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AN IMPROVEMENT FOR PARALLEL-ITERATED RUNGE-KUTTA-NYSTRÖM METHODS

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Abstract. This paper deals with predictor-corrector (PC) iteration of Runge-Kutta-Nyström (RKN) methods for integrating initial-value problems for special second-order, ordinary differential equations. We consider RKN correctors based on direct collocation techniques with high stage-order so that the stage-values can be used by extrapolation techniques to construct high-order predictior methods with no additional sequential right-hand side evaluations and without increasing number of processors. Having the high-order predictor in addition with the small convergence factor and error constant, the parallel-iterated PC methods based on direct collocation RKN correctors considered in this paper show the improved efficiency when they are compared to the PIRKN methods available in the literature.

1. Introduction

We will investigate a class of (explicit) predictor-corrector (PC) methods obtained by predictor-corrector iteration (or fixed point iteration) of Runge-Kutta-Nyström correctors for solving the initial-value problem (IVP) for non-stiff, special second-order, ordinary differential equations (ODEs)

$$\frac{d^2\mathbf{y}(t)}{dt^2} = f(\mathbf{y}(t)). \tag{1.1}$$

The efficiency of this class of the PC methods which are based either on Runge-Kutta correctors (for first-order ODEs), or on Runge-Kutta-Nyström correctors

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(for special second-order ODEs) depends on the accuracy of the predictions. In [4] and also in [11] by using a large number of processors, together with the approximation to the step point value, a whole block of approximations to the exact solutions at the off-step points is computed. This block of approximations can be used in the next step for obtaining a high-order predictor formula. In this paper, instead of block of approximations to the exact solutions at the off-step points as in [4] and [11] we use the block of approximations to the stage values and the step point value. Of course, we can not obtain high accurate prediction as in [4] and [11] because the stage order of the corrector methods is lower than the step point order. However no additional processors are needed in the implementation. As an analogue of the PIRKN methods proposed in [8] and [12], the PC methods constructed in this paper will be termed *improved PIRKN (IPIRKN) methods*. We restrict our consideration to the IPIRKN methods based on RKN correctors directly constructed for second-order ODEs (see [5]). This class of corrector methods of higher stage order, (see also [5]) can be used for generating higher order predictor formula. The IPIRKN methods based on this class of corrector methods have small error constants and small convergence factors with sufficiently large stability boundaries for nonstiff problems. Numerical experiments show the better performance of the IPIRKN methods in comparison with the PIRKN methods proposed in [8] and based on the same corrector methods.

For notational convenience, we assume that the equation (1.1) is a scalar equation. However, all considerations below can be straightforwardly extended to a system of ODEs, and therefore, also to nonautonomous equations.

2. Improved PIRKN methods

The starting point is a fully implicit s-stage RKN method of direct collocation type. We shall consider only the RKN methods based on Gauss-Legendre collocation points because the step point value which is different from stage values, gives a possiblity to generate high order predictor formula. For a scalar equation, this method assumes the form

$$\mathbf{Y}_n = y_n\mathbf{e} + hcy'_n + h^2Af(\mathbf{Y}_n), \tag{2.1a}$$

$$y_{n+1} = y_n + hy'_n + h^2\mathbf{b}^T f(\mathbf{Y}_n), y'_{n+1} = y'_n + h\mathbf{d}^T f(Y_n), \tag{2.1b}$$

where A is a $s \times s$ matrix , $\mathbf{b}, \mathbf{d}, \mathbf{c}, \mathbf{e}$ are s-dimensional vectors, $\mathbf{e} = (1, 1, \ldots, 1)^T$, and $\mathbf{Y}_n$ is the stage vector corresponding to the n-th step. Furthermore, we use the convention that for any given vector $\mathbf{v} = (v_j), f(\mathbf{v})$ denotes the vector with entries $f(v_j)$.

Consider the following fixed point iteration scheme:

$$\mathbf{Y}_n^{(0)} = V\mathbf{Y}_{n-1}{}^{(m)} + \mathbf{w}y_n, \tag{2.2a}$$

$$\mathbf{Y}_n^{(j)} = y_n\mathbf{e} + hcy'_n + h^2 Af(\mathbf{Y}_n^{(j-1)}), j = 1, \ldots, m, \tag{2.2b}$$

$$y_{n+1} = y_n + hy'_n + h^2\mathbf{b}^T f(\mathbf{Y}_n^{(m)}), y'_{n+1} = y'_n + h\mathbf{d}^T f(\mathbf{Y}_n^{(m)}) \tag{2.2c}$$

where V is $s \times s$ matrix , $\mathbf{w}$ is s-dimensional vector, both determined by the order conditions. Notice that the block vector $((Y_{n-1}^{(m)})^T, y_n)^T$ is already provided at the previous step, the s components of the vectors $\mathbf{Y}_n^{(0)}$ and $\mathbf{Y}_n^{(j)}$ can be computed in parallel, provided that only s processors are available, so that the computational time needed for one iteration of (2.2b) is equivalent to the time required to evaluate one right-hand side function on a sequential computer. The PC method (2.2) is of the same nature as the PIRKN methods considered in [8] and [12]. The amelioration here consists of higher order predictor formula (2.2a). Therefore, the method (2.2) is called the *improved* PIRKN (IPIRKN) method.

Treating the predictor formula (2.2a) as the predictor method, and (2.1) as the corrector method, (2.2) may be considered as a conventional PC method (in $P(CE)^m E$ mode). Assuming that the function $f(y)$ is Lipschitz continuous and that (2.2a) defines a q-order predictor formula (*i.e.*, $\mathbf{Y}_n^{(0)} - \mathbf{Y}_n = 0(h^{q+1})$), the following theorem easily follows (see also [8], [12])

THEOREM 2.1. *Let the generating corrector method (2.1) be of order p. Then on s-processor computers the IPIRKN method defined by (2.2a), (2.2b), (2.2c) represents an explicit RKN method of order* $p^* = min\{p, 2m + q + 1\}$ *requiring* $m + 1$ *sequential right-hand side evaluations per step*

REMARK 2.1. From Theorem 2.1, we see that by setting $m = [(p-q)/2]$, $[.]$ denoting the integer function, we have a IPIRKN method of maximum order $p^* = p$ (order of the corrector) with only $[(p-q+2)/2]$ sequential right-hand evaluations per step.

2.1. Order condition for the predictor methods

It is known that the s-stage high-order direct collocation RKN correctors based on Gauss-Legendre collocation points (Gauss-Legendre RKN correctors) have stage order $s+1$ (cf.[5]). Since the block of approximations in the predictor method defined by (2.2a) has dimension $s+1$, we can construct a predictor method of order s.

We now suppose that fixed stepsize is used in the integration process. The order condition for the predictor to be of order s is derived by replacing $Y_{n-1}^{(m)}, y_n$, and $Y_n^{(0)}, y_{n+1}$ by the exact solution values $y(t_{n-1}\mathbf{e}+h\mathbf{c}), y(t_n)$ and $y(t_n\mathbf{e}+h\mathbf{c}), y(t_{n+1})$ (see [4], [9]). Let us denote $\mathbf{a}^T = (\mathbf{c}^T, 1)$ and by requiring that the predictor method is of order s in h, we are led to the conditions

$$y(t_n\mathbf{e}^*+h\mathbf{a})-By(t_{n-1}\mathbf{e}^*+h\mathbf{a}) = y(t_n\mathbf{e}^*+h\mathbf{a})-By(t_n\mathbf{e}^*+h(\mathbf{a}-\mathbf{e}^*)) = O(h^{s+1}), \tag{2.3}$$

where $\mathbf{e}^*$ is a (s+1)-dimensional vector with unit entries. Using the relation

$$y(t\mathbf{e}^* + h\mathbf{x}) = exp(h\mathbf{x}\frac{d}{dt})y(t),$$

we obtain

$$[exp(h(\mathbf{a}+\mathbf{e}^*)\frac{d}{dt}) - Bexp(h\mathbf{a}\frac{d}{dt})]y(t_n) = O(h^{s+1})$$

yielding the conditions

$$(\mathbf{a}+\mathbf{e}^*)^j - Ba^j = 0, j = 0, 1, \ldots, s. \tag{2.4}$$

Let us define the matrices

$$P = (\mathbf{e}^*, (\mathbf{a}+\mathbf{e}^*), (\mathbf{a}+\mathbf{e}^*)^2, \ldots, (\mathbf{a}+\mathbf{e}^*)^s),\ Q = (\mathbf{e}^*, \mathbf{a}, \mathbf{a}^2, \ldots, \mathbf{a}^s)$$

where P and Q are $(s+1) \times (s+1)$ matrices. Then the condition (2.4) can be written in the form

$$P - BQ = O. \tag{2.4'}$$

Since the abscissas a_j defined in this paper are distinct, we can derive matrix $B = PQ^{-1} = (b_{ij})$. Matrix V and vector $\mathbf{w}$ in (2.2a) can be obtained by writing B in the form

$$B = \begin{pmatrix} & V & & \mathbf{w} \\ b_{s+1,1} & \cdots & b_{s+1,s} & b_{s+1,s+1} \end{pmatrix} \tag{2.5}$$

From Theorem 2.1. we deduce the following corollary:

COROLLARY 2.1. *Let the matrix V and vector* $\mathbf{w}$ *be defined according to (2.4) and (2.5), let p be the order of the corrector method (2.1). Then (2.2a), (2.2b) and (2.2c) define an IPIRKN method of order* $p^* = min\{p, 2m + s + 1\}$

REMARK 2.2. From Corollary 2.1, we see that by setting m = [(p-s)/2], we have an IPRKN method of maximum order $p^* = p$ (order of the corrector) with only [(p-s+2)/2] sequential right-hand side evaluations per step.

Specification of the parameters (A, $\mathbf{b}, \mathbf{d}, \mathbf{c}$) of the direct collocation corrector methods can be found in institute report version of [8]. In the following subsections, we will discuss the convergence, the stability and the error estimate of the IPIRKN methods.

2.2. Convergence boundaries

The convergence factors and convergence boundaries of the IPIRKN methods are indentical with those of the direct PIRKN methods studied in [8]. Here we briefly mention the most important results.

The rate of convergence was determined (in [8]) by using the test equation $y'' = \lambda y$, where λ runs through the eigenvalues of the Jacobian matrix $\partial \mathbf{f}/\partial \mathbf{y}$. For this equation, we obtain the iteration error equation

$$\mathbf{Y}_n^{(j)} - \mathbf{Y}_n = zA[\mathbf{Y}_n^{(j-1)} - \mathbf{Y}_n], \quad z := \lambda h^2, \ j = 1, \ldots, m.$$

Hence, with respect to the test equation, the rate of convergence is determined by the spectral radius $\rho(A)$ of the matrix A. We shall call $\rho(A)$ th *convergence*

factor of the PIRKN (and also IPIRKN) method. Requiring that $\rho(zA) < 1$, leads us to the convergence condition

$$|z| < \frac{1}{\rho(A)} \quad \text{or} \quad h^2 < \frac{1}{\rho(A)\rho(\partial f/\partial y)}. \tag{2.6}$$

This convergence condition is of the same form as the stability condition associated with RKN methods. In analogy with the notion of the *stability boundary*, we shall call $1/\rho(A)$ the *convergence boundary*. We refer to [8] for specification of the convergence boundaries for the various PIRKN methods.

2.3. *Stability boundaries*

The linear stability of the IPIRKN method (2.2) is investigated by again using the model equation $y'' = \lambda y$, where λ runs through the eigenvalues of $\partial \mathbf{f}/\partial \mathbf{y}$.

THEOREM 2.2. *For the equation $y'' = \lambda y$ the numerical solution obtained by the IPIRKN method { (2.2a), 2.2b), 2.2c)} satisfies the recursion*

$$\begin{pmatrix} Y_n^{(m)} \\ Y_{n+1} \\ hy'_{n+1} \end{pmatrix} = M_m(z) \begin{pmatrix} Y_{n-1}^{(m)} \\ Y_n \\ hy'_n \end{pmatrix} \tag{2.7}$$

where $M_m(z)$ is the amplification matrix

$$M_m(z) = \begin{pmatrix} (zA)^m V & (I-zA)^{-1}(I-(zA)^m)\mathbf{e} + (zA)^m \mathbf{w} & X_m(z) \\ z\mathbf{b}^T(zA)^m V & 1 + z\mathbf{b}^T((zA)^m w + (I-zA)^{-1}(I-(zA)^m))\mathbf{e} & g_m(z) \\ z\mathbf{d}^T(zA)^m V & z\mathbf{d}^T((zA)^m w + (I-zA)^{-1}(I-(zA)^m))\mathbf{e} & k_m(z) \end{pmatrix}$$

where

$$X_m(z) = (I-zA)^{-1}(I-(zA)^m)\mathbf{c}$$

$$g_m(z) = 1 + z\mathbf{b}^T(I-zA)^{-1}(I-(zA)^m)\mathbf{c}$$

$$k_m(z) = 1 + z\mathbf{d}^T(I-zA)^{-1}(I-(zA)^m)\mathbf{c}$$

PROOF. Applying the IPIRKN method (2.2) to the model equation, we obtain

$$\begin{aligned} Y_n^{(m)} &= y_n \mathbf{e} + hcy'_n + zAY_n^{(m-1)} \qquad (2.8a)\\ &= (I + zA + (zA)^2 + \ldots + (zA)^{m-1})(y_n \mathbf{e} + hcy'_n) + (zA)^m \mathbf{Y}_n^{(0)} \\ &= (zA)^m VY_{n-1}^{(m)} + ((I - zA)^{-1}(I - (zA)^m)\mathbf{e} + (zA)^m \mathbf{w})y_n \\ &\quad + (I - zA)^{-1}(I - (zA)^m)\mathbf{c}hy'_n \end{aligned}$$

$$\begin{aligned} y_{n+1} &= y_n + hy'_n + z\mathbf{b}^T Y_n^{(m)} \\ &= z\mathbf{b}^T (zA)^m VY_{n-1}^{(m)} + (1 + z\mathbf{b}^T((zA)^m \mathbf{w} + (I - zA)^{-1}(I - (zA)^m)\mathbf{e}))y_n \\ &\quad + (1 + z\mathbf{b}^T(I - zA)^{-1}(I - (zA)^m)\mathbf{c})hy'_n \qquad (2.8b) \end{aligned}$$

$$\begin{aligned} hy'_{n+1} &= hy'_n + z\mathbf{d}^T Y_n^{(m)} \\ &= z\mathbf{d}^T (zA)^m VY_{n-1} + z\mathbf{d}^T((zA)^m \mathbf{w} + (I - zA)^{-1}(I - (zA)^m)\mathbf{e})y_n \\ &\quad + (1 + z\mathbf{d}^T(I - zA)^{-1}(I - (zA)^m)\mathbf{c})hy'_n: \qquad (2.8c) \end{aligned}$$

Combining the relations (2.8a), (2.8b), (2.8c), the one-step recursion (2.7) of Theorem 2.2 is easily obtained.

Similar to the stablity consideration of PIRKN methods (cf. [8], [12]), the $(s + 2) \times (s + 2)$ matrix $M_m(z)$, which determines the stability behaviour of the IPIRKN methods, will be called the amplification matrix and its spectral radius $\text{æ}(M_m(z))$ the stability function. From (2.8) we see that if z satisfies the convergence condition (2.6), then the spectral radius of $M_m(z)$ converges to the spectral radius of the amplification matrix $M(z)$ of the corrector methods as $m \to \infty$ (see [5]), i.e.,

$$\text{æ}(M_m(z)) \to \text{æ}(M(z)) \quad \text{as} \quad m \to \infty$$

Hence, the *asymptotic* stability interval for $m \to \infty$ is the intersection on the negative z-axis of the stability interval $(fi_{corr}, 0)$ of the generating corrector and the region of convergence in the complex z-plane defined by (2.6). For the IPIRKN methods studied in this paper, where the corrector method is conditionally stable with the stability boundaries less than the convergence boundaries (see [8]), the asymptotic stability region coincides with the stability

region of the corrector methods. For finite m, the stability intervals are given by

$$(-fi(m);)) := \{z : æ(M_m(z)) \; ; \; 1; z \leq 0\}:$$

The stability boundaries $-fi(m)$ listed in Table 2.1 for the various IPIRKN and direct PIRKN methods proposed in [8] show that the stability behaviour of the new IPIRKN methods is more regular than that of PIRKN methods. The stability boundaries corresponding to the minimal value of m (for both family of methods) needed to reach the order of the corrector are indicated in bold face. By means of Table 2.1 we can select the number of iterations needed to achieve an acceptable stability region (the corresponding stability boundaries are underlined).

Table 2.1. Stability boundaries $fi(m)$ for a few m of the various direct PIRKN and IPIRKN methods

PC method	P	m=1	m=2	m=3	m=4	m=5	m=6	...	m=∞
Direct PIRKN	4	**6.83**	0.00	0.00	8.57	0.00	0.00	...	9.00
IPIRKN	4	**0.33**	0.81	2.03	8.76	2.61	2.71	...	9.00
Direct PIRKN	6	7.06	**0.00**	18.77	0.00	9.80	0.00	...	9.77
IPIRKN	6	**0.23**	2.69	3.17	9.36	3.89	3.97	...	9.77
Direct PIRKN	8	7.06	0.00	**9.51**	0.00	0.37	9.86	...	9.86
IPIRKN	8	0.05	**1.38**	4.40	10.23	16.95	9.15	...	9.86
Direct PIRKN	10	7.06	0.00	9.51	**0.00**	9.86	0.01	...	36.65
IPIRKN	10	0.01	**0.74**	3.33	7.37	12.05	19.45	...	36.65
Direct PIRKN	12	7.06	0.01	9.51	0.21	**9.86**	1.17	...	39.45
IPIRKN	12	0.00	0.39	**2.29**	6.02	11.49	18.71	...	39.45

2.4. The truncation error

We will investigate the truncation error of the IPIRKN method (2.2) with respect to the model test equation $y'' = \breve{y}$. Let us denote the step values associated with the corrector method (2.1) by u_{n+1} and u'_{n+1}, and define

$$E_m(z) := \tag{2.9}$$
$$\begin{pmatrix} -z\mathbf{b}^T(zA)^m V & z\mathbf{b}^T((I-zA)^{-1}(zA)^m\mathbf{e}-(zA)^m\mathbf{w}) & z\mathbf{b}^T(I-zA)^{-1}(zA)^m\mathbf{c} \\ -z\mathbf{d}^T(zA)^m V & z\mathbf{d}^T((I-zA)^{-1}(zA)^m\mathbf{e}-(zA)^m\mathbf{w}) & z\mathbf{d}^T(I-zA)^{-1}(zA)^m\mathbf{c} \end{pmatrix}$$

$$\mathbf{w}_{n+1} = \begin{pmatrix} u_{n+1} \\ hu'_{n+1} \end{pmatrix};\ \mathbf{v}_{n+1} = \begin{pmatrix} y_{n+1} \\ hy'_{n+1} \end{pmatrix}.$$

THEOREM 2.3. *For the equation $y'' = \breve{y}$ the iteration error defined by $w_{n+1} - v_{n+1}$ satisfies the relation*

$$\mathbf{w}_{n+1} - \mathbf{v}_{n+1} = E_m(z)\begin{pmatrix} \mathbf{Y}_{n-1}{}^{(m)} \\ \mathbf{v}_n \end{pmatrix} = O(h^{2m+s+3}).$$

PROOF. By means of (2.8) and (2.9) and the stability matrix M(z) of the RKN corrector methods (see [5]), we have the following representation

$$M_m(z) =$$
$$\begin{pmatrix} (zA)^m V & (I-zA)^{-1}(I-(zA)^m)\mathbf{e} + (zA)^m\mathbf{w} & (I-(zA)^{-1}(I-(zA)^m)\mathbf{c} \\ O_{2s} & M(z) & \end{pmatrix}$$
$$- \begin{pmatrix} O_{s,s+2} \\ E_m(z) \end{pmatrix};$$

where O_{ij} is $i \times j$ matrix, with zero entries. In view of the recursion (2.7) in Theorem 2.2, Theorem 2.3 easily follows.

The local truncation error of the IPIRKN methods can be written as the sum of the truncation error of the corrector and the iteration error:

$$\begin{pmatrix} y(t_{n+1}) \\ hy'(t_{n+1}) \end{pmatrix} - \mathbf{v}_{n+1} = \begin{pmatrix} y(t_{n+1}) \\ hy'(t_{n+1}) \end{pmatrix} - \mathbf{w}_{n+1} + E_m(z)\begin{pmatrix} \mathbf{Y}_{n-1}^{(m)} \\ v_n \end{pmatrix}.$$

Small trucation error of the direct RKN corrector methods and small convergence factor for the corresponding IPIRKN methods (see [8]) are two potential effects to expect that the truncation error of the IPIRKN methods is small.

3. Numerical experiments

In this section we report the numerical results obtained by the various direct PIRKN and IPIRKN methods. The absolute error obtained at the end of integration interval is presented in the form 10^{-d} (d may be interpreted as the number of correct decimal digits (NCD)). In order to see the efficiency of the various direct PIRKN and IPIRKN methods, we follow a dynamical strategy for determining the number of iterations in the successive steps (see [10]).

$$||\mathbf{Y}_n^{(m)} - \mathbf{Y}_n^{(m-1)}||_\infty \leq Ch^{p-1} \quad \text{and} \quad m \geq [(p-s)=2]; \tag{3.1}$$

where p and s denote the corrector order and the number of stages of the corrector methods, C is a problem- and method-dependent parameter. Furthermore, in the tables of results, N_{seq} denotes the total number of sequential right-hand side evaluations, and N_{steps} denotes the total number of integration steps. The following two problems possess exact solutions in closed form. Initial conditions are taken from the exact solutions.

3.1. Linear nonautonomous problem

As a first numerical test, we apply the various direct PIRKN and IPIRKN methods to the linear nonautonomous problems (cf. [7])

$$\begin{aligned} \frac{d^2y(t)}{dt^2} &= \begin{pmatrix} -2ff(t)+1 & -ff(t)+1 \\ 2(ff(t)-1) & ff(t)-2 \end{pmatrix} y(t); ff(t) \\ &= max(2cos^2(t); sin^2(t)); \ 0 \leq t \leq 20; \end{aligned} \tag{3.2}$$

with the exact solution $y(t) = (-sin(t); 2sin(t))^T$. Table 3.1 clearly shows that the higher order IPIRKN methods with higher order predictor formula are superior to the direct PIRKN methods of the same order. The lower order IPIRKN methods are only a bit more efficient than the direct PIRKN methods because the predictor formula defined by (2.2a) for these IPIRKN methods are only slightly more accurate than that for the direct PIRKN methods.

Table 3.1. Values of NCD /N_{seq} for problem (3.2) obtained by the various direct PIRKN and IPIRKN methods.

PC methods	P	N_{steps} = 80	N_{steps} =160	N_{steps} =320	N_{steps} =640	N_{steps} =1280	C
Direct PIRKN	4	5.1/237	6.4/477	7.6/958	8.8/1918	10.0/3835	10^{-1}
IPIRKN	4	5.2/228	6.4/456	7.6/913	8.8/1827	10.0/3654	10^{-1}
Direct PIRKN	6	8.0/320	9.9/640	11.7/1280	13.5/2559	15.3/5119	10^{-3}
IPIRKN	6	8.0/292	9.8/481	11.7/961	13.5/1921	15.3/3841	10^{-3}
Direct PIRKN	8	13.0/399	16.1/799	18.6/1600	22.1/3198	23.9/6398	10^{-4}
IPIRKN	8	12.5/320	15.5/636	19.2/1271	21.3/2542	23.8/5081	10^{-4}
Direct PIRKN	10	13.3/436	17.6/920	21.8/1881	24.5/3802		10^{-4}
IPIRKN	10	14.4/318	17.6/638	21.0/1278	24.2/2558		10^{-4}
Direct PIRKN	12	19.1/556	23.1/1117	26.2/2236			10^{-7}
IPIRKN	12	19.3/397	22.8/792	26.5/1583			10^{-7}

3.2. Nonlinear Fehlberg problem

For the second numerical example, we consider the orbit equation (see [1])

$$\frac{d^2y(t)}{dt^2} = \begin{pmatrix} -4t^2 & -2=r(t) \\ 2=r(t) & -4t^2 \end{pmatrix} y(t); r(t)$$

$$= \sqrt{y_1^2(t) + y_2^2(t)}; \sqrt{ß=2} \leq t \leq 10; \qquad (3.3)$$

with the exact solution $y(t) = (cos(t^2);\ sin(t^2))^T$. The results are reported in Table 3.2. For this nonlinear problem, similar to the previous linear problem, the superiority of the IPIRKN methods over the direct PIRKN methods is once again demonstrated.

Table 3.2. Values of NCD /N_{seq} for problem (3.3) obtained by the various direct PIRKN and IPIRKN methods.

PC methods	P	N_{steps} = 200	N_{steps} =400	N_{steps} =800	N_{steps} =1600	N_{steps} =3200	C
Direct PIRKN	4	2.4/581	3.6/1197	4.8/2400	6.0/4800	7.2/9600	10^2
IPIRKN	4	2.4/535	3.6/1070	4.8/2144	6.0/4298	7.2/8577	10^2
Direct PIRKN	6	4.9/773	6.7/1531	8.6/3095	10.4/6256	12.2/12647	10^3
IPIRKN	6	5.0/649	6.7/1278	8.5/2504	10.4/4860	12.2/9600	10^3
Direct PIRKN	8	7.6/1022	10.0/2029	12.4/4022	14.8/7956	17.2/15720	10^3
IPIRKN	8	7.6/808	10.0/1561	12.4/2996	14.8/5973	17.2/11946	10^3
Direct PIRKN	10	10.5/1234	13.6/2457	16.6/4891	19.6/9733	22.6/19325	10^3
IPIRKN	10	10.6/939	13.6/1801	16.6/3561	19.6/7092	22.6/14085	10^3
Direct PIRKN	12	13.4/1365	17.0/2742	20.7/5491	24.3/10938		10^4
IPIRKN	12	13.4/980	17.1/1939	20.7/3819	24.3/7458		10^4

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