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ON THE EXISTENCE OF BAYESIAN ESTIMATES IN NONLINEAR STATISTICAL MODELS WITH COMPACT PARAMETER SPACE

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1. Introduction

The linear statistical models are investigated in a large scale (see [1], [2]). However, there are many problems in physics, engineering, biometrics, and other areas of applications of probability theory and mathematical statistics where the assumption of linearity is not satisfied. Therefore, the investigation of nonlinear statistical models is an important problem in mathematical statistics (see [2], [3], [4]).

Let us consider the following statistical model

$$X = \varphi(\theta) + \varepsilon, \tag{1}$$

where

X is an observed n-dimensional random vector;

ε is a n-dimensional random error vector;

φ is a known function, $\varphi : \Theta \to \mathbf{R}^n$, $\Theta \subset \mathbf{R}^{p'}$ is a subset of a $p-$dimensional Euclidean space.

θ is an unknown parameter, $\theta \in \Theta$.

Model (1) is called a linear model if $\Theta = R^p$, $\varphi(\theta) = A\cdot\theta$, where A is a known $n \times p-$matrix of real coefficients, θ is a $p-$dimensional unknown parameter (see [1]).

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Model (1) is called a nonlinear model if either Θ is a nonlinear subset of $\mathbf{R}^p$ or φ is a nonlinear function (see [2], [4], [5]).

In [4], Bunker investigated the nonlinear models by the method of least squares. In this note, we shall investigate nonlinear models, in which compactness of Θ plays an important role. Namely, assuming that Θ is a compact subset of $\mathbf{R}^p$, we shall prove the existence of Bayesian estimates of the parameter $\theta \in \Theta$ by the functional analysis method (see [6]).

Let us denote by $\mathcal{B}_p$ the σ-algebra of all Borel sets in $\mathbf{R}^p$, $\mathcal{X}$ the range of the observed n-dimensional random vector X; $\mathcal{B}(\mathcal{X})$ is the σ-algebra of all Borel sets in $\mathcal{X}$, and $\overline{K}$ the closure of a set K.

1.1 DEFINITION. A function $h : (\mathcal{X}, \mathcal{B}(\mathcal{X})) \to (\mathbf{R}^p, \mathcal{B}_p)$ is called an estimate of the parameter $\theta \in \Theta \subset \mathbf{R}^p$ if it is a Borel measurable function.

A Borel measurable function h is called bounded if it satisfies the condition

$$\sup_{x \in \mathcal{X}} \| h(x) \|_{\mathbf{R}^p} < +\infty$$

A Borel measurable function $h : (\mathcal{X}, \mathcal{B}(\mathcal{X})) \to (\mathbf{R}^p, \mathcal{B}_p)$ is called an essentially bounded function if there exists a set $B \in \mathcal{B}(\mathcal{X})$ with $\mu(B) = 0$ such that

$$\sup_{x \in \mathcal{X} \setminus B} \| h(x) \|_{\mathbf{R}^p} < +\infty.$$

Let $B(\mathcal{X}, \mathbf{R}^p)$ denote the space of all bounded Borel measurable functions on $\mathcal{X}$. Clearly, it is a Banach space with the norm

$$\| h \|_{B(\mathcal{X}, \mathbf{R}^p)} = \sup_{x \in \mathcal{X}} \| h(x) \|_{\mathbf{R}^p}$$

and it forms a class of estimates of parameter $\theta \in \Theta \subset \mathbf{R}^p$.

Similarly, let $L^\infty(\mu, \mathbf{R}^p)$ denote the space of all Borel essentially bounded functions. It is a Banach space with the norm

$$\| h \|_\infty = \inf_{B:\mu(B)=0} \sup_{x\in\mathcal{X}\setminus B} \| h(x) \|_{\mathbf{R}^p},$$

and it forms a class of estimates of parameter $\theta \in \Theta \subset \mathbf{R}^p$.

1.2 Definition. Let us consider the parameter space $(\Theta, \mathcal{B}(\Theta))$, where Θ is a compact subset of $\mathbf{R}^p$, $\mathcal{B}(\Theta)$ is a σ-algebra in Θ. A probability measure τ in the measurable space $(\Theta, \mathcal{B}(\Theta))$ is called a priori distribution of $\theta \in \Theta \subset \mathbf{R}^p$.

Suppose $H : \mathcal{X} \times \Theta \to R^p \times \Theta$ is a function defined by setting $H(x, \theta) = (h(x), \theta)$ and $L : \mathbf{R}^p \times \Theta \to \bar{\mathbf{R}}^+ = [0, +\infty]$ is a given function. Then the compound function defined by

$$L(h(\cdot), \cdot) = L \circ H : \mathcal{X} \times \Theta \to \bar{\mathbf{R}}^+$$

is called a loss funstion.

Let us consider the measurable spaces $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$, $(\mathbf{R}^p, \mathcal{B}_p)$, $(\Theta, \mathcal{B}(\Theta))$, $(\bar{\mathbf{R}}^+, \mathcal{B}(\bar{\mathbf{R}}^+))$, where $\mathcal{B}(\bar{R}^+)$ is a σ-algebra of all Borel sets in $\bar{\mathbf{R}}^+$. Put

$$\mathcal{A} = \{A \times B : A \in \mathcal{B}(\mathcal{X}), B \in \mathcal{B}(\Theta)\},$$

$$\mathcal{C} = \{C \times B : C \in \mathcal{B}_p, B \in \mathcal{B}(\Theta)\}.$$

Let $\mathcal{B}(\mathcal{X}) \times \mathcal{B}(\Theta)$ denote the σ-algebra generated by $\mathcal{A}$, and $\mathcal{B}_p \times \mathcal{B}(\Theta)$ denote the σ-algebra generated by $\mathcal{C}$. Then we have

1.3 Proposition. *Let L be a $(\mathcal{B}_p \times \mathcal{B}(\Theta), \mathcal{B}(\bar{\mathbf{R}}^+))$-measurable function. Then the loss function $L(h(.), .)$ is a $(\mathcal{B}(\mathcal{X}) \times \mathcal{B}(\Theta), \mathcal{B}(\bar{\mathbf{R}}^+))$-measurable function.*

Recall ([7]) that for a random vector X there exists a conditional regular distribution $P^{X|\theta}$. We shall use the symbol Q_θ to denote $P^{X|\theta}$.

Assume that μ is a σ-finite measure in $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ and $Q_\theta \ll \mu$ for every θ. Then by the Radon-Nikodym theorem there exists a density function

$$f_\theta(x) = \frac{Q_\theta(dx)}{\mu(dx)}.$$

By Proposition 1.3, we can define Bayesian estimates.

1.4. DEFINITION. A functional

$$\psi : B(\mathcal{X}, \mathbf{R}^p) \longrightarrow \overline{\mathbf{R}}^+$$

$$\left(\psi : L^\infty(\mu, \mathbf{R}^p) \longrightarrow \overline{\mathbf{R}}^+\right)$$

is said to be a Bayesian risk function with a priori distribution τ if

$$\psi(h) = L(h(x), \theta) Q_\theta(dx) \tau(d\theta)$$

$$\left(\psi(h) = \int_\Theta \int_\mathcal{X} L(h(x), \theta) f_\theta(x) \mu(dx) \tau(d\theta)\right)$$

An estimate $\hat{h} \in B(\mathcal{X}, \mathbf{R}^p)$ ($\hat{h} \in L^\infty(\mu, \mathbf{R}^p)$) is said to be a Bayesian estimate of the parameter $\theta \in \Theta \subset \mathbf{R}^p$ with a priori distribution τ if

$$\psi(\hat{h}) = \inf_{h \in B(\mathcal{X}, \mathbf{R}^p)} \psi(h)$$

$$\left(\psi(\hat{h}) = \inf_{h \in L^\infty(\mu, \mathbf{R}^p)} \psi(h)\right).$$

2. Existence of Bayesian estimates in nonlinear models

In this section, we shall investigate the existence of Bayesian estimates in $B(\mathcal{X}, \mathbf{R}^p)$ and $L^\infty(\mu, \mathbf{R}^p)$.

2.1 THEOREM. *Let K be a class of all estimates of the parameter $\theta \in \Theta \subset \mathbf{R}^p$ satisfying the following conditions:*

(i) $h(\mathcal{X}) \subset \Theta, \quad \forall h \in K.$

(ii) $\forall \varepsilon > 0, \exists$ *finite partition* $\{E_i\}_{i=1}^m$ *of* $\mathcal{X}$ *and points* $x_i \in E_i$, $i = 1, 2, \ldots, m$ *such that:*

$$\sup_{x \in E_i} \| h(x) - h(x_i) \|_{\mathbf{R}^p} < \varepsilon, \quad \forall h \in K, \ \forall i = 1, 2, \ldots, m.$$

(iii) There exists $C > 0$ *such that:*

$$|L(y, \theta) - L(y', \theta)| \le C \| y - y' \|_{\mathbf{R}^p}, \quad \forall y, y' \in \mathbf{R}^p, \ \forall \theta \in \Theta.$$

Then K *is a relatively compact subset of the space* $B(\mathcal{X}, \mathbf{R}^p)$ *and in the class* $\bar{K}$ *there exists a Bayesian estimate.*

PROOF. Since K is a class of estimates of the parameter $\theta \in \Theta$, it follows that the members of K are Borel measurable functions. Since Θ is a compact subset of $\mathbf{R}^p$, by (i), we have

$$\sup_{x \in \mathcal{X}} \| h(x) \|_{\mathbf{R}^p} < +\infty.$$

Accordingly, $K \subset B(\mathcal{X}, \mathbf{R}^p)$. Now let us consider the function $\phi : B(\mathcal{X}, \mathbf{R}^p) \longrightarrow \mathbf{R}^{p \cdot m}$ defined by $\phi(h) = (h(x_1), h(x_2), \ldots, h(x_m))$. By (i), there exists $C' > 0$ such that

$$\| h(x_i) \|_{\mathbf{R}^p} \le C', \quad \forall i = 1, \ldots, m, \ \forall h \in K.$$

Since

$$\| \phi(h) \|_{\mathbf{R}^{p \cdot m}} = \max_{1 \le i \le m} \| h(x_i) \|_{\mathbf{R}^p},$$

it follows that

$$\| \phi(h) \|_{\mathbf{R}^{p \cdot m}} \le C', \ \forall h \in K.$$

Thus $\phi(K)$ is a bounded set of $\mathbf{R}^{p \cdot m}$. Since $\mathbf{R}^{p \cdot m}$ is a finite dimensional space, then $\phi(K)$ is a totally bounded set of $\mathbf{R}^{p \cdot m}$, i.e. there exists s balls $B(t_j, \varepsilon)$, $j = 1, 2, \ldots, s$ such that

$$\phi(K) \subset \bigcup_{j=1}^{s} B(t_j, \varepsilon), \quad t_j \in \mathbf{R}^{p.m}.$$

It follows that for every $h \in K$, there exists j such that

$$\| \phi(h) - t_j \|_{\mathbf{R}^{p \cdot m}} < \varepsilon.$$

This means that

$$\| h(x_i) - t_{ji} \|_{\mathbf{R}^p} < \varepsilon, \quad \forall i = 1, 2, \ldots, m. \tag{2}$$

Moreover, we can choose the balls $B(t_1, \varepsilon), B(t_2, \varepsilon), \ldots, B(t_s, \varepsilon)$ such that

$$\phi(K) \cap B(t_j, \varepsilon) \neq \emptyset, \quad \forall j = 1, 2, \ldots, s.$$

Consequently, for every $B(t_j, \varepsilon)$ there exists $h_j \in K$, such that $\phi(h_j) \in B(t_j, \varepsilon)$. This shows that $\| \phi(h_j) - t_j \|_{\mathbf{R}^{p.m}} < \varepsilon$, $\forall j = 1, \ldots, s$. Therefore,

$$\| h_j(x_i) - t_{ji} \|_{\mathbf{R}^p} < \varepsilon, \quad \forall i = 1, \ldots, m. \tag{3}$$

From (2) and (3), we obtain

$$\| h(x_i) - h_j(x_i) \|_{\mathbf{R}^p} < 2\varepsilon, \quad \forall i = 1, \ldots, m. \tag{4}$$

Next we want to prove that $K \subset \bigcup_{j=1}^{s} B(h_j, 4\varepsilon)$. Let $h \in K$ be given arbitrarily. For every $i = 1, 2, \ldots, m$ and for any $x \in E_i$, $i = 1, \ldots, m$, we have

$$\| h(x) - h_j(x) \|_{\mathbf{R}^p} \leq \| h(x) - h(x_i) \|_{\mathbf{R}^p} + \| h(x_i) - h_j(x_i) \|_{\mathbf{R}^p} + \| h_j(x_i) - h_j(x) \|_{\mathbf{R}^p} .$$

Since $h, h_j \in K$, by (ii) we obtain,

$$\sup_{x \in E_i} \| h(x) - h(x_i) \|_{\mathbf{R}^p} < \varepsilon, \quad \forall i = 1, \ldots, m. \tag{5}$$

$$\sup_{x \in E_i} \| h_j(x) - h_j(x_i) \|_{\mathbf{R}^p} < \varepsilon, \quad \forall i = 1, \ldots, m. \tag{6}$$

From (4), (5), (6), we have

$$\sup_{x \in E_i} ||h(x) - h_j(x)||_{\mathbf{R}^p} < 4\varepsilon, \quad \forall i = 1, \ldots, m.$$

Finally, for any $x \in \mathcal{X} := \bigcup_{i=1}^{m} E_i$ we obtain that

$$\sup_{x\in\mathcal{X}} \| h(x) - h_j(x) \|_{\mathbf{R}^p} = \sup_{x\in \bigcup_{i=1}^{s} E_i} \| h(x) - h_j(x) \|_{\mathbf{R}^p}$$

$$= \max_{1\le i\le m} \sup_{x\in E_i} \| h(x) - h_j(x) \|_{\mathbf{R}^p} < 4\varepsilon.$$

This means that

$$K \subset \bigcup_{j=1}^{s} B(h_j, 4\varepsilon)$$

and K is a totally bounded set of $B(\mathcal{X}, \mathbf{R}^p)$. Since $B(\mathcal{X}, \mathbf{R}^p)$ is a complete metric space, it follows that K is a relatively compact subset of $B(\mathcal{X}, \mathbf{R}^p)$.

Now, we shall prove that in the class of estimates $\bar{K}$ there exists a Bayesian estimate. First, by virtue of (i) we have $h(\mathcal{X}) \subset \Theta$, $\forall h \in \bar{K}$. Indeed, for any $h \in \bar{K}$ there exists a sequence $(h_n) \subset K$, such that

$$\| h_n - h \|_{B(\mathcal{X},\mathbf{R}^p)} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.$$

Since

$$\| h_n(x) - h(x) \|_{\mathbf{R}^p} \le \| h_n - h \|_{B(\mathcal{X},\mathbf{R}^p)}$$

we obtain that

$$\| h(x) - h_n(x) \|_{\mathbf{R}^p} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.$$

On the other hand, it follows from (i) that $h_n(\mathcal{X}) \subset \Theta$, $\forall n \in N$. Thus, we have

$$\| h_n(x) - h(x) \|_{\mathbf{R}^p} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty$$

and

$$h_n(x) \subset \Theta, \quad \forall n \in N, \ \forall x \in \mathcal{X}.$$

Since Θ is a closed set, we obtain that $h(x) \in \Theta$, $\forall x \in \mathcal{X}$. This means that, $h(\mathcal{X}) \subset \Theta$, $\forall h \in \bar{K}$.

Next, consider the functional $\psi : B(\mathcal{X}, \mathbf{R}^p) \to \bar{R}^+$ defined by

$$\psi(h) = \int_{\Theta}\int_{\mathcal{X}} L(h(x), \theta) Q_\theta(dx) \tau(d\theta).$$

We claim that ψ is a continuous function on $B(\mathcal{X}, R^p)$. Indeed, take any $h \in B(\mathcal{X}, \mathbf{R}^p)$ and suppose that

$$\| h_n - h \|_{B(\mathcal{X},\mathbf{R}^p)} \longrightarrow 0 \quad \text{as } n \to \infty,$$

where $(h_n) \subset B(\mathcal{X}, \mathbf{R}^p)$. Then, by (iii) we obtain that

$$\begin{aligned} |\psi(h_n) - \psi(h)| &\leq \int_\Theta \int_\mathcal{X} |L(h_n(x), \theta) - L(h(x), \theta)|\, Q_\theta(dx)\tau(d\theta) \\ &\leq \int_\Theta \int_\mathcal{X} C. \| h_n(x) - h(x) \|_{\mathbf{R}^p} .Q_\theta(dx)\tau(d\theta) \\ &\leq \int_\Theta \int_\mathcal{X} C. \| h_n - h \|_{B(\mathcal{X},\mathbf{R}^p)} .Q_\theta(dx)\tau(d\theta) \\ &= C. \| h_n - h \|_{B(\mathcal{X},\mathbf{R}^p)} \int_\Theta \int_\mathcal{X} Q_\theta(dx)\tau(d\theta) \\ &= C. \| h_n - h \|_{B(\mathcal{X},\mathbf{R}^p)} \longrightarrow 0 \quad \text{as } n \to \infty. \end{aligned}$$

Thus, ψ is a continuous function on $B(\mathcal{X}, \mathbf{R}^p)$. Hence ψ is a continuous function on compact subset $\bar{K}$ of $B(\mathcal{X}, \mathbf{R}^p)$. Consequently, there exists $\hat{h} \in \bar{K}$ such that

$$\psi(\hat{h}) = \inf_{h \in \bar{K}} \psi(h).$$

By Definition 1.4, $\hat{h}$ is a Bayesian estimate and the proof is complete.

2.2 THEOREM. *Let K be a class of all estimates of the parameter $\theta \in \Theta \subset \mathbf{R}^p$ satisfying the following conditions:*

(i) $h(\mathcal{X}) \subset \Theta$ (mod μ), $\forall h \in K$,

(ii) $\forall \varepsilon > 0, \exists$ *finite partition* $\{E_i\}_{i=1}^m \subset \mathcal{X}$, *and points* $x_i \in E_i, i = 1, \ldots, m$. *such that*

(a) $\exists C' : \| h(x_i) \|_{\mathbf{R}^p} \leq C', \forall h \in K, \forall i = 1, 2, \ldots, m.$

(b) $\forall h \in K, \exists B \in \mathcal{B}(\mathcal{X}), \mu(B) = 0 : \sup_{x \in E_i} \| h(x) - h(x_i) \|_{\mathbf{R}^p} < \varepsilon, \forall i = 1, 2, \ldots, m.$

(iii) There exists $C > 0$ such that:

$$|L(y,\theta) - L(y',\theta)| \leq C \parallel y - y' \parallel_{\mathbf{R}^p}, \ \forall y, y' \in \mathbf{R}^p, \forall \theta \in \Theta.$$

Then K is a relativevity compact subset of the space $L^\infty(\mu, \mathbf{R}^p)$ and in the class $\bar{K}$ there exists a Bayesian estimate.

PROOF. Since Θ is a compact subset of $\mathbf{R}^p$, by (i) we have

$$\sup_{x \in \mathcal{X} \setminus B} ||h(x)||_{R^p} < +\infty,$$

where $B \in \mathcal{B}(\mathcal{X})$, $\mu(B) = 0$, $\forall h \in K$. Therefore $K \subset L^\infty(\mu, \mathbf{R}^p)$. Now, let us consider the function $\Phi : L^\infty(\mu, \mathbf{R}^p) \to \mathbf{R}^{pm}$ defined by

$$\Phi(h) = \big(h(x_1), h(x_2), \cdots, h(x_n)\big), \quad h \in L^\infty(\mu, \mathbf{R}^p).$$

Arguing similarly as in the proof of Theorem 2.1 we can show that for every ball $B(t_j, \varepsilon)$ there exists a function $h_j \in K$, such that

$$\Phi(h_j) \in B(t_j, \varepsilon), \quad \forall j = 1, 2, \cdots, s.$$

Hence,

$$||h(x_i) - h_j(x_i)||_{R^p} < 2\varepsilon, \quad \forall h \in K, \ \forall i = 1, 2, \cdots, m. \tag{7}$$

We claim that $K \subset \bigcup_{j=1}^{s} B(h_j, 4\varepsilon)$. Indeed, let us take any $h \in K$. For every $i = 1, 2, \cdots, m$ and for every $x \in E_i \setminus B$, $\mu(B) = 0$, $i = 1, \cdots, m$, we have

$$||h(x) - h_j(x)||_{\mathbf{R}^p} \leq ||h(x) - h(x_i)||_{\mathbf{R}^p} + ||h(x_i) - h_j(x_i)||_{\mathbf{R}^p} + ||h_j(x_i) - h(x)||_{\mathbf{R}^p}.$$

Since $h, h_j \in K$, by (b) we have

$$\sup_{x \in E_i \setminus B} ||h(x) - h(x_i)||_{\mathbf{R}^p} < \varepsilon, \quad \forall i = 1, \cdots, m. \tag{8}$$

$$\sup_{x \in E_i \setminus B} ||h_j(x) - h_j(x_j)||_{\mathbf{R}^p} < \varepsilon, \quad \forall i = 1, \cdots, m. \tag{9}$$

From (7), (8) and (9), we obtain

$$\sup_{x\in E_i\setminus B} ||h(x)-h_j(x)||_{\mathbf{R}^p} < 4\varepsilon, \quad \forall i=1,\cdots,m.$$

Next, for any $x \in \mathcal{X}\setminus B$, where $\mathcal{X} = \bigcup_{i=1}^{m} E_i$, we have

$$\sup_{\mathcal{X}\setminus B} ||h(x)-h_j(x)||_{\mathbf{R}^p} = \max_{1\le i\le m} \sup_{x\in E_i\setminus B} ||h(x)-h_j(x)||_{\mathbf{R}^p} < 4\varepsilon.$$

Hence,

$$\inf_{B:\mu(B)=0} \sup_{x\in\mathcal{X}\setminus B} ||h(x)-h_j(x)||_{\mathbf{R}^p} < 4\varepsilon.$$

This means that $||h-h_j||_\infty < 4\varepsilon$. Consequently,

$$K \subset \bigcup_{j=1} B(h_j, 4\varepsilon).$$

Thus, K is a totally bounded set of $L^\infty(\mu,\mathbf{R}^p)$. Since $L^\infty(\mu,\mathbf{R}^p)$ is a complete metric space, it follows that K is a relatively compact subset of $L^\infty(\mu,\mathbf{R}^p)$.

Next we shall prove the existence of Bayesian estimates in the class of estimates $\bar{K}$. First, we see that $h(\mathcal{X}) \subset \Theta \pmod{\mu}$, $\forall h \in \bar{K}$. Indeed, for any $h \in \bar{K}$ there is a sequence $(h_n) \subset K$, such that $||h_n - h||_\infty \to 0$ as $n \to \infty$. Since

$$||h_n(x)-h(x)||_{\mathbf{R}^p} \le ||h_n - h||_\infty \pmod{\mu},$$

we obtain that

$$||h_n(x)-h(x)||_{\mathbf{R}^p} \to 0 \pmod{\mu}, \quad \text{as } n\to\infty.$$

On the other hand, it follows from (i) that $h_n(\mathcal{X}) \subset \Theta \pmod{\mu}$, $\forall n \in N$. Let us set

$$A = \{x\in\mathcal{X} : ||h_n(x)-h(x)||_{\mathbf{R}^p} \to 0\}$$

$$B_n = \{x\in\mathcal{X} : h_n(x)\in\theta\},$$

$$B = \{x\in\mathcal{X} : h_n(x)\in\Theta,\ \forall n\in N\} = \bigcap_{n\in N}\{x\in\mathcal{X} : h_n(x)\in\Theta\} = \bigcap_{n\in N} B_n.$$

Then,

$$A \cap B = \{x \in \mathcal{X} : ||h_n(x) - h(x)||_{\mathbf{R}^p} \to 0 \text{ as } n \to \infty, \text{ and } h_n(x) \in \Theta, \ \forall n \in N\}.$$

Since Θ is a closed subset of $\mathbf{R}^p$, we see that, if $x \in A \cap B$, then $h(x) \in \Theta$. This shows that

$$A \cap B \subset \{x \in \mathcal{X} : h(x) \in \Theta\}.$$

Hence $\{h(x) \in \Theta\}^c \subset \{A \cap B\}^c = A^c \cup B^c$. Since $\mu(A^c \cup B^c) \leq \mu(A^c) + \mu(B^c) = 0$, we obtain that

$$\mu(\{h(x) \in \Theta\}^c) = 0.$$

This means that

$$h(\mathcal{X}) \subset \Theta \ (\mathrm{mod}\, \mu), \quad \forall h \in \bar{K}.$$

Finally, consider the function $\psi : L^\infty(\mu, \mathbf{R}^p) \to \bar{\mathbf{R}}^+$ defined by

$$\psi(h) = \int_\Theta \int_{\mathcal{X}} L(h(x), \theta) f_\theta(x) \mu(dx)\, \tau(d\theta).$$

We see that ψ is a continuous function on $L^\infty(\mu, \mathbf{R}^p)$. In fact, take any $h \in L^\infty(\mu, \mathbf{R}^p)$ and suppose that $||h_n - h||_\infty \to 0$, as $n \to \infty$, where $(h_n) \subset L_\infty(\mu, \mathbf{R}^p)$. Then by (iii) we have

$$\begin{aligned}
|\psi(h_n) - \psi(h)| &\leq \int_\Theta \int_{\mathcal{X}} |L(h_n(x), \theta) - L(h(x), \theta)| \cdot f_\theta(x) \mu(dx)\, \tau(d\theta) \\
&\leq C \int_\Theta \int_{\mathcal{X}} ||h_n(x) - h(x)||_{R^p} \cdot f_\theta(x) \mu(dx)\, \tau(d\theta) \\
&\leq C \int_\Theta \int_{\mathcal{X}} ||h_n - h||^\infty \cdot f_\theta(x) \mu(dx)\, \tau(d\theta) \\
&= C||h_n - h||_\infty \int_\Theta \int_{\mathcal{X}} Q_\theta(dx) \tau(d\theta) \\
&= C||h_n - h||_\infty \longrightarrow 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Thus, ψ is a continuous function on $L^\infty(\mu, \mathbf{R}^p)$. Therefore, ψ is a continuous function on the compact subset $\bar{K}$ of $L^\infty(\mu, \mathbf{R}^p)$. Consequently, there exists

$\hat{h} \in \bar{K}$ such that

$$\psi(\hat{h}) = \inf_{h \in \bar{K}} \psi(h).$$

By Definition 1.4, $\hat{h}$ is a Bayesian estimate. The proof is complete.

2.3. REMARK. In [5] we have considered the following model

$$X = A\theta + \varepsilon, \tag{1'}$$

where A is a unknown $n \times p$-matrix of coefficients, θ is a unknown parameter, $\theta \in \theta$, and Θ is a compact subset of $\mathbf{R}^p$. In [5] we proved the existence of Bayesian estimates for local parameter $\theta \in \Theta \subset \mathbf{R}^p$ in the class of estimates $\bar{K} \subset L^\infty(\mu, \mathbf{R}^p)$. Clearly, the model (1') is a particular case of (1).

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