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ON SUBGROUPS OF THE GENERAL LINEAR GROUP OVER A COMMUTATIVE VON NEUMANN REGULAR RING

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Abstract. Let R be a commutative von Neumann regular ring. In this paper we study the lattice of subgroups of the general linear group $G = GL_n(R)$ containing the group $D = D_n(R)$ of diagonal matrices. We show that if R satisfies some conditions (see Definition 2.4 in the text), then for every such subgroup H there is a uniquely determined D-net σ of ideals such that $G(\sigma) \leq H \leq N(\sigma)$, where $N(\sigma)$ is the normalizer of a D-net subgroup $G(\sigma)$.

1. Introduction

Let R be an associative ring with identity, $G = GL_n(R)$ the general linear group of matrices of order $n \geq 2$ over R, and $D = D_n(R)$ the subgroup of all invertible diagonal matrices of G. We are interested in intermediate subgroups H between D and G. If R is a field which has at least seven elements, then the lattice of all such subgroups H is described in [2]. Later, this result was extended to semilocal rings (even non-commutative) in [3]. In this paper we consider some other generalizations of the results in [2]. In fact, we study the analogous problem for commutative von Neumann regular rings.

Note that a commutative regular ring R contains only a finite number of idempotents if and only if it is isomorphic to a direct product of a finite number of fields. Therefore, such rings are semilocal and the problem was solved (see [3], Theorem 4).

It is clear that the structure of the lattice of intermediate subgroups H depends essentially on properties of R. Therefore, in this paper we also establish some special properties of commutative von Neumann regular rings which are userful for our purpose.

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As in [2]-[4], the classification of intermediate subgroups H between D and G we are interested in is given in term of the concepts of nets and net subgroups. Namely, under some assumptions, for every such subgroup H there is a uniquely determined D-net of ideals σ of order n such that $G(\sigma) \leq H \leq N(\sigma)$, where $N(\sigma)$ is the normalizer of a D-net subgroup $G(\sigma)$ in G.

In this paper we use the following notaions: R is an associative ring with identity 1; R^* the group of units of R; I the unit matrix (whose order is clear from the context); e_{ij} the matrix with a single nonzero 1 in the (i,j)-position; $t_{ij}(x)$ the elementary transvection $i + xe_{ij}$, $x \in R$, $i \neq j$; $d_r(u)$ the diagonal matrix $I + (u-1)e_{rr}$, $u \in R^*$, $1 \leq r \leq n$.

2. Some results on commutative von Neumann regular rings

The purpose of this section is to introduce von Neumann unit-regular rings, especially, commutative ones. We give a necessary and sufficient condition for subrings of a direct product of fields to be regular. We also introduce a special class of commutative von Neumann regular rings which we need later.

DEFINITION 2.1. *A ring R with identity is von Neumann regular (resp. unit-regular) if for every $x \in R$ there exists $y \in R$ (resp. $y \in R^*$) such that $xyx = x$.*

For von Neumann regular rings we have the following result.

THEOREM 2.1(see [6, Corollary 4.2]). *Any commutative von Neumann regular ring is unit-regular.*

DEFINITION 2.2. *A ring R is Dedekind finite if every one-sided invertible element in R is two-sided invertible.*

It is clear that any von Neumann unit-regular ring is Dedekind finite.

DEFINITION 2.3. *A ring R satisfies the first stable range condition if the following holds: for every x and y in R with $Rx + Ry = R$ there exists t in R such that $x + ty$ is left invertible (i. e. $R(x + ty) = R$).*

Note that in Definition 2.3 the requirement that $x + ty$ must be two-sided invertible is not really strong (see [8, Theorem 2.6]). For brevity, L. N. Vaserstein calls such a ring *B-ring* [8]. For our use we note the following:

THEOREM 2.2(see [8, Theorem 5.3]). *A von Neumann regular ring R is von Neumann unit-regular if and only if it is a B-ring.*

Now let R be a commutative von Neumann regular ring. Since Jacabson radical of R is zero, R is a subring of some direct product of fields. Here we make some remarks on von Neumann regular rings which are subrings of some direct product of fields.

Let R be a subring of a direct product of fields

$$P = \prod_{x \in \Omega} K_\alpha,$$

where K_α are fields for all $\alpha \in \Omega$. Then every element x in P can be written in the form $x = (x_\alpha)$, $x_\alpha \in K_\alpha$ for all $\alpha \in \Omega$. Let $1 = (a_\alpha)$ be the identity of R. it is clear that $a_\alpha = 0$ or $a_\alpha = 1$ for every $\alpha \in \Omega$. Put

$$\Omega_0 = \{\alpha \in \Omega; a_\alpha = 0\}; \quad \Omega_1 = \{\alpha \in \Omega; a_\alpha = 1\}.$$

Then $\Omega = \Omega_0 \cup \Omega_1$ and for any $x = (x_\alpha) \in R$, we have $x_\alpha = 0$ for all $\alpha \in \Omega_0$.

THEOREM 2.3. *Let R be a subring with identity of a direct product of fields*

$$P = \prod_{\alpha \in \Omega} K_\alpha.$$

Then R is von Neumann regular if and only if for any $x = (x_\alpha)$ in R there exists a unit $y = (y_\alpha)$ in R such that $y_\alpha = x_\alpha$ for all $x_\alpha \neq 0$.

PROOF. Suppose that R is von Neumann regular and $x = (x_\alpha)$ is an arbitrary element in R. Since R is commutative, it is unit-regular (Theorem 2.1). We may write x in the form $x = ye$, where e is an idempotent and $y = (y_\alpha)$ is a unit in R. It is clear that $e = (e_\alpha)$ is an idempotent if and only if $e_\alpha = 0$ or $e_\alpha = 1$ for every $\alpha \in \Omega$. Hence $x_\alpha = y_\alpha$ for all $x_\alpha \neq 0$.

Conversely, suppose that for any $x = (x_\alpha)$ in R, there exists a unit $y = (y_\alpha)$ in R such that $y_\alpha = x_\alpha$ for all $x_\alpha \neq 0$. Put $e = xy^{-1} \in R$. It is obvious that $e = (e_\alpha)$, where

$$e_\alpha = \begin{cases} 1 & \text{for } x_\alpha \neq 0; \\ 0 & \text{for } x_\alpha = 0. \end{cases}$$

Therefore, e is an idempotent in R and $x = ye$ is a product of a unit and an idempotent in R. This proves that R is von Neumann regular.

Next, we state an interesting corollary of Theorem 2.3 which plays an important role in the proof of Proposition 2.1 and Lemma 2.1.

COROLLARY 2.1. *Let R be a von Neumann subring with identity of a direct product of fields*

$$P = \prod_{x \in \Omega} K_\alpha.$$

Then an element $x = (x_\alpha)$ in R is invertible if and only if $x_\alpha \neq 0$ for all $\alpha \in \Omega_1$.

DEFINITION 2.4. *Let R be a von Neumann regular ring. We say that R satisfies the condition (Φ) if every sum of squares of units in R is also a unit.*

Note that there are many rings satisfying the condition (Φ). For example, any von Neumann subring of an arbitrary direct product

$$P = \prod_{x \in \Omega} K_\alpha,$$

where $K_\alpha \cong R$ (the field of real numbers) for all $\alpha \in \Omega$, satisfies the condition (Φ).

For commutative von Neumann regular rings satisfying the condition (Φ) we have the following results which are useful for our problem.

PROPOSITION 2.1. *Let R be a commutative von Neumann regular ring. Then the following conditions are equivalent:*

(i) R satisfies the condition (Φ),

(ii) Every residue class field of R satisfies the condition (Φ),

(iii) R is a subring of some direct product

$$P = \prod_{x \in \Omega} K_\alpha,$$

where K_α are fields satisfying the condition (Φ) for all $\alpha \in \Omega$.

PROOF. (i) $\Rightarrow$ (ii). Every commutative von Neumann regular ring is considered as a subring of the direct product

$$P = \prod_{x \in \Omega} K_\alpha,$$

where K_α are residue class fields of R. Furthermore, R contains the identity 1_P and all projections $p_\alpha : R \to K_\alpha$ are projective.

Suppose that there exists $\beta \in \Omega$ such that K_β does not satisfies the condition (Φ), i.e. there are nonzero elements $a_{1\beta}, a_{2\beta}, ..., a_{n\beta}$ in K_β such that

$$a_{1\beta}^2 + \cdots + a_{n\beta}^2 = 0. \tag{1}$$

We will prove that R does not satisfy the condition (Φ). Since the projection p_β is surjective, there exist $x_j = (x_{j\alpha})$ in R ($1 \leq j \leq n$) such that

$$x_{j\beta} = a_{j\beta} \text{ for all } 1 \leq j \leq n. \tag{2}$$

Moreover, by Theorem 2.3 we can find units $y_j = (y_{j\alpha})$ in R ($1 \leq j \leq n$) such that

$$y_{i\alpha} = x_{j\alpha} \text{ for all } x_{j\alpha} \neq 0. \tag{3}$$

Put $y = (y_\alpha) = y_1^2 + \cdots + y_n^2$. It follows from (1) - (3) that

$$y_\beta = a_{1\beta}^2 + \cdots + a_{n\beta}^2 = 0.$$

Therefore y is not a unit in R by Corollary 2.1. This proves that R does not satisfy the condition (Φ).

(ii) $\Rightarrow$ (iii). It is obvious.

(iii) $\Rightarrow$ (i). Let $x_j = (x_{j\alpha})$ ($1 \leq j \leq n$) be arbitrary units in R. Put

$$x = (x_\alpha) = x_1^2 + \cdots + x_n^2.$$

According to Corollary 2.1, $x_{j\alpha} \neq 0$ for all $1 \leq j \leq n$ and $\alpha \in \Omega_1$. Hence $x_\alpha \neq 0$ for all $\alpha \in \Omega_1$ by the condition (Φ). In view of Corollary 2.1 $x \in R$ which proves the propostion.

LEMMA 2.1. *Let R be a commutative von Neumann regular ring satisfying the condition (Φ). Then for any element $x_1, ..., x_n$ in R, there exists an unit u in R such that $x_1 + u, x_2 + u, ..., x_n + u$ are also units.*

PROOF. By Proposition 2.1 R is a subring of some direct product

$$P = \prod_{\alpha \in \Omega} K_\alpha,$$

where K_α are fields satisfying the condition (Φ) for all $\alpha \in \Omega$. On the other hand, note that elements $m1$ (m are nonzero integers) are always invertible in any von Neumann regular ring satisfying the condition (Φ). Put

$$u = (x_1 + 1)^2 + (x_2 + 1)^2 + \cdots + (x_n + 1)^2 + (3/2)^2.$$

Then $u = (u_\alpha)$, where

$$u_\alpha = (x_{1\alpha} + 1)^2 + (x_{2\alpha} + 1)^2 + \cdots + (x_{n\alpha} + 1)^2 + (3/2)^2,$$

for all $\alpha \in \Omega$. Hence, by the condition (Φ) for K_α, we have $u_\alpha \neq 0$ for all $\alpha \in \Omega$. According to Corollary 2.1 u is unit. We have

$$x_j + u = (x_{j\alpha} + u_\alpha),$$

where

$$x_{j\alpha} + u_\alpha = (x_{1\alpha}+1)^2 + (x_{2\alpha}+1)^2 + \cdots + (x_{j\alpha}+3/2)^2 + \cdots + (x_{n\alpha}+1)^2 + 1^2 \neq 0,$$

for all $\alpha \in \Omega$. Therefore, all elements $x_1 + u, x_2 + u, ..., x_n + u$ are invertible in R. This completes the proof of the lemma.

3. Nets and net subgroups over a commutative von Neumann regular ring

Let R be an arbitrary associated ring with identity 1.

DEFINITION 3.1. *For a natural number n we consider a square array $\sigma = (\sigma_{ij})$, $1 \le i, j \le n$, of order n whose elements are two-sided ideals σ_{ij} of R. We call this array a net of ideals in R of order n if*

$$\sigma_{ir}\sigma_{rj} \subseteq \sigma_{ij}, \tag{4}$$

for all values of indices i, j and r. If in addition $\sigma_{ii} = R$ for all $i = 1, 2, \ldots, n$ then a net is called a D-net.

For each net σ we denote by $M(\sigma)$ the following collection

$$M(\sigma) = \{a = (a_{ij});\ a_{ij} \in \sigma_{ij} \text{ for all } i \text{ and } j\}.$$

Note that in view of (4) $M(\sigma)$ is a subring of $M_n(R)$. It is clear that the identity matrix I is contained in $M(\sigma)$ if and only if σ is a D-net. The set $I + M(\sigma) = \{I + a; a \in M(\sigma)\}$ is a multiplicative system. For a D-net σ, $M(\sigma) = I + M(\sigma)$.

DEFINITION 3.2. *Let σ be an arbitrary net of ideals in R of order n. The maximal subgroup of $G = GL_n(R)$ contained in $I + M(\sigma)$ is called the net subgroup of G corresponding to the net σ and is denoted by $G(\sigma)$. If σ is a D-net, then $G(\sigma)$ is called a D-net subgroup.*

LEMMA 3.1. *Let R be a unit-regular ring and σ be an arbitrary net of ideals of order n in R. Then*

$$G(\sigma) = G \cap (I + M(\sigma)).$$

PROOF. Suppose that R is unit-regular. It is clear that R/A is unit-regular for every ideal A of R. Then R/A is a B-ring (Theorem 2.2) and hence for every $n \ge 1$, $M_n(R/A)$ is a B-ring (see [8, Theorem 2.4]). Moreover, it is well-known that $M_n(R/A)$ is regular. Hence $M_n(R/A)$ is unit-regular (Theorem 2.2) and it is Dedekind finite. The proof is now completed by Theorem 1 of [4].

LEMMA 3.2. *Let R be a commutative regular ring whose residue class fields all have characteristic different from 2. Suppose that σ is an arbitrary net of ideals of order n in R. Then a matrix $a = (a_{ij})$ in $GL_n(R)$ is contained in the normalizer $N(\sigma)$ of the net subgroup $G(\sigma)$ if and only if*

$$a_{ir}\sigma_{rs}a'_{sj} \subseteq \sigma_{ij}, \tag{5}$$

for all possible values of the subcripts i, j, r and s (here a'_{ij} are the elements of the inverse matrix $a^{-1} = (a'_{ij})$).

PROOF. Suppose that the inclusions (5) hold for a matrix $a = (a_{ij})$. Let us consider an arbitrary matrix $b = (b_{ij})$ from $G(\sigma)$. Then

$$b_{ij} = \delta_{ij} + b^*_{ij}, \quad b^*_{ij} \in \sigma_{ij}.$$

Put $c = aba^{-1}$. Since

$$c_{ij} = \sum_{r,s} a_{ir}\sigma_{rs}a'_{sj} = \delta_{ij} + \sum_{r,s} a_{ir}b^*_{rs}a'_{sj} \in \delta_{ij} + \sigma_{ij},$$

we have $c \in I + M(\sigma)$. Therefore $c \in G(\sigma)$ which implies $a \in N(\sigma)$.

Conversely, suppose that $a \in N(\sigma)$. We have to prove the inclusion (5) for all i, j, r and s.

If $r \neq s$ and $z \in \sigma_{rs}$, then $t_{rs}(z) \in G(\sigma)$. By our assumption

$$at_{rs}(z)a^{-1} = e + \sum_{i,j} a_{ir}za'_{sj}e_{ij} \in G(\sigma).$$

Hence $a_{ir}za'_{sj} \in \sigma_{ij}$.

If $r = s$, then $\sigma_{rr} = R$. By our assumption and Corollary 2.1 the element 2 is invertible in R. Then, by Theorem 7 of [5] every element of R may be represented as a sum of two units. Now, let us consider an arbitrary element z in R. Then the element $z + 2$ may be written in the form $z + 2 = x' + y'$ with $x', y' \in R^*$. Therefore

$$z = x + y, \text{ with } 1 + x, 1 + y \in R^*, \tag{6}$$

where $x = x' - 1$ and $y = y' - 1$. By (6) it is sufficient to prove that $a_{ir} z a'_{sj} \in \sigma_{ij}$ for every $z \in R$ with $1 + z \in R^*$. It is clear that $d_r(1+z) \in G \cap (e + M(\sigma))$. By Theorem 2.1 and Lemma 3.1 $d_r(1+z) \in G(\sigma)$. Therefore $ad_r(1+z)a^{-1} \in G(\sigma)$. Hence $a_{ir} z a'_{sj} \in \sigma_{ij}$. The proof of the lemma is now completed.

4. Lemmas on transvections

Let R be a unit regular ring, in which the element 2 is invertible. According to Theorem 7 of [5] every element of R is a sum of two units. Let H be a subgroup of $G = GL_n(R)$, containing the group $D = D_n(R)$ of diagonal matrices. For each pair of distinct subcripts i and j we put

$$\sigma_{ij} = \{x \in R; t_{ij}(x) \in H\}.$$

Using the above property for elements in R we can show that all σ_{ij} are ideals in R. Next, it is easy to check that by setting, in addition, $\sigma_{ii} = R$ for all $i = 1, 2, ..., n$, we obtain a D-net $\sigma = (\sigma_{ij})$ of ideals of order n in R. We call it the D-net associated to H. Note that the condition $x \in \sigma_{ij}$ is equivalent to the condition that the transvection $t_{ij}(x)$ is contained in H. In the following we prove two lemmas on transvections which play an important role in the proof of Theorem 5.1.

LEMMA 4.1. *Let R be a unit-regular ring in which the element 2 is invertible and H a subgroup of G containing D. If for some r $(1 \leq r \leq n)$ in the matrix $a = (a_{ij})$ from H all elements a_{rj} with $r \neq j$ are zeros, then $t_{ir}(a_{ir}) \in H$ for all $i \neq r$.*

PROOF. In the inverse matrix $a^{-1} = (a'_{ij})$ we also have $a'_{rj} = 0$ for $j \neq r$ and $a_{rr} a'_{rr} = 1$. Write the identity 1 in the form $1 = u + v$, with $u, v \in R^*$. It is easy to verify the following formula

$$[d_i(u), [a, d_r(v^{-1})]] = t_{ir}(-v a_{ir} u a'_{rr}).$$

Since $a \in H$, it follows $-v a_{ir} u a'_{rr} \in \sigma_{ir}$. Hence $t_{ir}(a_{ir}) \in H$.

Now, let $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ be elements in R such that $y_1x_1 + y_2x_2 + \cdots + y_nx_n = 1$. If for $z \in R$ the element $1+z$ is invertible, then so is the matrix $a(z) = (\delta_{ij} + x_izy_j)$, and its inverse is

$$a(-z(1+z)^{-1}) = (\delta_{ij} - x_iz(1+z)^{-1}y_j).$$

LEMMA 4.2. *Let R be a commutative regular ring satisfying the condition (Φ). If for all $z \in R$ for which $1+z \in R^*$ the matrices $a(z)$ are contained in a subgroup H, $D \leq H \leq G$, then $t_{ij}(x_izy_j) \in H$ (i.e. $x_izy_j \in \sigma_{ij}$) for all $1 \leq i,j \leq n$.*

PROOF. Let us fix a natural number r between 1 and n and set for brevity $x = x_r$ and $y = y_r$. Similar to the proof of Lemma 4 of [3], it is sufficient to show that for any z (with $1+z \in R^*$) R contains elements u, v and w such that the following elements are invertible:

$$u, 1+uz, 1+uxzy; \tag{7}$$

$$v, 1+vz, 1+vxzy, u-v; \tag{8}$$

$$1 - xvz(1+vz)^{-1}y; \tag{9}$$

$$w, 1+w\bar{z}, 1+wx\bar{z}y, 1-w; \tag{10}$$

$$v - uw(1+uxzy)^{-1}(1+x\bar{z}y)(1+wx\bar{z}y)^{-1}(1+vxzy), \tag{11}$$

where $\bar{z} = -vz(1+vz)^{-1}$. Applying Lemma 2.1 to elements z and xzy, we find $u' \in R^*$ such that $u'+z \in R^*$ and $u'+xzy \in R^*$. Putting $u = (u')^{-1}$ we see that elements (7) are invertible.

Similarly, we may apply Lemma 2.1 to elements $z, xzy, -u^{-1}, z - xzy$ in order to find $v' \in R^*$ such that $v'+z, v'+xzy, v'-u^{-1}, v'+z-xzy$ are invertible. If we set $v = (v')^{-1}$, then elements (8) are invertible. On the other hand, since

$$1 - xvz(1+vz)^{-1}y = v(1+vz)^{-1}(v'+z-xzy),$$

the element (9) is also invertible.

Finally, by Lemma 2.1 there exists an element $w' \in R^*$ such that the following elements are invertible:

$$w' + \bar{z}, w' + x\bar{z}y, w' - 1, w' + x\bar{z}y - uv^{-1}(1 + uxzy)^{-1}(1 + x\bar{z}y)(1 + vxzy).$$

Set $w = (w')^{-1}$, then elements (10) are invertible. Since the element (11) may be represented in the form of a product of units in R

$$vw(1 + wx\bar{z}y)^{-1}(w' + x\bar{z}y - uv^{-1}(1 + uxzy)^{-1}(1 + x\bar{z}y)(1 + vxzy)),$$

it is also invertible. The proof of the lemma is now completed.

5. Inclusion in the normalizer

Now we are ready to state the main result of this paper.

THEOREM 5.1. *Let R be a commutative von Neumann regular ring satisfying the condition (Φ). Let H be a subgroup of $GL_n(R)$, $n \geq 2$, containing the group $D_n(R)$ of diagonal matrices. If σ is the D-net associated to H, then*

$$G(\sigma) \leq H \leq N(\sigma), \tag{12}$$

where $N(\sigma)$ is the normalizer of the D-net subgroup $G(\sigma)$. Conversely, if (12) holds for some D-net σ, then σ is the D-net associated to H.

PROOF. Let $a = (a_{ij}) \in H$. By Lemma 3.2 we have to verify that $a_{ir}\sigma_{rs}a'_{sj} \subseteq \sigma_{ij}$ for all r, s and $i \neq j$. First, we consider the case $r = s$. Since R is commutative, it is sufficient to prove that $a_{ir}a'_{rj} \in \sigma_{ij}$. For any $z \in R$ for which $1 + z \in R^*$ the matrix $b = ad_r(1 + z)a^{-1}$ is contained in H, and its elements are determined by the formula $b_{ij} = \delta_{ij} + a_{ir}za'_{rj}$. Note that R is unit-regular (Theorem 2.1). Hence we may apply Lemma 4.2 to $x_i = a_{ir}$ and $y_i = a'_{ri}$ $(1 \leq i \leq n)$ in order to obtain what is needed.

Now let $r \neq s$ and z be an arbitrary element in σ_{rs}. The matrix

$$b = (b_{ij}) = at_{rs}(z)a^{-1} = (\delta_{ij} + a_{ir}za'_{sj})$$

is contained in H, and its inverse is $b^{-1} = (b'_{ij}) = (\delta_{ij} - a_{ir}za'_{sj})$. As above we can show that $b_{ij}b'_{jj} \in \sigma_{ij}$, i. e.

$$a_{ir}za'_{sj}(1 - a_{jr}za'_{sj}) \in \sigma_{ij}. \tag{13}$$

Changing z by $-z$ we also obtain

$$-a_{ir}za'_{sj}(1 + a_{jr}za'_{sj}) \in \sigma_{ij}. \tag{14}$$

From (13) and (14) we have $2a_{ir}za'_{sj} \in \sigma_{ij}$. Since $2 \in R^*$, it follows that $a_{ir}za'_{sj} \in \sigma_{ij}$. The first assertion is proved.

Finally, note that the indentity 1 in any commutative von Neumann regular ring satisfying the condition (Φ) is always represented as a sum of two units. Therefore the last assertion of the theorem is obvious by Proposition 5 of [2].

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