ACTA MATHEMATICA VIETNAMICA
Volume 20, Number 1, 1995, pp. 43-53

THE IRREDUCIBLE MODULAR REPRESENTATIONS OF SEMIGROUPS OF ALL MATRICES

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Abstract. Let $M_n = M(n, F_q)$ be the semigroup of all $n \times n$ matrices over the field F_q of q elements. By using Dickson's invariants we construct a complete set of q^n distinct irreducible $F_q[M_n]$ modules, called H_β and give isomorphisms between them and the modules $F^{\underline{\alpha}}$ which were constructed in [3] by the "Weyl module" construction.

1. Introduction.

Let p be a prime number and F_p the field of p elements. Let $H \cong (F_p)^n$ be a p-abelian group, BH_+ its classifying space with a disjoint basepoint. The problem of finding a stable splitting BH_+ into indecomposable wedge summands leads to the study of modular representations of $M(n, F_p)$, where $M(n, F_p)$ is the semigroup of all matrices over F_p.

A decomposition of the identity in $F_p[M(n, F_p)]$ into orthogonal idempotents is shown to induce a stable splitting of BH_+. It is also known that up to homotopy type, the summands constructed from a primitive orthogonal idempotent decomposition are indecomposable and are in one to one correspondence with the irreducible representations of $M(n, F_p)$. Moreover, the multiplicities of these summands are closely related to the dimensions of corresponding irreducible representations .

Now let $M_n = M(n, F_q)$ be the semigroup of all $n \times n$ matrices over the field F_q of q elements. In [3] Harris and Kuhn noted that the "Weyl module" construction of irreducible $F_q[GL(n, F_q)]$-modules in fact constructs $F_q[M_n]$-modules. They followed the above construction as given by James and Kerber and constructed the complete set of q^n irreducible $F_q[M_n]$-modules, called

Received June 18, 1994; in revised form April 20, 1994.

$F^{(\alpha_1,\dots,\alpha_n)}$, $0 \le \alpha_i - \alpha_{i+1} \le q-1$ for $i = 1,\dots,n-1$ and $0 \le \alpha_n \le q-1$. However, up to now there is no standard method to determine the dimensions of these irreducible $F_q[M_n]$-modules.

In this paper, let $F_q[x_1,\dots,x_n]$ be the commutative polynomial algebra in n indeterminants, $x_1,\dots,x_n$, over F_q and let M_n act on $F_q[x_1,\dots,x_n]$ in the usual way. By using Dickson's invariants we construct a complete set of q^n distinct irreducible $F_q[M_n]$ modules, called $H_{(\beta_1,\dots,\beta_n)}$, $0 \le \beta_i \le q-1$ for $i = 1,\dots,n$. Particularly, when F_q is the field of two elements, the dimensions of modules $H_{(0,\dots,0,1,0,\dots,0)}$, where 1 stands at the i-th position, are determined.

To state the main results we recall that the Dickson invariants $L_n = L_n(x_1, \dots, x_n)$ are defined by

$$L_n = \begin{vmatrix} x_1 & \cdots & x_n \\ x_1^q & \cdots & x_n^q \\ \vdots & \ddots & \vdots \\ x_1^{q^{n-1}} & \cdots & x_n^{q^{n-1}} \end{vmatrix}.$$

Then $\sigma . L_n = (\det \sigma) L_n$ for $\sigma \in M_n$. For $\beta = (\beta_1,\dots,\beta_n)$, we denote

$$L^\beta = \prod_{i=1}^{n} L_i^{\beta_i} \in F_q[x_1,\dots,x_n].$$

THEOREM 1.1. *Let H_β be the $F_q[M_n]$-submodule of $F_q[x_1,\dots,x_n]$ generated by L^β. Then*

$$\{H_\beta : \beta = (\beta_1,\dots,\beta_n) : 0 \le \beta_i \le q-1 \text{ for } i = 1,\dots,n\}$$

is a complete set of q^n distinct irreducible modules.

Denote $GL_n = GL(n, F_q)$. If H is an $F_q[M_n]$-module, we shall denote by $res^{M_n}_{GL_n} H$ the $F_q[GL_n]$-module obtained by restriction of the set of operators on H from $F_q[M_n]$ to $F_q[GL_n]$.

COROLLARY 1.2. *$\{H'_{(\beta_1,\dots,\beta_n)} = res^{M_n}_{GL_n}(H_{(\beta_1,\dots,\beta_n)}) : 0 \le \beta_i \le q-1$ for $i = 1,\dots,n-1$ and $1 \le \beta_n \le q-1\}$ is a set of $(q-1)q^{n-1}$ distinct irreducible $F_q[GL_n]$-modules.*

THEOREM 1.3. *For* $1 \le i \le n, 0 \le \alpha_i - \alpha_{i+1} \le q-1, 0 \le \alpha_n \le q-1$, *we have*

$$F^{(\alpha_1,\dots,\alpha_n)} \cong H_{(\alpha_1-\alpha_2,\dots,\alpha_{n-1}-\alpha_n,\alpha_n)} \text{ as } F_q[M_n]-\textit{modules}.$$

When F_q is the field of two elements we have

PROPOSITION 1.4.

$$\dim\ H_{(0,\dots,0,1,0,\dots,0)} = C_n^i,$$

where 1 stands at the i-th position

ACKNOWLEDGEMENT. I would like to express my warmest thanks to my adviser Professor Huynh Mui for his inspiring guidance and generous help in completing this paper.

2. The "Weyl module" construction.

PROPOSITION 2.1 [3,6.3]. *There are* $F_q[M_n]$*-modules* W^α, *defined for* $\alpha = (\alpha_1,\dots,\alpha_n)$ *with* $\alpha_i - \alpha_{i+1} \ge 0$, $i = 1,\dots,n$ *and* $\alpha_n \ge 0$ *having the following properties:*

i) W^α *is a submodule of* $V^{\otimes m}$, *where* $V = (F_q)^n$ *and* $m = \alpha_1 + \cdots + \alpha_n$.

ii) Let $\alpha + 1 = (\alpha_1+1,\dots,\alpha_n+1)$. *Then* $W^{\alpha+1} = W^\alpha \otimes (det)$.

iii) Let $e_{n-1} = \begin{pmatrix} I_{n-1} & 0 \\ 0 & 0 \end{pmatrix}$. *Then* $e_{n-1}W^{(\alpha_1,\dots,\alpha_{n-1},0)} = W^{(\alpha_1,\dots,\alpha_{n-1})}$ *as* $F_q[M_{n-1}]$*-modules.*

PROPOSITION 2.2 [3,6.4]. *There exists a bilinear form* $\phi^\alpha : W^\alpha \times W^\alpha \longrightarrow F_q$ *such that* $\phi^\alpha(mx,y) = \phi^\alpha(x,m^ty)$ *for* $m \in M_n$, *and* $x,y \in W^\alpha$ *(*m^t *is the transpose of m).*

Let $W^\alpha_\perp = \{\omega \in W^\alpha : \phi^\alpha(\omega,v) = 0 \text{ for all } v \in W^\alpha\}$ and $F^\alpha = W^\alpha / W^\alpha_\perp$. These are $F_q[M_n]$-modules.

PROPOSITION 2.3 [3, 6.6]. *Let* $\alpha = (\alpha_1,\dots,\alpha_n)$ *be such that* $0 \le \alpha_i - \alpha_{i+1} \le q-1$ *for* $i = 1,\dots,n-1$ *and* $0 \le \alpha_n \le q-1$. *Then*

i) $F^{\alpha+1} = F^\alpha \otimes (det)$,

ii) $e_{n-1}F^{(\alpha_1,\dots,\alpha_{n-1},0)} = F^{(\alpha_1,\dots,\alpha_{n-1})}$ *as* $F_q[M_{n-1}]$*-modules.*

THEOREM 2.4 [3,6.1]. *$\{F^{(\alpha_1,\dots,\alpha_n)} : 0 \le \alpha_i - \alpha_{i+1} \le q-1$ for $i = 1,\dots,n-1$ and $0 \le \alpha_n \le q-1\}$ is a complete set of irreducible $F_q[M_n]$-modules.*

3. Proof of Theorem 1.1 and Corollary 1.2.

According to Theorem 2.4, $F_q[M_n]$ has a complete set of q^n irreducible modules. Therefore it suffices to prove that the modules H_β are irreducible and distinct.

Let T_n be the Sylow p-subgroup of GL_n consisting of all upper triangular matrices with 1 on the main diagonal, where p is the characteristic of F_q and B_n the subgroup of GL_n consisting of all upper triangular matrices. For each nonnegative integer m, the homogeneous polynomials of degree m form a subspace P_m in $F_q[x_1,\dots,x_n]$. P_m is a $F_q[M_n]$-submodule of $F_q[x_1,\dots,x_n]$.

If x is a T_n invariant in P_m, then the one dimensional space spanned by x will be an irreducible $F_q[T_n]$ module. In [4] Huynh Mui showed that

$$F_q[x_1,x_2,\dots,x_n]^{T_n} = F_q[V_1,\dots,V_n],$$

where $V_i = V_i(x_1,\dots,x_i) = \prod_{\alpha_1,\dots,\alpha_{i-1}\in F_q}(\alpha_1 x_1 + \dots + \alpha_{i-1}x_{i-1} + x_i)$.

We note that if $\sigma = (a_{jk})$ is an element of B_n then $\sigma.V_i = a_{ii}V_i$ for $i = 1,\dots,n$. Let $i_1,\dots,i_n$ be nonnegative integers and $M_{i_1,\dots,i_n}$ the one-dimensional space spanned by $V_1^{i_1}\cdots V_n^{i_n}$. Then $M_{i_1,\dots,i_n}$ is a $F_q[B_n]$-module. It is obvious that $M_{i_1,\dots,i_n} \cong M_{i'_1,\dots,i'_n}$ as $F_q[B_n]$-modules if and only if $i_j = i'_j \bmod (q-1)$ for $1 \le j \le n$.

Set $L_{j_1,\dots,j_k} = L_k(x_{j_1},\dots,x_{j_k})$. Let $\beta = (\beta_1,\dots,\beta_n)$ with $0 \le \beta_i \le q-1$ for $i = 1,\dots,n$. We define U_β by induction as follows

Set $U_{(\beta_1,0,\dots,0)} = P_{\beta_1}$. Suppose that $U_{(\beta_1,\dots,\beta_{\ell-1},0,\dots,0)}$ are defined for $2 \le \ell \le n$. Then we put

$$U_{(\beta_1,\dots,\beta_\ell,0,\dots,0)} = \sum_{\substack{h_{j_1\dots j_\ell}\ge 0 \\ \sum_{1\le j_1<\dots<j_\ell\le n} h_{j_1\dots j_\ell}=\beta_\ell}} \prod_{1\le j_1<\dots<j_\ell\le n} L_{j_1\dots j_\ell}^{h_{j_1\dots j_\ell}} U_{(\beta_1,\dots,\beta_{\ell-1},0,\dots,0)}.$$

Obviously U_β are $F_q[M_n]$-modules.

LEMMA 3.1. *Let* $\beta = (\beta_1, \dots, \beta_n)$ *with* $0 \le \beta_i \le q-1$ *for* $i = 1, \dots, n$. *Let* $f(x_1, \dots, x_n) \in U_\beta$ *be a* T_n *invariant, then* $f(x_1, \dots, x_n) = aL^\beta$ *for some* $a \in F_q$.

PROOF. Let ℓ be the largest integer in $\{1, \dots, n\}$ such that $\beta_\ell \neq 0$. Since $f(x_1, \dots, x_n)$ is a T_n invariant, $f(x_1, \dots, x_n) \in F_q[V_1, \dots, V_n]$. We note that if $g(x_1, \dots, x_n) \in U_\beta$ and for $1 \le i \le n$, it is easy to show that $\deg_{x_i} g < q^\ell$. Therefore $\deg_{x_i} f < q^\ell$. As $\deg_{x_i} V_i = q^{i-1}$, it implies $f(x_1, \dots, x_n) \in F_q[V_1, \dots, V_\ell]$. Taking $\sigma = \begin{pmatrix} I_\ell & 0 \\ 0 & 0 \end{pmatrix} \in M_n$, we get $\sigma.f(x_1, \dots, x_n) = f(x_1, \dots, x_n)$. On the other hand, $f(x_1, \dots, x_n) \in U_\beta$ so $\sigma.f(x_1, \dots, x_n) = L_\ell^{\beta_\ell}.u'$ for some $u' \in U_{(\beta_1, \dots, \beta_{\ell-1}, 0, \dots, 0)}$. From this we have $f(x_1, \dots, x_n) = L_\ell^{\beta_\ell}.u'$. In [4] Mui showed that $L_\ell = V_1 \cdots V_\ell$. Therefore $L_\ell^{\beta_\ell}$ is a T_n invariant, so u' is also in $U_{(\beta_1, \dots, \beta_{\ell-1}, 0, \dots, 0)}$.

Repeat this procedure for u' and so on. Finally we have

$$f(x_1, \dots, x_n) = a \prod_{i=1}^{\ell} L_i^{\beta_i}$$

for some $a \in F_q$ and the lemma is proved.

We note that $L_i = V_1 \cdots V_i$. Hence

$$L^\beta = L^{\beta_1, \dots, \beta_n} = V_n^{\beta_n} V_{n-1}^{\beta_n + \beta_{n-1}} \cdots V_1^{\beta_n + \dots + \beta_2 + \beta_1}.$$

3.2 THE MODULES H_β ARE IRREDUCIBLE. For fixed $\beta = (\beta_1, \dots, \beta_n)$, note that H_β is a $F_q[M_n]$-submodule of U_β. Let N be a nonzero $F_q[M_n]$-submodule of H_β. We consider N as an $F_q[T_n]$-module by restriction of the set of operators on N from $F_q[M_n]$ to $F_q[T_n]$. Then N contains a one dimensional trivial $F_q[T_n]$-submodule (see [1], Ch.8, Exercise 1) which is spanned by some T_n invariant $f(x_1, \dots, x_n) \neq 0$ in U_β. According to Lemma 3.1 $f(x_1, \dots, x_n) = aL^\beta$ for some $a \neq 0$, $a \in F_q$. Since $aL^\beta \in N$ and N is a $F_q[M_n]$-module, $N = H_\beta$. From this we conclude that H_β is irreducible.

To prove that modules H_β are distinct we need the following notations and lemmas.

Let N_i be the set of elements of M_n with rank $\le i$, for $0 \le i \le n$.

DEFINITION 3.3. *Let M be a $F_q[M_n]$-module, $M \neq \{0\}$. For $0 \le i \le n$, M is i singular if and only if $N_{i-1}M = \{0\}$ and $N_iM \neq \{0\}$.*

LEMMA 3.4. *Let U, V be two $F_q[M_n]$-modules, U is i singular and V is j singular. If $i \neq j$ then $U \not\cong V$ as $F_q[M_n]$-modules.*

PROOF. It immediately follows from the assumptions on U and V.

LEMMA 3.5. *Let $\beta = (\beta_1, \dots, \beta_n)$ with $0 \le \beta_i \le q-1$ for $i = 1, \dots, n$. Then*

(i) $H_{(\beta_1,\dots,\beta_n)} = H_{(\beta_1,\dots,\beta_{n-1},0)} \otimes (det)^{\beta_n}$.

(ii) Let $e_{n-1} = \begin{pmatrix} I_{n-1} & 0 \\ 0 & 0 \end{pmatrix}$. Then $e_{n-1}H_{(\beta_1,\dots,\beta_{n-1},0)} = H_{(\beta_1,\dots,\beta_{n-1})}$ as $F_q[M_{n-1}]$-modules.

PROOF. To prove the assertion i) we note that $L^{(\beta_1,\dots,\beta_n)} = L_n^{\beta_n} L^{(\beta_1,\dots,\beta_{n-1},0)}$. For $\sigma \in M_n$, we have $\sigma.L^{(\beta_1,\dots,\beta_n)} = L_n^{\beta_n}((\det\sigma)^{\beta_n}\sigma.L^{(\beta_1,\dots,\beta_{n-1},0)})$. Therefore

$$H_{(\beta_1,\dots,\beta_n)} \subset L_n^{\beta_n} H_{(\beta_1,\dots,\beta_{n-1},0)}.$$

If rank $\sigma < i$, then $\sigma.x_1, \dots, \sigma.x_i$ is linearly dependent. By definition of L_i, this implies that $\sigma.L_i = L(\sigma.x_1, \dots, \sigma.x_i) = 0$. For $\tau \in M_n$, $\tau.L^{(\beta_1,\dots,\beta_{n-1},0)}) \neq 0$ then the first l columns of the matrix τ are linearly independent, where l is the largest integer in $\{1, \dots, n\}$ such that $\beta_l \neq 0$. Choose $\tau' \in GL_n$ such that the first l columns of matrix τ' equal to the first l columns of τ. Then $\tau'.L^{(\beta_1,\dots,\beta_{n-1},0)}) = \tau.L^{(\beta_1,\dots,\beta_{n-1},0)}$. So

$$L_n^{\beta_n}(\tau.L^{(\beta_1,\dots,\beta_{n-1},0)}) = (\det\tau')^{-\beta_n}(\tau'.L^{(\beta_1,\dots,\beta_{n-1},\beta_n)}).$$

From this it follows that

$$H_{(\beta_1,\dots,\beta_n)} = L_n^{\beta_n} H_{(\beta_1,\dots,\beta_{n-1},0)}.$$

Let $\{u_1, \dots, u_r\}$ be a basis of $H_{(\beta_1,\dots,\beta_{n-1},0)}$, then $\{L_n^{\beta_n}u_1, \dots, L_n^{\beta_n}u_r\}$ is a basis of $H_{(\beta_1,\dots,\beta_n)}$. Define $\eta : H_{(\beta_1,\dots,\beta_n)} \longrightarrow H_{(\beta_1,\dots,\beta_{n-1},0)} \otimes (\det)^{\beta_n}$ on these basis vectors by $\eta(L_n^{\beta_n}u_i) = u_i \otimes \iota$ where $\{\iota\}$ is a basis of $(\det)^{\beta_n}$ and extend to $H_{(\beta_1,\dots,\beta_n)}$ by linearity. Now let $\sigma \in M_n$, $\sigma.u_i = \sum_k \alpha_{ki}u_k$, $\alpha_{ki} \in F_q$. We have

$$\eta(\sigma(L_n^{\beta_n} u_i) = \eta(\det\sigma)^{\beta_n} L_n^{\beta_n}(\sigma.u_i))$$
$$= (\det\sigma)^{\beta_n}(\sum_k(\alpha_{ki}(u_k \otimes \iota)) = (\det\sigma)^{\beta_n}(\sigma.u_i \otimes \iota).$$

On the other hand,

$$\sigma.\eta(L_n^{\beta_n} u_i) = \sigma.(u_i \otimes \iota) = \sigma.u \otimes (\det\sigma)^{\beta_n}\iota.$$

So η is a $F_q[M_n]$-homomorphism. A dimension count then shows that η is a $F_q[M_n]$-isomorphism and the assertion i) follows.

The assertion ii) immediately follows from the definition.

As a Corollary of Lemma 3.5 we have

LEMMA 3.6. *Let $\beta = (\beta_1, \dots, \beta_n)$ with $0 \leq \beta_i \leq q-1$ for $i = 1, \dots, n$ and l the largest integer in $\{1, \dots, n\}$ such that $\beta_l \neq 0$. Then H_β are l singular.*

PROOF. For $\sigma \in N_{l-1}$, as in the proof of Lemma 3.5, $\sigma.L_{j_1,\dots,j_l} = 0$ with $1 \leq j_1 < \cdots < j_l \leq n$. Therefore $N_{l-1}.H_\beta = 0$. On the other hand, taking $e_l = \begin{pmatrix} I_l & 0 \\ 0 & 0 \end{pmatrix}$ we have $e_l L^\beta = e_l L^{(\beta_1,\dots,\beta_l,-,\dots,0)} = L^{(\beta_1,\dots,\beta_l)}$. So $N_l.H_\beta \neq 0$ and the lemma follows.

3.7 THE MODULES H_β ARE DISTINCT. Let H_β, $H_{\beta'}$ as in Theorem 1.1 such that $H_\beta \cong H_{\beta'}$ as $F_q[M_n]$-modules. We have to prove $\beta = \beta'$. Let l be the largest integer in $\{1, \dots, n\}$ such that $\beta_l \neq 0$. Then H_β is l singular by Lemma 3.6. According to Lemma 3.4 $H_{\beta'}$ is also l singular. Therefore $\beta'_l \neq 0$ and $\beta'_i = 0$ for $l < i \leq n$. From the proof in 3.2 we see that H_β and $H_{\beta'}$ contain respectively unique minimal $F_q[B_n]$-submodules $M_{\beta_1+\dots+\beta_l,\dots,\beta_{l-1}+\beta_l,\beta_l}$ and $M_{\beta'_1+\dots+\beta'_l,\dots,\beta'_{l-1}+\beta'_l,\beta'_l}$. So $H_\beta \cong H'_\beta$ as $F_q[B_n]$ modules. Therefore $M_{\beta_1+\dots+\beta_l,\dots,\beta_{l-1}+\beta_l,\beta_l} \cong M_{\beta'_1+\dots+\beta'_l,\dots,\beta'_{l-1}+\beta'_l,\beta'_l}$ as $F_q[B_n]$-modules. Thus $\beta_l = \beta'_l \mod (q-1)$. As $0 < \beta_l, \beta'_l \leq q-1$ it implies $\beta_l = \beta'_l$. Let h be the largest integer $(h < l)$ such that $\beta_h \neq \beta'_h$. It implies that $\beta_h = 0$ and $\beta'_h = q-1$ or $\beta'_h = 0$ and $\beta_h = q-1$.

Suppose that $\beta_h = 0$ and $\beta'_h = q-1$. Let $\eta : H_\beta \cong H'_\beta$ be a $F_q[M_n]$-isomorphism. Then $\eta : M_{\beta_1+\cdots+\beta_l,\dots,\beta_{l-1}+\beta_l,\beta_l} \cong M_{\beta'_1+\cdots+\beta'_l,\dots,\beta'_{l-1}+\beta'_l,\beta'_l}$ as $F_q[B_n]$-modules. Therefore

$$\eta(\prod_{\substack{1\leq i\leq l\\ i\neq h}} L^{\beta_i}_{1,\dots,i}) = aL^{q-1}_{1,\dots,h} \prod_{\substack{1\leq i\leq l\\ i\neq h}} L^{\beta'_i}_{1,\dots,i}$$

for some $a \neq 0$, $a \in F_q$. Taking $\sigma = (a_{i,j}) \in M_n$ such that $a_{k,k} = 1, k \neq h, h+1, a_{h,h} = a_{h+1,h+1} = 0, a_{h,h+1} = -1, a_{h+1,h} = 1$ and $a_{i,j} = 0$ at other positions, we have

$$\eta(\sigma.(\prod_{\substack{1\leq i\leq l\\ i\neq h}} L^{\beta_i}_{1,\dots,i})) = aL^{q-1}_{1,\dots,h-1,h} \prod_{\substack{1\leq i\leq l\\ i\neq h}} L^{\beta'_i}_{1,\dots,i},$$

and

$$\sigma.\eta(\prod_{\substack{1\leq i\leq l\\ i\neq h}} L^{\beta_i}_{1,\dots,i}) = aL^{q-1}_{1,\dots,h-1,h+1} \prod_{\substack{1\leq i\leq l\\ i\nu h}} L^{\beta'_i}_{1,\dots,i}.$$

From this we see that η is not a $F_q[M_n]$-homomorphism. This contradiction shows that there does not exist any h $(h < l)$ such that $\beta_h \neq \beta'_h$. In other words $\beta = \beta'$.

If $\beta_h = q-1$ and $\beta'_h = 0$, the proof is entirely analogous to the above proof and Theorem 1.1 is proved.

3.8 PROOF OF COROLLARY 1.2. Let $H'_{(\beta_1,\dots,\beta_n)}$ be a $F_q[GL_n]$-module as in Corollary 1.2. Then $\sigma \in M_n \backslash GL_n$, $\sigma.L_n = (\det)L_n = 0$. Therefore $\sigma.H_{(\beta_1,\dots,\beta_n)} = 0$. Let N be a $F_q[GL_n]$-submodule of $H'_{(\beta_1,\dots,\beta_n)}$. N can be considered as a $F_q[M_n]$-submodule of $H_{(\beta_1,\dots,\beta_n)}$ by extending the set of operators from $F_q[GL_n]$ to $F_q[M_n]$ with $\sigma \in M_n \setminus GL_n$. Then $\sigma.N = 0$. Since $H_{(\beta_1,\dots,\beta_n)}$ is an irreducible $F_q[M_n]$-module, $N = \{0\}$ or $N = H_{(\beta_1,\dots,\beta_n)}$. On the other hand, $H_{(\beta_1,\dots,\beta_n)} = H'_{(\beta_1,\dots,\beta_n)}$ as F_q- spaces. Hence $N = \{0\}$ or $N = H'_{(\beta_1,\dots,\beta_n)}$. Thus $H'_{(\beta_1,\dots,\beta_n)}$ is irreducible. Now let $H'_\beta \cong H'_{\beta'}$ as $F_q[GL_n]$-modules. Then $H_\beta \cong H_{\beta'}$ as $F_q[M_n]$-modules and therefore $\beta = \beta'$. So the modules $H'_{(\beta_1,\dots,\beta_n)}$ are distinct and the corollary is proved.

4. Proof of Theorem 1.3.

LEMMA 4.1. *Let* $\alpha = (\alpha_1, \dots, \alpha_n)$ *with* $0 \le \alpha_i - \alpha_{i+1} \le q-1$ *for* $i-1, \dots, n-1$ *and* $0 \le \alpha_n \le q-1$. *Let* ℓ *be the largest integer in* $\{1, \dots, n\}$ *such that* $\alpha_\ell \ne 0$. *Then* F^α *defined in Section 2 are* ℓ *singular.*

PROOF. Let $e_i = \begin{pmatrix} I_i & 0 \\ 0 & 0 \end{pmatrix} \in M_n$ for $0 \le i \le n$. By Proposition 2.3, and induction we have $e_\ell F^{(\alpha_1,\dots,\alpha_n)} = F^{(\alpha_1,\dots,\alpha_\ell)}$ and $e_j F^{(\alpha_1,\dots,\alpha_n)} = 0$ for $j < l$. From this it implies $N_\ell F^{(\alpha_1,\dots,\alpha_n)} \ne 0$. On the other hand, for each $\sigma \in N_{\ell-1}$, rank $\sigma = r < \ell$. Then $\sigma = \sigma_1 e_r \sigma_2$ for some $\sigma_1, \sigma_2 \in GL_n$, $\sigma.F^{(\alpha_1,\dots,\alpha_n)} = \sigma_1 e_r \sigma_2.F^{(\alpha_1,\dots,\alpha_n)} = \sigma_1 e_r.F^{(\alpha_1,\dots,\alpha_n)} = 0$. Therefore $N_{\ell-1} F^{(\alpha_1,\dots,\alpha_n)} = 0$ and the lemma follows.

4.2 PROOF OF THEOREM 1.3. Because the modules H_β in Theorem 1.1 and F^α in Theorem 2.4 respectively form a complete set for $F_q[M_n]$, they are isomorphic to each other up to some permutation. By Lemma 3.4 we need only to establish isomorphisms between the ℓ singular modules H_β and the ℓ singular modules F^α for $0 \le \ell \le n$. We shall prove this by induction on the ℓ.

For $\ell = 0$, $H_{(0,\dots,0)}$ and $F^{(0,\dots,0)}$ are nothing but the trivial one-dimensional module. Therefore the theorem is proved. For $\ell > 0$, suppose that the theorem is true for the ℓ' singular modules with $\ell' < \ell$.

Let $f : F^{(\alpha_1,\dots,\alpha_\ell,0\dots,0)} \cong H_{(\beta_1,\dots,\beta_\ell,0\dots,0)}$ be an isomorphism between the ℓ singular modules. Let $e_\ell = \begin{pmatrix} I_\ell & 0 \\ 0 & 0 \end{pmatrix} \in M_n$. Since f is a $F_q[M_n]$- isomorphism we have

$$f : e_\ell F^{(\alpha_1,\dots,\alpha_\ell,0\dots,0)} \cong e_\ell H_{(\beta_1,\dots,\beta_\ell,0\dots,0)}.$$

Then iteratedly applying Proposition 2.3 and Lemma 3.5 we get $f : F^{(\alpha_1,\dots,\alpha_\ell)} \cong H_{(\beta_1,\dots,\beta_\ell)}$. By induction assumption we have

$$F^{(\alpha_1-\alpha_\ell,\dots,\alpha_{\ell-1}-\alpha_\ell,0)} \cong H_{(\alpha_1-\alpha_2,\dots,\alpha_{\ell-1}-\alpha_\ell,0)}.$$

Then

$$F^{(\alpha_1-\alpha_\ell,\dots,\alpha_{\ell-1}-\alpha_\ell,0)} \otimes (\det)^{\alpha_l} \cong H_{(\alpha_1-\alpha_2,\dots,\alpha_{\ell-1}-\alpha_\ell,0)} \otimes (\det)^{\alpha_\ell},$$

$$F^{(\alpha_1,\dots,\alpha_{\ell-1},\alpha_\ell)} \cong H_{(\alpha_1-\alpha_2,\dots,\alpha_{\ell-1}-\alpha_\ell,\alpha_\ell)}.$$

From this we conclude that

$$H_{(\beta_1,\dots,\beta_\ell)} \cong H_{(\alpha_1-\alpha_2,\dots,\alpha_{\ell-1}-\alpha_\ell,\alpha_\ell)}.$$

Therefore $\beta_1 = \alpha_1 - \alpha_2, \dots, \beta_{\ell-1} = \alpha_{\ell-1} - \alpha_\ell, \beta_\ell = \alpha_\ell$ and the theorem is proved.

4.3 Proof of Proposition 1.4. Let U be the F_q-vector space generated by the set

$$\{L_{k_1,\dots,k_i} : 1 \le k_1 < \cdots < k_i \le n\}.$$

Obviously U is a $F_2[M_n]$-module. As $L_i = L_{1,\dots,i} \in U$, we have $H_{(0,\dots,0,1,0,\dots,0)} \subset U$. Let $k_1, \dots, k_i$ be such that $1 \le k_1 < \cdots < k_i \le n$. Take $\sigma = (a_{j\ell}) \in M_n$ such that $a_{jk_j} = 1$, $1 \le j \le i$ and $a_{j\ell} = 0$ at the other positions. Then $\sigma.L_{1,\dots,i} = L_{k_1,\dots,k_i}$, and therefore $H_{(0,\dots,0,1,0,\dots,0)} = U$.

Now we show that the set

$$\{L_{k_1,\dots,k_i} : 1 \le k_1 < \cdots < k_i \le n\}$$

is independent.

Let

$$\sum_{1\le k_1<\cdots<k_i\le n} a_{k_1\dots k_i} L_{k_1,\dots,k_i} = 0.$$

For each tuple $(k_1, \dots, k_i)$ with $1 \le k_1 < \cdots < k_i \le n$, take $\sigma_{k_1\dots k_i} = (a_{uv}) \in M_n$ such that $a_{uv} = 1$ with $u = v = k_1, \dots, k_i$ and $a_{uv} = 0$ at the other positions. Then

$$\sigma_{k_1\dots k_i}.(\sum_{1\le k_1<\cdots<k_i\le n} a_{k_1\dots k_i} L_{k_1,\dots,k_i}) = a_{k_1\dots k_i} L_{k_1,\dots,k_i} = 0.$$

So $a_{k_1\dots k_i} = 0$. Thus the set

$$\{L_{k_1,\dots,k_i} : 1 \le k_1 < \cdots < k_i \le n\}$$

is independent. Therefore

$$\dim H_{(0,\dots,0,1,0,\dots,0)} = \dim U = C_n^i$$

and the proposition is proved.

References

[1] C. W. Curtis and I. Reiner, "Representation Theory of Finite Groups and Associative Algebras," Interscience, New York, 1962.

[2] D. J. Glover, *A Study of Certain Modular Representations*, J. Algebra **51** (1978), 425 - 475.

[3] J. C. Harris and N. J. Kuhn, *Stable decompositions of classiffying spaces of finite abelian p-groups*, Math. Proc. Camb. Phil. Soc. **103** (1988), 427.

[4] Huỳnh Mùi, *Modular invariant theory and cohomology algebras of symmetric groups*, J. Fac. Sci. Univ. Tokyo Sec. IA Math. **22** (1975), 319-369.

[5] G. James and A. Kerber, "The Representation Theory of the Symmetric Group," Addison-Wesley, 1981.

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