.\|_k$ which converges to z. Since E_p is a Hilbert space we can find $f \in H(Z_h)$ such that

$$\sup |f(x, \omega_{kp}(z_n))| = \infty,$$

what is impossible, because $f\omega_p \in H(D \times E_k)$.

b) General case. Let $p_1 > p$ such that $f(D \times U_{p_1} \cap E)$ is contained and relatively compact in a Stein open subset of G, where $U_{p_1} = \{x \in E : \|x\|_{p_1} \leq 1\}$. By the Liouville theorem, it follows that

$$f(z, x + \text{ Ker } \|.\|_{p_1}) = f(z, x),$$

for $(z, x) \in D \times U_{p_1} \cap E$. Hence, the unique principle implies that the relation holds for all $(z, x) \in D \times E$. As in [8] put

$$J = \{j \in \mathbb{N} : \|e_j\|_{p_1} \neq 0\},$$

and write

$$E = E^1 \oplus E^2,$$

where E^1 is the subspace of E with a Schauder basis $\{e_j, j \in J\}$ and a continuous norm $\|.\|_{p_1}|_{E^1}$, and $E^2 = \text{ Ker } \|.\|_{p_1}$.

Using a) to $f|_{D\times E^1}$, we can find $k > p_1$ and a holomorphic function h^1 on $D \times E_k^1$ with values in G such that

$$f|_{D\times E^1} = h^1(id \times \omega_q)|_{D\times E^1}.$$

It is easy to see that $E_k = E_k^1 \oplus E_k^2$. Consequently, setting $h(z,x) = h^1(z,x^1)$ for $x = (x^1, x^2) \in E_k$, we get a holomorphic function h on $D \times E_k$, with values in G, for which $f = h(id \times \omega_k)$.

2) To complete the proof it remains to check that the necessity of the theorem holds for every complex Lie group. Given G a complex Lie group. By virtue of [11] there exists a Stein morphism θ from G onto a torus S. Since S has an universal cover which is a Euclidean space, by Meise and Vogt [5] the necessity of the theorem holds for S and hence holds for G. The theorem is completely proved.

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