ACTA MATHEMATICA VIETNAMICA
Volume 19, Number 1, 1994, pp. 25-30

VERTEXSET CONTAINED IN LONGEST DOMINATING CYCLES IN GRAPHS

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Abstract. Let G be an undirected and simple graph. A cycle C in G is called a *dominating cycle* if $V(G) - V(C)$ is an independent set of vertices in G. G is called *dominable* if G contains a dominating cycle C, and we say G is *dominated* by C. In this paper we establish sets of vertices which are contained in any longest dominating cycle in G.

Introduction

We consider only finite undirected graphs without loops or multiple edges. Our terminology and notation are standard except as indicated. A good reference for undefined term is [2]. Herein $V(G), \alpha$ and $\omega(G)$ will denote the vertex set, the independence number and the number of components of a graph G, respectively. δ will denote the minimum valence of the vertices of G. We let $n = |V(G)|$ throughout the paper.

Following Chvátal [4] we define a graph G to be *1-tough* if $\omega(G-S) \leq |S|$ for every subset S of $V(G)$ with $\omega(G-S) > 1$. A cycle C in G is called a *dominating cycle* if $V(G) - V(C)$ consists of independent vertices. G is called a *dominable graph* if G contains a dominating cycle C and we say that G is dominated by C. A hamiltonian cycle is also a dominating cycle. Clearly, 1-tough condition is a necessary condition for the existence of a hamiltonian cycle in a given graph. Dominable graphs are studied by some authors (see [1] - [3]). The length $\ell(C)$ of a longest cycle C in a graph G called the circumference of G will be denoted by $c(G)$. A dominating cycle C is called a longest dominating cycle if for any dominating cycle C' we have $\ell(C) \geq \ell(C')$. The length of a longest dominating cycle in G will be denoted by $c_d(G)$. We know that $c(G)$ and $c_d(G)$ are not always the same and, in fact, the difference $c(G) - c_d(G)$ can be made

Received April 12, 1993; in revised form January 26, 1994

arbitrarily large. Examples for such graphs first appeared in [5]. A cycle C' is called an extended cycle of C if $V(C) \subset V(C')$. Clearly, every extended cycle of a dominating cycle C is also a dominating cycle. A (dominating) cycle C is a maximal (dominating) cycle if there exists no (dominating) cycle C' such that C' is an extended cycle of C. For an integer a we denote $V(\geq a) = \{v \in V(G) | d(v) \geq a\}$ and $n(\geq a) = |V(\geq a)|$. We shall establish some lower bounds for the lengths of maximal dominating cycles in G.

THEOREM 1. *Let G be a dominable graph. Then*

(a) Any dominating cycle C has a length $\ell(C) \geq n - \alpha$.

(b) If C is a maximal dominating cycle and G is 2-connected nonhamiltonian. Then $\ell(C) \geq 4$. Moreover, if G is a 1-tough graph, then $\ell(C) \geq 6$.

(c) $V(\geq \alpha) \subseteq V(C)$ and $\ell(C) \geq \max(n - \alpha, n(\geq \alpha))$ for any maximal dominating cycle C.

For a dominable graph we can establish the vertex set which is contained in every longest dominating cycle.

THEOREM 2. *Let G be a dominable graph with $\delta \geq 2$ and*

$$T = V\left(\geq \max\left\{\frac{n-\delta-1}{2}, \alpha - 1\right\}\right).$$

Then $c_d \geq |T| - 1$ and every longest dominating cycle C avoids at most one vertex of T. If $c_d \leq n-2$, then $c_d \geq |T|$ and $V(C) \supseteq T$ for every longest dominating cycle C in G.

We give an example for an odd number $n \geq 15$ by constructing the graph G_n from $\overline{K}_{\frac{n-1}{2}} \cup K_{\frac{n-5}{2}} \cup K_3$ by joining every vertex in $K_{\frac{n-5}{2}}$ to all vertices in $\overline{K}_{\frac{n-1}{2}} \cup K_3$ and by adding a matching between the vertices in K_3 and three vertices in $\overline{K}_{\frac{n-1}{2}}$. By the graph G_n and by the Petersengraph we can see that Theorem 1 and Theorem 2 are both best possible.

Notations and auxiliary results

If C is a cycle of G with a given orientation, and if u, v are vertices on C, then $u \overrightarrow{C} v$ denotes the consecutive vertices on C from u to v in the direction specified by C. The same vertices, in reverse order, are given by $v \overleftarrow{C} u$. We use u^+ to denote the successor of u on C and u^- to denote its predecessor. If $v \in V(G)$ and $H \subseteq V(G)$, then $N_H(v)$ is the set of all vertices in H adjacent to v. We denote $|N_H(v)|$ by $d_H(v)$. We write short $N(v)$ for $N_G(v)$. If $A \subseteq V(C)$, then $A^+ = \{v^+ | v \in A\}$. The set A^- is analogously defined.

In what follows a maximal dominating cycle C with a direction on C is fixed. Let $v_0 \in V(G) - V(C)$, set $A = N(v_0)$ and let $\{v_1, v_2, ..., v_k\}$ be the vertices of A, occurring on C in consecutive order. A path B joining two different vertices v and u on C is called an arc if $V(B) \cap (V(C) \cup \{v_0\}) = \{u, v\}$. For $i \in \{1, ..., k\}$ we set $u_i = v_i^+$, $w_i = v_{i+1}^-$, $L_i = v_i \overrightarrow{C} w_i$ and $H_i = L_i \cup N(u_i) - \{v_{i+1}\}$.

The following lemmas hold for the case $\ell(C) \leq n - 1$ and will be used to facilitate the proof of Theorem 1 and Theorem 2. In each lemma we assume G is a connected graph and C is a maximal dominating cycle except as indicated.

LEMMA 1. *$N(p) \cap N(p)^+ = N(p) \cap N(p)^- = \emptyset$ for any vertex $p \in V(G) - V(C)$.*

PROOF. Suppose, to the contrary, that there exists some vertex $p \in V(G) - V(C)$ and some $v \in N(p)$ such that $v^+ \in N(p)$. Then $pv^+ \overrightarrow{C} vp$ would be an extended dominating cycle of C, a contradiction. Thus, $N(p) \cap N(p)^+ = \emptyset$. Similarly, $N(p) \cap N(p)^- = \emptyset$.

LEMMA 2. *There exists no arc between the vertices of A^+. Similarly, there exists no arc between the vertices of A^-.*

PROOF. Suppose otherwise that there exists some arc B joining some u_i with $u_j (i \neq j)$. Then $v_0 v_i \overleftarrow{C} u_j B u_i \overrightarrow{C} v_j v_0$ would be an extended cycle of C, a contradiction. Thus, there exists no arc between the vertices of A^+ and, similarly, there exists no arc between the vertices of A^-.

LEMMA 3. *If $u_i = w_i$ and B is an arc joining u_i with a vertex z on C, then $\{v_0, z^+\} \cup A^+$ is an independent set of vertices.*

PROOF. If suffices to show that there exists no edge joining z^+ with any vertex of $\{v_0\} \cup A^+$ since $\{v_0\} \cup A^+$ is an independent set of vertices by Lemma 1 and Lemma 2. But $v_0 z^+ \notin E(G)$ since, otherwise, $v_0 v_{i+1} \overrightarrow{C} z B u_i \overleftarrow{C} z^+ v_0$ would be an extended cycle of C, a contradiction. Now, suppose the otherwise that $z^+u_j \in E(G)$ for some u_j. Then $v_0v_j \overleftarrow{C} z^+u_j \overrightarrow{C} u_iBz \overleftarrow{C} v_{i+1}v_0$ when $u_j \in z \overrightarrow{C} u_i$ and $v_0v_j \overleftarrow{C} u_iBz \overleftarrow{C} u_jz^+ \overrightarrow{C} v_iv_0$ when $u_j \in u_i \overrightarrow{C} z$ would be an extended cycle of C, a contradiction. Thus, Lemma 3 is true.

Proof of theorems

1. PROOF OF THEOREM 1. It is easy to see that (a) of Theorem 1 is trivial and (b) of Theorem 1 follows from Lemma 2. Now, we suppose, to the contrary of (c), that there exists a vertex v with $d(v) \geq \alpha$ and a maximal dominating cycle C such that $v \in V(G) - V(C)$. By Lemma 2, $\{v\} \cup N(v)^+$ is an independent set of at least $\alpha + 1$ elements, a contradiction.

2. PROOF OF THEOREM 2. To prove Theorem 2 it suffices to prove that if there exist a vertex v_0 with $d(v_0) \geq \max\left(\frac{n-\delta-1}{2}, \alpha - 1\right)$ and a longest dominating cycle C such that $v_0 \in V(G) - V(C)$ then $\ell(C) = n - 1$.

Suppose otherwise that there exist a longest dominating cycle C of length $\ell(C) \leq n-2$ and a vertex $v_0 \in V(G) - V(C)$ with $d(v_0) \geq \max\left(\frac{n-\delta-1}{2}, \alpha - 1\right)$. Let $\{p_1, ..., p_t\}$ be the vertices of $V(G) - V(C) - \{v_0\}$ $(t \geq 1)$. By Lemma 2 we can derive the next claim.

CLAIM 1: $d(v_0) = \alpha - 1 \geq \frac{n-\delta-1}{2}$.

CLAIM 2. *If $N(v_i^{++}) \cap \{p_1, ..., p_t\} \neq \emptyset$ for some i, then $N(u_i) \cap \{p_1, ..., p_t\} \neq \emptyset$. Similarly, if $N(v_{i+1}^{--}) \cap \{p_1, ..., p_t\} \neq \emptyset$, then $N(w_i) \cap \{p_1, ..., p_t\} \neq \emptyset$.*

PROOF. Suppose that there exists some $p \in N(v_i^{++})$. By Lemma 2, and by Claim 1, there exists some j such that $u_jp \in E(G)$. By Lemma 1, $i \neq j$. Since $C' : v_j \overleftarrow{C} v_i^{++}pu_j \overrightarrow{C} v_iv_0$ is a longer cycle than C, C' is not dominating. It follows that $N(u_i) \cap \{p_1, ..., p_t\} \neq \emptyset$.

CLAIM 3. $N(u_i) \cap \{p_1, ..., p_t\} = \emptyset$ *for any* $u_i = w_i$.

PROOF. Suppose otherwise that there exist some $u_i = w_i$, say $u_1 = w_1$, and a vertex p such that $p \in N(u_1)$. By Lemma 3, and by Claim 1, $N(p) - \{u_1\} \subseteq A$, since $\{v_0\} \cup A^+$ is a maximal independent set of vertices. By Lemma 1, $N(p) \subseteq$

$A - \{v_1, v_2\} \cup \{u_1\}$. By Claim 2, $|H_i| > 2$ if $u_i = w_i$ and $v_i p \in E(G)$. Hence, $|N_{L_i}(p)| \leq |H_i| - 2$ for any $i > 1$. Moreover, if equality holds for some $i \neq 1$, then the following condition is satisfied

(*) *v_i and p are not adjacent iff $u_i = w_i$ and $H_i = L_i$.*

From Lemma 2 it follows that $H_1 ... H_k$ are pairwise disjoint, hence

(**) $d(p) \leq (n-1) - 2d(v_0)$.

In fact (**) is an equality by Claim 1. Hence, condition (*) is satisfied. Since v_2 and p are not adjacent, $u_2 = w_2$ and $L_2 = H_2$. Now, by considering the inverse direction on C and by applying the same argument, we get that $v_3 p \notin E(G)$. By repeating this argument several times, we easily conclude that $u_i = w_i$ for any i and $N(p) = \{u_1\}$, which contradicts the hypothesis that $\delta \geq 2$. Thus Claim 3 is true.

By Lemma 1, $|N_{L_i}(p_1)| \leq |L_i| - 1$ for any i. We claim that $|N_{L_i}(p_1)| \leq |H_i|$ - 2 for any i. By Claim 3, u_i and p_1 are not adjacent if $u_i = w_i$. Moreover, $H_i \neq L_i$ if $u_i = w_i$ and $v_i p_1 \in E(G)$ because of Claim 2. Hence, $|N_{L_i}(p_1)| \leq |H_i| - 2$ for $u_i = w_i$. If $u_i \neq w_i$ and $u_i p_1 \in E(G)$, then $\{v_i, v_i^{++}\} \cap N(p_1) = \emptyset$, implying that $L_i - \{u_i\} = N_{L_i}(p_1)$ if $|N_{L_i}(p_1)| = |L_i| - 1$. But, in this case, $H_i \neq L_i$ by Claim 2. Hence, we conclude that $|N_{L_i}(p_1)| \leq |H_i| - 2$ for any i, and therefore $d(p_1) \leq (n-2) - 2\,d(v_0)$, which contradicts Claim 1. This contradiction completes our proof.

ACKNOWLEDGEMENT. The author wishes to thank the referee for many helpful suggestions.

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