ACTA MATHEMATICA VIETNAMICA
Volume 19, Number 1, 1994, pp. 31–39

A LINEAR PROGRAMMING APPROACH TO SOLVING A JOINTLY CONSTRAINED BILINEAR PROGRAMMING PROBLEM WITH SPECIAL STRUCTURE

TRAN VU THIEU

Abstract. In this paper we shall deal with the following nonconvex optimization problem: (P) Minimize $c^T z$, subject to $z \in D$ and $z_i = x_i y_i$ for all $i = 1, \ldots, p \leq n$, $x \in X, y \in S$, where D, S are polyhedrons in $\mathbb{R}^n$, $\mathbb{R}^p$ respectively, $X = \{x \in \mathbb{R}^p \,|\, 0 < a \leq x \leq A\}$; a, A are p-vectors; c is an n-vector. It is shown that (P) can be reduced to a linear program whose constraints can explicitly be given in some special cases

1. Introduction

In this paper we shall deal with the following nonconvex optimization problem

(P) Minimize $c^T z$, subject to $z \in D$ and

$$z_i = x_i y_i \quad \text{for all } i = 1, \ldots, p \leq n,\ x \in X,\ y \in S, \tag{1}$$

where D, S are polyhedrons in $\mathbb{R}^n, \mathbb{R}^p$ respectively, $X = \{x \in \mathbb{R}^p \,|\, 0 < a \leq x \leq A\}$; a, A are p-vectors; c is an n-vector.

Problem (P) was studied in [2] and [4] when $p = n$ and $S = \{y \in \mathbb{R}^n \,|\, 0 \leq b \leq y \leq B\ , \alpha \leq d^T y \leq \beta\}$; b, B, d are n-vectors; α, β are real numbers. The presence of the constraints $z_i = x_i y_i$ $(i = 1, \ldots, p)$ destroys the linearity of the problem and makes it nonconvex with respect to the variables x and y. In fact (P) can be regarded as a jointly constrained bilinear programming problem (see [1]) and has the rank p structure as defined in [5]. In general, solution techniques developed for global optimization (see e.g. [3]), although of interest by their own right, seem to be inefficient for (P).

Received May 3, 1993
This work was supported in part by the National Basic Research Program in Natural Sciences, Vietnam.

Exploiting the special structure of the problem, we will show in §2 that (P) is equivalent to a linear program of the variable z with the constraint $z \in D$ and other additional constraints on z instead of (1). Also, we will show in §3 that in some special cases the constraints of the equivalent program can explicitly be given. Therefore, instead of solving (P) we can solve its equivalent linear program. After obtaining an optimal solution z to the later, x and y can easily be defined from z by solving a system of linear inequalities or by direct computation in the cases considered in §3. This solution technique is extremely efficient for problems in which n may be large but p is not too big. Since, however, the amount of constraints of the equivalent program tends to quickly increasing along with p, it may not be efficient for problems with larger p. To overcome this drawback a branch-and-bound algorithm for solving (P) will be developed in a subsequent paper.

2. The equivalent program

In this section we will construct a linear program equivalent to (P). Introducing additional variables $x_j = 1, y_j = z_j$ for $j > p$, if necessary, we may assume from now on that $p = n$.

Definition 1. Let $x,\ y,\ z \in \mathbf{R}^n$. We say that z is a component product (more briefly, a product) of x and y if $z_i = x_i y_i$ for all $i = 1, \ldots, n$. For convenience, we shall denote this product by the symbol $\bullet$ and write $z = x \bullet y$.

Definition 2. Let S be a subset of $\mathbf{R}^n$. Given a point $x \in \mathbf{R}^n$, we denote by $x \bullet S$ the set of all products $z = x \bullet y$ with $y \in S$.

We have the following property.

Proposition 1. *For any $x \in \mathbf{R}^n$:*

(i) *$x \bullet S$ is convex if S is convex.*

(ii) *$x \bullet S$ is a polytope (a polyhedron, resp.) if S is so.*

Proof. (i) Assume that S is convex and $z^k = x \bullet y^k$ for some $y^k \in S,\ k = 1, 2$. Consider $z = \lambda z^1 + (1-\lambda)z^2,\ 0 \le \lambda \le 1$. Upon simple computation, we get $z = x \bullet y$ where $y = \lambda y^1 + (1-\lambda)y^2 \in S$ because of the convexity of S. Thus, $x \bullet S$ is convex.

(ii) Assume now that S is a polytope with vertices $y^1, \ldots, y^q$, i.e. $S = co\{y^1, \ldots, y^q\}$, the convex hull of $y^1, \ldots, y^q$. It can easily be verified that $x \bullet S = co\{x \bullet y^1, \ldots, x \bullet y^q\}$ which shows that $x \bullet S$ is also a polytope.

The case where S is a polyhedron is proved in a similar way.

REMARK 1. The converse of the above statement holds if $x_i \neq 0$ for all $i = 1, \ldots, n$, since $S = x^{-1} \bullet (x \bullet S)$, where $x^{-1} = (1/x_1, \ldots, 1/x_n)$.

DEFINITION 3. Let $X \subset \mathbf{R}^n$ be a positive rectangle which is defined by

$$X = \{x \in \mathbf{R}^n \,|\, 0 < a_i \leq x_i \leq A_i, i = 1, \ldots, n\},$$

where a_i and A_i are given positive numbers. Let S be a subset of $\mathbf{R}^n_+$, the non-negative orthant of $\mathbf{R}^n$. We denote by G the set of all products $z = x \bullet y$ with $x \in X, y \in S$ and write $G = X \bullet S$.

It is natural to ask whether G is a convex set (a polyhedron or a polytope) if S is so. The affirmative answer is given in the following

PROPOSITION 2. *Under the stated hypotheses on X and S, $G = X \bullet S$ is convex if S is convex. Furthermore, if S is a polyhedron (a polytope, resp.), G is also a polyhedron (a polytope, resp.).*

PROOF. By Proposition 1 it suffices to show that

$$G = co\left(\bigcup_{x \in vert(X)} x \bullet S\right),$$

where $vert(X)$ denotes the vertex set of X and $co(S)$ stands for the convex hull of S.

To do this let $z \in G$. This means that $z = x \bullet y$ with $x \in X$ and $y \in S$. Since x can be expressed as a convex combination of the vertices of X, $z = x \bullet y$ is a convex combination of $z^k = x^k \bullet y$ with $x^k \in vert(X)$. Hence, $z \in co\left(\bigcup_{x \in vert(X)} x \bullet S\right)$.

Conversely, suppose $z \in co\left(\bigcup_{x \in vert(X)} x \bullet S\right)$. This means that $z = \sum_k \lambda_k x^k \bullet y^k$ with $x^k \in vert(X)$, $y^k \in S, \lambda_k \geq 0, \sum_k \lambda_k = 1$. Denote $z^k =$

$x^k \bullet y^k$. Since $x^k \in X$ and $y_i^k \geq 0$, we have $a_i y_i^k \leq z_i^k \leq A_i y_i^k$ for all $i = 1, \ldots, n$. It follows that

$$\frac{z_i^k}{A_i} \leq y_i^k \leq \frac{z_i^k}{a_i} \text{ for all } i = 1, \ldots, n$$

and, hence,

$$\frac{z_i}{A_i} \leq \sum_k \lambda_k y_i^k \leq \frac{z_i}{a_i} \text{ for all } i = 1, \ldots, n \tag{2}$$

Define $\quad y_i = \sum_k \lambda_k y_i^k, \quad$ and

$$x_i = \begin{cases} z_i / y_i & \text{if } y_i \neq 0, \\ \text{any number in } [a_i, A_i] & \text{if } y_i = 0. \end{cases} \tag{3}$$

for all i=1, ...,n. Obviously, $z = x \bullet y$. From (2), (3) and the convexity of S it follows that $x = (x_1, \ldots, x_n) \in X$, $y = (y_1, \ldots, y_n) \in S$. Thus, $z \in G$, completing the proof.

REMARK 2. It is easy to see that Proposition 2 may be no longer valid if the rectangle X is replaced by any polytope in $\mathbb{R}_+^n$ or if S is not contained in $\mathbb{R}_+^n$.

Assume now that S is a polyhedron in $\mathbb{R}_+^n$. By Proposition 2, $G = X \bullet S$ is also a polyhedron. The question is how to describe G by a system of linear inequalities. To this end, we denote

$$X_k = \{x \in \mathbb{R}^n \,|\, a_k \leq x_k \leq A_k, x_j = 1 \text{ for all } j \neq k\}$$

and define

$$S_0 = S, \ S_k = X_k \bullet S_{k-1}, \ k = 1, \ldots, n.$$

It can easily be seen that $G = S_n$. Therefore, to determine the linear constraints defining G we need only to know how to describe S_k by a system of linear inequalities from the given linear constraints that define S_{k-1}. To do this let S_{k-1} be defined by the system

$$\sum_{j=1}^{n} e_{ij} y_j \leq f_i, \ i \in I_{k-1}, \ y \geq 0. \tag{4}$$

Denote $I_k^+ = \{i \in I_{k-1} \,|\, e_{ik} > 0\}$, $I_k^- = \{i \in I_{k-1} \,|\, e_{ik} < 0\}$ and $I_k^0 = \{i \in I_{k-1} \,|\, e_{ik} = 0\}$. It is easily seen that $y \in S_{k-1}$ if and only if

$$\sum_{j\neq k} e_{ij}y_j \leq f_i,\ i \in I_k^0,\ y \geq 0, \tag{5}$$

$$\max\{(f_i - \sum_{j\neq k} e_{ij}y_j)/e_{ik} \mid i \in I_k^-\} \leq y_k \leq \min\{(f_i - \sum_{j\neq k} e_{ij}y_j)/e_{ik} \mid i \in I_k^+\}. \tag{6}$$

PROPOSITION 3. *S_k is defined by the following system of inequalities*

$$\sum_{j\neq k} e_{ij}z_j \leq f_i,\ i \in I_k^0,\ z \geq 0, \tag{7}$$

$$\frac{z_k}{A_k} \leq \min\{(f_i - \sum_{j\neq k} e_{ij}z_j)/e_{ik} \mid i \in I_k^+\}, \tag{8}$$

$$\max\{(f_i - \sum_{j\neq k} e_{ij}z_j)/e_{ik} \mid i \in I_k^-\} \leq \frac{z_k}{a_k}, \tag{9}$$

$$\max\{(f_i - \sum_{j\neq k} e_{ij}z_j)/e_{ik} \mid i \in I_k^-\} \leq \min\{(f_i - \sum_{j\neq k} e_{ij}z_j)/e_{ik} \mid i \in I_k^+\}. \tag{10}$$

PROOF. Suppose $z \in S_k$. From the definition of S_k and X_k it follows that there exist $x_k \in [a_k, A_k]$ and $y \in S_{k-1}$ such that $z_k = x_k y_k$ and $z_j = y_j$ for all $j \neq k$. From (5), (6) it follows that

$$\sum_{j\neq k} e_{ij}z_j \leq f_i,\ i \in I_k^0,\ z \geq 0,$$

$$\max\{(f_i - \sum_{j\neq k} e_{ij}z_j)/e_{ik} \mid i \in I_k^-\} \leq \frac{z_k}{x_k} \leq \min\{(f_i - \sum_{j\neq k} e_{ij}z_j)/e_{ik} \mid i \in I_k^+\},$$

$$\frac{z_k}{A_k} \leq \frac{z_k}{x_k} \leq \frac{z_k}{a_k},$$

which show that (7)–(10) hold.

Suppose now that z satifies (7)–(10). Relations (8)–(10) show that there exists a number t satisfying

$$\max\{(f_i - \sum_{j\neq k} e_{ij}z_j)/e_{ik} \mid i \in I_k^-\} \leq t \leq \min\{(f_i - \sum_{j\neq k} e_{ij}z_j)/e_{ik} \mid i \in I_k^+\}, \tag{11}$$

$$\frac{z_k}{A_k} \leq t \leq \frac{z_k}{a_k}. \tag{12}$$

Set $y_j = z_j$ for all $j \neq k, y_k = t, x_k = z_k/t$. From (5), (6), (7), (11), (12) it follows that $y \in S_{k-1}, x_k \in [a_k, A_k]$ and hence $z \in S_k$, completing the proof.

REMARK 3. Direct computation shows that the system (7)–(10) is equivalent to the following one which is linear

$$\sum_{j\neq k} e_{ij}z_j \le f_i,\ i \in I_k^0,\ z \ge 0, \tag{13}$$

$$\frac{e_{ik}}{A_k} z_k + \sum_{j\neq k} e_{ij}z_j \le f_i,\ i \in I_k^+, \tag{14}$$

$$\frac{e_{ik}}{a_k} z_k + \sum_{j\neq k} e_{ij}z_j \le f_i,\ i \in I_k^-, \tag{15}$$

$$\sum_{j\neq k} \left(\frac{e_{rj}}{e_{rk}} - \frac{e_{sj}}{e_{sk}}\right) z_j \le \frac{f_r}{e_{rk}} - \frac{f_s}{e_{sk}},\ r \in I_k^+,\ s \in I_k^-. \tag{16}$$

(The first inequality (13) is the same as (5), while the inequalities (14), (15) are obtained from (4) when replacing e_{ik} by e_{ik}/A_k if $i \in I_k^+$ and by e_{ik}/a_k if $i \in I_k^-$. Also, the last inequality (16) follows from (4) by adding the r-th and s-th inequalities ($r \in I_k^+, s \in I_k^-$), divided by e_{rk} and $-e_{sk}$ respectively). Note, however, that the number of inequalities in (16) may be large, although among them there may be redundant constraints for S_k.

The above results show that Problem (P) can be converted into the program

(L) Minimize $c^T z$, subject to $z \in D$ and $z \in G = X \bullet S$,

which is linear since D, S are polyhedrons. Furthermore, the constraints that define G can be computed from the ones defining S by applying n times Proposition 3.

Thus, instead of solving (P) we can solve (L). After obtaining an optimal solution to (L) we can compute two vectors $x \in X, y \in S$ such that $z_i = x_i y_i$ for all $i = 1, \dots, n$ by solving the system

$$y \in S \text{ and } y \in Y_z = \{y \in \mathbb{R}^n |\ \frac{z_i}{A_i} \le y_i \le \frac{z_i}{a_i}, i = 1, \dots, n\}.$$

REMARK 4. If $\min_{y\in S} y_i > 0$ for all $i = 1, \dots, n$, then we can eliminate the variable x from (1) by setting $x_i = z_i/y_i, i = 1, \dots, n$, and Problem (P) can directly be converted into the following one

(Q) Minimize $c^T z$, subject to $z \in D, y \in S$,

$$z_i - A_i y_i \leq 0 \text{ and } a_i y_i - z_i \leq 0 \text{ for all } i = 1, \ldots, n.$$

In this case solving (P) is not matter of concern.

3. Some special cases

In this section we shall examine some special cases of (P) which allow us to describe explicitly the constraints of the equivalent program (L). In addition, this program can efficiently be solved by a certain relaxation of its constraints.

Case 1. S is simply a non-negative rectangle, denoted by Y, which is defined by

$$Y = \{y \in \mathbb{R}^n | 0 \leq b_i \leq y_i \leq B_i, i = 1, \ldots, n\}, \tag{17}$$

where b_i and B_i are given non-negative numbers. Then it can easily be verified that $G = X \bullet Y$ is the rectangle

$$Z = \{z \in \mathbb{R}^n | a_i b_i \leq z_i \leq A_i B_i, i = 1, \ldots, n\}. \tag{18}$$

Case 2. Assume that aside from the constraints in (17), S has a linear constraint of the form

$$e_1 y_1 + \ldots + e_n y_n \leq f. \tag{19}$$

Denote $I^+ = \{i | \ e_i > 0\}$, $I^- = \{i | \ e_i < 0\}$ and $I^0 = \{i | \ e_i = 0\}$. We shall assume that

$$\sum_{i \in I^+} e_i b_i + \sum_{i \in I^-} e_i B_i \leq f, \tag{20}$$

which means that there exists at least a point y satisfying (17) and (19).

For each $I \subset \{1, \ldots, n\}$, $I \cap (I^+ \cup I^-) \neq \emptyset$ we define the constraint

$$\sum_{i \in I^+ \cap I} \frac{e_i}{A_i} z_i + \sum_{i \in I^- \cap I} \frac{e_i}{a_i} z_i \leq f - \sum_{i \in I^+ \setminus I} e_i b_i - \sum_{i \in I^- \setminus I} e_i B_i. \tag{21}$$

Proposition 4. *Assume that S is defined by (17), (19) and Z has the form (18). Then $G = X \bullet Y$ is the set of all $z \in Z$ satisfying all the constraints in (21).*

PROOF. Let $z \in G$, i.e. $z = x \bullet y$ with $x \in X, y \in S$. Obviously $z \in Z$. For $i \in I^+$, since $(z_i/A_i) \leq (z_i/x_i) = y_i$ we have $(e_i/A_i)z_i \leq e_i y_i$. For $i \in I^-$, since $(z_i/a_i) \geq (z_i/x_i) = y_i$ we have $(e_i/a_i)z_i \leq e_i y_i$. Hence, for any $I \subset \{1,\ldots,n\}$, $I \cap (I^+ \cup I^-) \neq \emptyset$ we have

$$\sum_{i\in I^+\cap I} \frac{e_i}{A_i} z_i + \sum_{i\in I^-\cap I} \frac{e_i}{a_i} z_i \leq \sum_{i\in I^+\cap I} e_i y_i + \sum_{i\in I^-\cap I} e_i y_i \leq$$

$$\leq f - \sum_{i\in I^+\setminus I} e_i y_i - \sum_{i\in I^-\setminus I} e_i y_i \leq f - \sum_{i\in I^+\setminus I} e_i b_i - \sum_{i\in I^-\setminus I} e_i B_i.$$

Thus, z satisfies all constraints in (21).

Conversely, suppose that $z \in Z$ and z satisfies all constraints in (21). Setting

$$I_1 = \{i \in I^+ \cup I^0 | \frac{z_i}{A_i} \geq b_i\}, \ I_2 = \{i \in I^- | \frac{z_i}{a_i} \leq B_i\},$$

we define vectors $x \in X$, $y \in S$ such that $z = x \bullet y$ as follows

a) for $i \in I_1$ set $x_i = A_i$, $y_i = z_i/A_i$;
b) for $i \in (I^+ \cup I^0) \setminus I_1$ set $x_i = z_i/b_i$, $y_i = b_i$ (note that $b_i > 0$ for all $i \in (I^+ \cup I^0) \setminus I_1$);
c) for $i \in I_2$ set $x_i = a_i$, $y_i = z_i/a_i$;
d) for $i \in I^- \setminus I_2$ set $x_i = z_i/B_i$, $y_i = B_i$ ($i \in I^- \setminus I_2$ implies $B_i > 0$).

Direct computation shows that $a_i \leq x_i \leq A_i$, $b_i \leq y_i \leq B_i$ and $z_i = x_i y_i$ for all $i = 1,\ldots,n$. If $(I_1 \cap I^+) \cup I_2 = \emptyset$ then from (20) follows $\sum_i e_i y_i \leq f$. Otherwise, since z satisfies the constraint (21) for $I = I_1 \cup I_2$ we have

$$\sum_{i\in I^+\cap I} \frac{e_i}{A_i} z_i + \sum_{i\in I^-\cap I} \frac{e_i}{a_i} z_i \leq f - \sum_{i\in I^+\setminus I} e_i b_i - \sum_{i\in I^-\setminus I} e_i B_i$$

or, equivalently, $\sum_i e_i y_i \leq f$. Thus, $z = x \bullet y$ with $x \in X$, $y \in S$, i.e. $z \in G$, as was to be proved.

Proposition 4 remains valid if instead of (19) we consider the constraint

$$e_1 y_1 + \ldots + e_n y_n \geq g, \tag{19'}$$

provided that (20) and (21) are respectively replaced by

$$\sum_{i\in I^+} e_i B_i + \sum_{i\in I^-} e_i b_i \geq g, \tag{20'}$$

$$\sum_{i\in I^+\cap I} \frac{e_i}{a_i} z_i + \sum_{i\in I^-\cap I} \frac{e_i}{A_i} z_i \geq g - \sum_{i\in I^+\setminus I} e_i B_i - \sum_{i\in I^-\setminus I} e_i b_i. \qquad (21')$$

Case 3. S is defined by (17) and by a pair of constraints of the form

$$g \leq e_1 y_1 + \cdots + e_n y_n \leq f. \qquad (22)$$

PROPOSITION 5. *(see [4]) Let S be defined by (17) and (22). Then $G = X \bullet Y$ is the set of all $z \in Z$ satisfying constraints (21) and (21') for all $I \subset \{1, \ldots, n\}$, $I \cap (I^+ \cup I^-) \neq \emptyset$.*

In this special case a finite relaxation algorithm was developed in [4] for solving (P). It heavily relies on the specific structure of S and cannot be extended to the case considered in this paper.

REFERENCES

[1] F. Al- Khayyal and J.E. Falk, *Jointly constrained biconvex programming*, Mathematics of Operations Research **8** (1983), 273–286.

[2] N.C. Binh and T.C. Tu, *Un algorithme de résolution d'un problème de programmation non linéaire*, Séminaire d'Analyse Convexe. Montpellier. Exposé n^o3 (1987).

[3] R. Horst and H. Tuy, "Global Optimization. Deterministic Approaches," Springer–Verlag, Berlin–Heidelberg–New York, 1990.

[4] T.V. Thieu, *A note on the solution of a special class of nonconvex optimization problems*, Journal of Mathematics **22** (1 & 2) (1994), 38–46.

[5]. Y. Yajima and H. Konno, *Efficient algorithms for solving rank two and rank three bilinear programming problems*, Journal of Global Optimization **1** (1991), 155–171.

INSTITUTE OF MATHEMATICS
P.O. BOX 631 BO HO, HANOI, VIETNAM