ACTA MATHEMATICA VIETNAMICA
Volume 19, Number 2, 1994, pp. 137-148

ON THE SUBDIFFENTIAL OF AN UPPER ENVELOPE OF CONVEX FUNCTIONS

M. VOLLE

Abstract. We extend a Rockafellar's result for the subdifferential of the upper envelope $j = \sup_{1\leq i\leq n} f$, of a finite collection $f_1, \cdots, f_r$ of convex proper functionals on a locally convex Hausdorff topological space X. Assumming that $f_1, \cdots, f_{n-1}$ are finite and continuous at a point x_0 of X where f_n is finite, we show that, for any point x of X such that $f(x)$ is finite.

$$(*)\qquad \partial f(x) = \text{co}\{\partial f_k(x) : f_k(x) = f(x)\} + \sum_{i=1}^{n} N(\text{dom } f_i, x),$$

where co stands for the convex hull and $N(\text{dom } f_i, x)$ for the normal cone to the domain dom f_i of f_i at x. We also give an application of $(*)$ to asymptotical analysis related to a result by Choquet, and prove that $(*)$ remains true when the epigraph of the Legendre-Fenchel conjugate of f is weak* complete and pointed, and the f_i are lower-semicontinuous.

1. Introduction

Among the classical rules of subdifferential calculus ([4] [12] [13] [14]...) one of the most important occurs when the case of an upper envelope of convex functions is considered. Let us recall that given a Hausdorff locally convex topological space X with dual X^*, $h: X \to \mathbb{R} \cup \{+\infty\}$ a convex function, $x \in \text{dom } h := \{x \in X : h(x) < +\infty\}$, the subdifferential of h at x is defined as follows:

$$\partial h(x) = \{y \in X^*;\ \forall u \in X : h(u) - h(x) \geq \langle u - x, y\rangle\}.$$

We now consider a finite collection $f_1, \cdots, f_n$ of convex functions on X with valued in $\mathbb{R} \cup \{+\infty\}$, and set $f = \sup_{1\leq i\leq n} f_i$ for the upper envelope of the f_i. Assuming that f is finite and continuous at a given point x of X, there exists

Received February 15th, 1994
1991 Mathematics Subject Classification. 49 J 52.
Key words: Subdifferential calculus, convex duality, asymptotical analysis.

a formula for the subdifferential of f at x. This formula (see [4] [6] [16]...) says that

$$\partial f(x) = \text{ co } \{\partial f_i(x);\ f_i(x) = f(x)\}, \tag{1}$$

where co denotes the convex hull. There is a more general formula due to Rockafellar [15, Theorem 4] that requires the nonvoidness of $\partial f_i(x)$ and also a qualification condition depending on x. Our purpose is to show that a general formula holds *at each point of the space*, and without assuming the nonvoidness of $\partial f_i(x)$, under the classical condition below

$$\text{There exists} \quad x_0 \in X \quad \text{such that} \quad f_1 \cdots, f_{n-1} \quad \text{(are finite and)}$$
$$\text{continuous at } x_0 \text{ and } f_n(x_0) \in \mathbb{R}. \tag{2}$$

Very simple examples (see [17]) show that (2) is not sufficient to ensure (1) neither the assumption of [15, Theorem 4]. Nevertheless, (2) leads to the classical Moreau-Rocafellar rule

$$\partial(f_1 + \cdots + f_n)(x) = \partial f_1(x) + \cdots + \partial f_n(x), \tag{3}$$

for all $x \in X$ (see e.g. [6, Theorem 1, p.200]), where the addition in the second member is the algebraic sum of sets.

We are going to show that (2) is also useful for computing the subdifferential of f at any point where f is finite. For doing this, we shall work in the spirit of [15] [11], and also use the normal cone to the domain of a functional (see e.g. [6]). Recall that the normal cone of a convex subset C of a locally convex space U, with dual U^*, at a point $x \in C$ is given by

$$N(C, x) = \{v \in U^*;\ \forall u \in C : \langle u - x, v\rangle \leq 0\}.$$

By introducing the indicator function I_C of C,

$$I_C(u) = 0 \text{ if } u \in C, \text{ and } I_C(u) = +\infty \text{ if } u \in U \setminus C,$$

the normal cone of C at x coincides with the subdifferential of I_C at x :

$$N(C, x) = \partial I_C(x).$$

Given a convex function $g : X \to \mathbb{R} \cup \{+\infty\}$, there is a fundamental relation between the subdifferential of g at $x \in \operatorname{dom} g$ and the normal cone to the epigraph

$$E(g) = \{(x, r) \in X \times \mathbb{R};\ g(x) \leq r\}$$

of g at $(x, g(x))$; namely (see e.g. [3]),

$$\partial g(x) = \{y \in X^*;\ (y, -1) \in N(E(g), (x, g(x)))\}. \tag{4}$$

We need to complete the above relation by the following observations.

LEMMA(see, for instance, Durier [5, Lemma 3]). *Let g be a convex function on X with values in $\mathbb{R} \cup \{+\infty\}, x \in \operatorname{dom} g$ and $(x, t) \in E(g)$. Then, for any $(y, s) \in X^* \times \mathbb{R}$,*

a) $(y, s) \in N(E(g), (x, t)) \Rightarrow s \leq 0$

For any $(y, s) \in X^* \times] - \infty, 0[$,

b) $(y, s) \in N(E(g), (x, t)) \Rightarrow t = g(x)$ and $-\frac{y}{s} \in \partial g(x)$.

For any $y \in X^*$,

c) $(y, 0) \in N(E(g), (x, t)) \Rightarrow y \in N(\operatorname{dom} g, x)$.

PROOF. a) Let $(y, s) \in N(E(g), (x, t))$. As $(x, t + 1)$ belongs to $E(g)$, we have

$$0 \geq \langle x - x, y\rangle + s(t + 1 - t) = s.$$

b) Let $(y, s) \in N(E(g), (x, t))$, with $s < 0$. As $N(E(g), (x, t))$ is a cone, we have $(-\frac{y}{s}, -1) \in N(E(g), (x, t))$. On the other hand, as $(x, g(x)) \in E(g)$.

$$0 \geq \langle x - x, -\frac{y}{s}\rangle - (g(x) - t) = t - g(x),$$

so that $t = g(x)$. Therefore, $(-\frac{y}{s}, -1) \in N(E(g), x, g(x))$, and, by (4), $-\frac{y}{s} \in \partial g(x)$.

c) Let $(y, 0) \in N(E(g), (x, t))$. For any $u \in \operatorname{dom} g, (u, g(u)) \in E(g)$, so that

$$0 \geq \langle u - x, y\rangle + 0(g(u) - t) = \langle u - x, y\rangle,$$

or, in other words, $y \in N(\operatorname{dom} g, x)$.

2. A general formula

Let us return to the convex functions $f_1, \cdots, f_n : X \to \mathbb{R} \cup \{+\infty\}$ verifying the condition (2), and to their upper envelope $f = \sup_{1 \leq i \leq n} f_i$.

THEOREM. *Let $f_1, \cdots, f_n$ be convex functions on the locally convex space X with values in $\mathbb{R} \cup \{+\infty\}$. Assuming the existence of $x_0 \in \operatorname{dom} f_n$ such that $f_1, \ldots, f_{n-1}$ are finite and continuous at x_0, we have, for any $x \in \operatorname{dom} f$,*

$$\partial f(x) = \operatorname{co} \{\partial f_k(x);\ f_k(x) = f(x)\} + \sum_{i=1}^{n} N(\operatorname{dom} f_i, x).$$

PROOF. Condition (2) amounts to saying that

$$\operatorname{int} E(f_1) \cap \ldots \cap \operatorname{int} E(f_{n-1}) \cap E(f_n) \neq \emptyset,$$

and we then have ([17, Proposition 1, p.205], see also (3)) for any $x \in \operatorname{dom} f$,

$$N(E(f), (x, f(x))) = \sum_{i=1}^{n} N(E(f_i), (x, f(x))). \tag{5}$$

Let us take $y \in \partial f(x)$. By (4) and (5), we have

$$(y, -1) = \sum_{i=1}^{n} (y_i, s_i),$$

with $(y_s, s_i) \in N(E(f_i), (x, f(x)))$, for any $i \in \{1, \cdots, n\}$.

By Part a) of the lemma, all the s_i are nonpositive. Hence there exist $p \in \{1, \cdots, n\}$, $i_1, \cdots, i_p \in \{1, \cdots, n\}$, and possibly points $j_1, \cdots, j_{n-p} \in \{1, \cdots, n\}$ such that $s_{i_k} < 0$ for any $k \in \{1, \cdots, p\}$, $\sum_{i=1}^{n} s_{i_k} = -1, s_{j_\ell} = 0$ for any $\ell \in \{1, \cdots, n-p\}$. By Parts b) and c) of the lemma we then have

$$-\frac{y_{i_k}}{s_{i_k}} \in \partial f_{i_k}(x),\ f_{i_k}(x) = f(x) \quad \text{for each} \quad k \in \{1, \cdots, p\},$$

$$y_{j_\ell} \in N(\operatorname{dom} f_{j_\ell}, x) \quad \text{for each} \quad \ell \in \{1, \cdots, n-p\}.$$

Therefore,

$$y \in \sum_{k=1}^{p} -s_{i_k}\left(\frac{y_{i_k}}{-s_{i_k}}\right) + \sum_{\ell-p}^{n-p} N(\operatorname{dom} f_{i_\ell}, x),$$

and, a fortiori,

$$y \in \operatorname{co}\{\partial f_k(x) : f_k(x) = f(x)\} + \sum_{i=1}^{n} N(\operatorname{dom} f_i, x).$$

It turns out that the reverse inclusion

$$\partial f(x) \supset \operatorname{co}\{\partial f_k(x) : f_k(x) = f(x)\} + \sum_{i=1}^{n} N(\operatorname{dom} f_i, x)$$

always holds. To see this, set K to be the set of indices $k \in \{1, \cdots, n\}$ such that $f_k(x) = f(x)$ and consider, for each $k \in K$, $y_k \in \partial f_k(x), \lambda_k \geq 0$ with $\Sigma_{k \in K} \lambda_k = 1$, and also $z_i \in N(\operatorname{dom} f_i, x)$ for each $i \in \{1, \cdots, n\}$. It must be proved that $\sum_{k \in K} \lambda_k y_k + \sum_{i=1}^{n} z_i$ belongs to $\partial f(x)$. Now we have, for any $k \in K, u \in X$,

$$f(u) - f(x) \geq f_k(u) - f_k(x) \geq \langle u - x, y_k \rangle,$$

and consequently,

$$f(u) - f(x) \geq \langle u - x, \sum_{k \in K} \lambda_k y_k \rangle.$$

Moreover, for any $u \in \operatorname{dom} f = \cap_{i=1}^{n} \operatorname{dom} f_i$, we have

$$\langle u - x, z_i \rangle \leq 0, \ \forall i \in \{1, \cdots, n\}.$$

It follows that for any $u \in X$,

$$f(u) - f(u) \geq \langle u - x, \sum_{k \in K} \lambda_k y_k \rangle + \sum_{i=1}^{n} \langle u - x, z_i \rangle.$$

In other words, $\sum_{k \in K} \lambda_k y_k + \sum_{i=1}^{n} z_i \in \partial f(x)$.

REMARK. In the case when $f_1, \cdots, f_n$ are finite and continuous at x, one has $x = \cap_{i=1}^{n} \operatorname{int} \operatorname{dom} f_i$, so that for each $i \in \{1, \cdots, n\}$, $N(\operatorname{dom} f_i, 0) = \{0\}$. We then recover the classical formula (1).

3. Application

Now we give an application of the previous theorem to the asymptotical analysis of a closed convex hull. Given nonvoid closed convex subsets $C_1, \cdots, C_n$ of the locally convex Hausdoff space U with dual U^*, we are going to apply the theorem in the case when $X = U^*$ is equipped with a topology compatible with the duality between U^* and U. We also take $f_i = \mathfrak{S}_{C_i}, 1 \leq i \leq n$, where, for any nonvoid subset A of $U, \mathfrak{S}_A$ denotes the support function of A which is defined for any $v \in U^*$ by

$$\mathfrak{S}_A(v) = \sup\{\langle u, v\rangle;\ u \in A\}.$$

We then have (see e.g. [9])

$$\overline{\text{co}}\ A = \partial \mathfrak{S}_A(0).$$

The domain of $\mathfrak{S}_A$ is the so called barrier of A

$$b(A) := \text{ dom } \mathfrak{S}_A.$$

When A is closed and convex, the negative polar cone of $b(A)$ is known to be the asymptotic cone of A :

$$N(b(A), 0) = \text{ as } A := \bigcap_{\lambda > 0} \lambda(A - a), \quad \text{for every} \quad a \in A.$$

By assuming that

$$\mathfrak{S}_{C_1}, \cdots, \mathfrak{S}_{C_{n-1}} \quad \text{are finite and continuous at a point} \\ \text{where} \quad \mathfrak{S}_{C_n} \quad \text{is finite,} \tag{6}$$

and by virtue of the relation

$$\sup_{1 \leq i \leq n} \mathfrak{S}_{C_i} = \mathfrak{S}_{\overline{\text{co}} \bigcup_{i=1}^n C_i}, \tag{7}$$

the theorem, which we apply at the origin, says that

$$\overline{\text{co}} \bigcup_{i=1}^{n} C_i = \partial(\sup_{1\leq i\leq n} \mathfrak{S}_{C_i})(0) = \text{co}\ \{\bigcup_{i=1}^{n} \partial \mathfrak{S}_{C_i}(0)\} + \sum_{i=1}^{n} N(b(C_i), 0)$$
$$= \text{co}\ \{\bigcup_{i=1}^{n} C_i\} + \sum_{i=1}^{n} \text{ as } C_i.$$

We can then state

COROLLARY 1. *Assume that $C_1, \cdots, C_n$ are closed convex subsets of a locally convex Hausdorff space satisfying the condition (6). Then*

$$\overline{co} \bigcup_{i=1}^{n} C_i = co \bigcup_{i=1}^{n} C_i + \sum_{i=1}^{n} \text{ as } C_i.$$

REMARK. Assuming that U^* is barrelled, (6) is equivalent to (cf. [14])

$$\text{int } b(C_1) \cap \cdots \cap \text{ int } b(C_{n-1}) \cap b(C_n) \neq \emptyset.$$

That is, in fact, equivalent to the following assertion (cf. [7, Proposition 3, p.206]): For any $x_1 \in$ as $C_1, \cdots, x_n \in$ as C_n such that $\sum_{i=1}^{n} x_i = 0$, one has $x_1 = \cdots = x_n = 0$.

The formula given in Corollary 1 has been established by Choquet [2, Corollary 6] in a slightly different context: The closed convex sets $C_1, \cdots, C_n$ were assumed to be included in a given weakly complete pointed (i.e. containing no line) convex set. Let us interpret condition (6) in such a framework. For doing this, we have to use the Legendre-Fenchel transformation. Recall that the Fenchel conjugate of a function $g : U^* \to \mathbb{R} \cup \{+\infty\}$ is given, for any $u \in U$, by

$$g^*(u) = \sup_{v \in U^*} \{\langle u, v\rangle - g(v)\}.$$

Of course, an analogous notion holds for the functions defined on U. By the Moreau-Fenchel duality Theorem, any convex lower-semicontinous functional $f : U \to \mathbb{R} \cup \{+\infty\}$ coincides with its bi-conjugate : $f = f^{**}$. In particular, the

Fenchel transform of the support function of any subset A of U is nothing but the indicator function of the closed convex hull of A :

$$(\mathfrak{S}_A)^* = I_{\overline{co}A}.$$

The proposition below is a step toward the assumption made by Choquet. It involves weakly locally compact pointed closed convex sets and their support functions ([1], [8]).

PROPOSITION 1. *Let $A_1, \cdots, A_n$ be nonvoid subsets of a locally convex space U whose dual U^* is equipped with the Mackey topology. The following properties are equivalent :*

a) $\mathfrak{S}_{A_1}, ..., \mathfrak{S}_{A_n}$ are finite and continuous at a point of U^*,

b) $\overline{co} \bigcup_{i=1}^n A_i$ is locally compact and pointed.

PROOF. Let us observe that a) amounts to saying that $\sup_{1\leq i \leq n} \mathfrak{S}_{A_i}$ is finite and continuous at a point of U^*. Now, by (7) and [1, Corollary 1.15] this property is equivalent to b).

As a consequence of Corollary 1 and Proposition 1 we get

COROLLARY 2. *Let $C_1, \cdots, C_n$ be nonvoid closed convex Hausdorff space. Assume that $C_1, \cdots, C_n$ are included in a weakly locally compact pointed convex set. Then*

$$\overline{co} \bigcup_{i=1}^n C_i = co \bigcup_{i=1}^n C_i + \sum_{i=1}^n as\, C_i.$$

In fact one can also deduce the above corollary from the result of Choquet ([2, Corollary 6]) by noticing the following

PROPOSITION 2. *Every pointed closed locally compact convex subset of a locally convex Hausdorff space is weakly complete.*

PROOF. Let A be a pointed closed locally compact (hence weakly locally compact) convex subset of the locally convex Hausdorff space U. By [1, Corollary 1.15] there exists $v \in U^*$ such that for any $r \in \mathbb{R}, \{a \in A : \langle a, v\rangle \geq r\}$ is weakly compact. Now if we consider a generalized Cauchy sequence $(u)_{i\in I}$ for the weak topology with values in A, there exist $i_0 \in I$ and $r \in \mathbb{R}$ such that

$$\forall i \geq i_0;\ \langle u_i, v\rangle \geq r.$$

Therefore, all the u_i with $i \geq i_0$ belong to a weakly compact subset U and the generalized sequence $(u_i)_{i\in I}$ has a cluster point in A which is also the limit of $(u_i)_{i\in I}$.

It is tempting to apply Choquet's formula to the epigraphs of convex functions for obtaining subdifferential calculus formulas. Such a method has been partially applied in [10] for the subdifferential of the sum of two convex functions. Here we consider the case of the supremum of a finite collection of convex functions.

THEOREM bis. *Let $f_1, \cdots, f_n$ be lower-semicontinuous convex proper functions on the locally convex space X. We assume that the epigraph of the Fenchel transform of the upper envelope $f = \sup_{1\leq i\leq n} f_i$ is pointed and weak* complete. At any point $x \in$ dom f, we then have*

$$\partial f(x) = \text{ co } \{\partial f_k(x) : f_k(x) = f(x)\} + \sum_{i=1}^{n} N(dom\ f_i, x).$$

PROOF. In the proof of Theorem, we have yet observed that the inclusion $\supset$ always holds. Let us prove the other inclusion. Take $x \in \operatorname{dom} f$ and $y \in \partial f(x)$. The functional $f^* - \langle x, \cdot\rangle$ is bounded from below on X^* by the real number $-f(x)$ and reaches its infimum at the point y. It follows that

$$(y, -f(x)) \in E(f^* - \langle x, \cdot\rangle). \tag{8}$$

Let us notice that $E(f^* - \langle x, \cdot\rangle)$ coincides with the weak* closed convex hull of the $E(f_i^* - \langle x, \cdot\rangle)$, $1 \leq i \leq n$. Now, as $E(f^*)$ is weak* complete and pointed, the same is true for $E(f^* - \langle x, \cdot\rangle)$. By [2, Corollary 6] we then have

$$E(f^* - \langle x, \cdot\rangle) = \text{co} \bigcup_{1\leq i\leq n} E(f_i^* - \langle x, \cdot\rangle) + \sum_{i=1}^{n} \text{as } E(f_i^* - \langle x, \cdot\rangle). \tag{9}$$

At this stage let us recall (see e.g. [9]) that setting $\varphi_i := f_i^* - \langle x, \cdot \rangle$ for each $i \in \{1, \cdots, n\}$, as $E(\varphi_i)$ is the epigraph of the asymptotic functional

$$\text{as}\, \varphi_i := \mathfrak{S}_{\operatorname{dom} \varphi_i^*}. \tag{10}$$

By the lower-semicontinuity of the f_i we also have

$$\varphi_i^*(\cdot) = f_i(x + \cdot) \quad \text{for any} \quad i \in \{1, \cdots, n\}. \tag{11}$$

As $\varphi_1, \cdots, \varphi_n$ are bounded from below (by $-f(x)$), as φ_i takes only non negative values :

$$\varphi_i \geq 0 = \text{ as}\, \varphi_i(0) \quad \text{for any} \quad i \in \{1, \cdots, n\}. \tag{12}$$

From (10), (11) we deduce that

$$\{z \in X^*;\ \text{as}\, \varphi_i(z) = 0\} = N(\operatorname{dom} \varphi_i^*, 0) = N(\operatorname{dom} f_i, x). \tag{13}$$

Now, by (8) and (9), there exist $(y_1, s_1), \ldots, (y_n, s_n)$ respectively in $E(\varphi_1), \ldots, E(\varphi_n)$, $\lambda_1 \geq 0, \ldots, \lambda_n \geq 0$, with $\sum_{i=1}^n \lambda_i = 1, (z_1, t_1), \ldots, (z_n, t_n)$ respectively in $E(\text{as}\, \varphi_1), \ldots, E$ (as φ_n) such that

$$(y, -f(x)) = \sum_{i=1}^n \lambda_i (y_i, s_i) + \sum_{i=1}^n (z_i, t_i).$$

In particular,

$$\begin{aligned} -f(x) &= \sum_{i=1}^n \lambda_i s_i + \sum_{i=1}^n t_i \\ &\geq \sum_{i=1}^n \lambda_i \varphi_i(y_i) + \sum_{i=1}^n t_i \\ &\geq \sum_{i=1}^n \lambda_i (f^*(y_i) - \langle x, y_i \rangle) + \sum_{i=1}^n t_i \\ &\geq \sum_{i=1}^n \lambda_i (-f(x)) + \sum_{i=1}^n t_i = -f(x) + \sum_{i=1}^n t_i. \end{aligned}$$

It follows that $\sum_{i=1}^n t_i \leq 0$. But by (12) all the t_i are non negative. Hence we have

$$t_1 = \cdots = t_n = 0,$$

and, by (13),

$$z_i \in N(\operatorname{dom} f_i, x) \quad \text{for any } i = \{1, \cdots, n\}.$$

It remains to show that for any $i = \{1, \cdots, n\}$,

$$\lambda_i > 0 \Rightarrow \varphi_i(y_i) = -f(x).$$

For otherwise we have

$$-f(x) = \sum_{i=1}^{n} \lambda_i s_i \geq \sum_{i=1}^{n} \lambda_i \varphi_i(y_i) > \sum_{i=1}^{n} \lambda_i(-f(x)) = -f(x),$$

which is absurd.

To conclude the proof it suffices to observe that $\varphi_i(y_i) = -f(x)$ entails, by using Fenchel inequality,

$$-f(x) \leq -f_i(x) \leq f_i^*(y_i) - \langle x, y_i \rangle = \varphi_i(y_i) = -f(x),$$

that means

$$f_i(x) = f(x) \quad \text{and} \quad y_i \in \partial f_i(x).$$

ACKNOWLEDGEMENT. The author is grateful to Michel Valadier for helpful remarks.

REFERENCES

[1] C. Castaing, M. Valadier, *Convex Analysis and Measurable Multifunctions*, Lecture Notes in Math. 580, Springer-Verlag, 1977.

[2] G. Choquet, *Ensembles et cônes convexes faiblement complets*, C.R.A.S. Paris **354**(1962), 1908-1910.

[3] F. H. Clarke, *Optimization and Nonsmooth Analysis*, Wiley, 1983.

[4] A. Dubovicki, A. Milyutin, *Extremum problems in the presence of restrictions*, Comput. Math. and Math. Phys. **5**(1965), 1-80.

[5] R. Durier, *On locally polyhedral convex functions*, International Series of Numerical Mathematics, **84** (1988), 55-66, Birkhäuser Verlag.

[6] A. D. Ioffe, V. L. Levin, *Subdifferential of convex functions*, Trans. Moscow Math. Soc. **26**(1972), 1-72.

[7] A. D. Ioffe, V. M. Thihomirov, *Theory of Extremal Problems*, North-Holland, 1979.

[8] J. -L. Joly, *Une famille de topologies et de convergences sur l'ensemble des fonctionnelles convexes*, Thèse d'état, Grenoble, 1970.

[9] P. -J. Laurent, *Approximation et Optimisation*, Hermann, 1972.

[10] C. Lescarret, *Sur la sous-différentiabilité d'une somme de fonctionnelles convexes semi-continuous inférieurement*, C.R.A.S. Paris **262**(1966), 443-446.

[11] B. Mordukhovich, *Nonsmooth analysis with nonconvex generalized differentials and adjoint mappings*, Dokl. Akad. Nauk BSSR **28**(1984), 976-979.

[12] J. -J. Moreau, *Fonctionnelles convexes*, Collège de France 1966.
[13] B. N. Pshenichnyi, *Convex programming in a normalized space*, Cybernetics **1**(1965), 46-57.
[14] R. T. Rockafellar, *Extension of Fenchel's duality theorem for convex function*, Duke Math. J. **33**(1966), 81-89.
[15] R. T. Rockaffelar, *Directionally lipschizian functions and subdifferential calculus*, Proc. London Math. Soc. (3) **39**(1979), 331-355.
[16] M. Valadier, *Sous différentiel d'une borne supérieure et d'une somme continue de fonctions convexes*, C.R.A.S. Paris **268**(1969), 39-42.
[17] M. Volle, *Sous différentiel d'une enveloppe supérieure de fonctions convexes*, C.R.A.S. Paris **317**(1993), 845-849.

UNIVERSITY OF AVIGNON
DEPARTMENT OF MATHEMATICS
33, RUE LOUIS PASTEUR
84000 AVIGNON, FRANCE