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ON THE ROCKAFELLAR DERIVATIVE OF MARGINAL FUNCTIONS AND APPLICATIONS

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Abstract. In this paper, some evaluations of the directional Rockafellar derivative of a function defined by $\varphi_F(x) = \inf \{f(x,y) \mid y \in F(x)\}$ via the directional Clarke and Rockafellar derivative of f are shown. These evaluations are applied to find necessary optimality conditions for generalized optimal control problems governed by hemivariational inequalities.

1. Introduction

Let us consider the following function

$$\varphi_F(x) = \inf \{f(x,y) \mid y \in F(x)\}\ , \tag{1.1}$$

where f is a function defined on $X \times Y$, F is a set-valued map from a Banach space X into a finite dimensional space Y.

The directional derivatives of φ_F under different assumptions have been already studied in many works, see e.g. [1], [3], [5]–[7], [9], [11]–[14]. In [12] Pschenichnyi has investigated the case where f is a continuous convex function and the graph of F is a convex closed set. For the convex case see also Hogan [9]. For the differentiable case, we refer the reader to Beresnev and Pschenichnyi [1], Dem'yanov [5], Dem'yanov and Malozemov [6]. In [7] Hiriart and Urruty has shown various evaluations of Clarke's generalized gradients of φ_F in the case in which φ_F is locally Lipschitz. These results were generalized by Minchenko [11] for the case of reflexive Banach spaces, and by Thibault [14] in terms of Kruger-Mordukhovich's subdifferentials.

In the present paper we treat some cases when φ_F is not locally Lipschitz. The paper is structured as follows. Section 2 is devoted to evaluating the

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Rockafellar derivative of φ_F via the Clarke derivative of f. Section 3 treats the case of f being directionally Lipschitz. Finally, in Section 4, an application to find necessary optimality conditions for generalized optimal control problems governed by hemivariational inequalities is given.

2. An evaluation of the Rockafellar derivative of φ_F

For convenience of the reader, let us recall some notions of nonsmooth analysis (see e.g. [4]).

DEFINITION 2.1. Let φ be Lipschitz near $x_0 \in X$. The directional Clarke derivative of φ at x_0 with respect to d, denoted by $\varphi^0(x_0;d)$, is defined by

$$\varphi^0(x_0;d) = \limsup_{\substack{x\to x_0\\ \lambda\downarrow 0}} \frac{\varphi(x+\lambda d)-\varphi(x)}{\lambda} \tag{2.1}$$

DEFINITION 2.2. Let φ be an extended-real-valued function on X. The directional Rockafellar derivative of φ at x_0 with respect to d, denoted by $\varphi^\uparrow(x_0;d)$, is defined by

$$\varphi^\uparrow(x_0;d) = \lim_{\varepsilon\downarrow 0} \limsup_{\substack{(x,\alpha)\downarrow_\varphi x_0\\ \lambda\downarrow 0}} \inf_{\omega\in d+\varepsilon B} \frac{\varphi(x+\lambda\omega)-\alpha}{\lambda}. \tag{2.2}$$

Here, $(x,\alpha)\downarrow_\varphi x_0$ means that $(x,\alpha)\in \operatorname{epi}\varphi$, $x\to x_0$ and $\alpha\to\varphi(x_0)$; B stands for the open unit ball.

When φ is lower semicontinuous at x_0, the limit (2.2) is of the form

$$\varphi^\uparrow(x_0;d) = \lim_{\varepsilon\downarrow 0} \limsup_{\substack{x\downarrow_\varphi x_0\\ \lambda\downarrow 0}} \inf_{\omega\in d+\varepsilon B} \frac{\varphi(x+\lambda\omega)-\varphi(x)}{\lambda}, \tag{2.3}$$

where $x\downarrow_\varphi x_0$ means that $x\to x_0$ and $\varphi(x)\to\varphi(x_0)$.

We recall [4] that when φ is finite at x_0, the set of subgradients of φ at x_0 is defined by

$$\partial\varphi(x_0) = \{x^*\in X^* \mid (x^*,-1)\in N_{\mathrm{epi}\varphi}(x_0,\varphi(x_0))\}, \tag{2.4}$$

where $N_{\mathrm{epi}\varphi}(x_0,\varphi(x_0))$ is the normal cone to $\mathrm{epi}\varphi$ at $(x_0,\varphi(x_0))$.

Note that when $\varphi^{\uparrow}(x_0;0) > -\infty$, (2.4) can be expressed by

$$\overline{\partial}\varphi(x_0) = \{x^* \in X^* \mid \ < x^*, d > \le \varphi^{\uparrow}(x_0;d),\ \forall d \in X\}. \tag{2.5}$$

Throughout this paper we shall deal with functions defined by (1.1) where f is an extended-real-valued function on $X \times Y$. To evaluate the Rockafellar derivative of φ_F via the Clarke derivative of f, we set

$$M_F(x) = \{y \in F(x) \mid \varphi_F(x) = f(x,y)\},$$

and denote by $\mathcal{F}$ the graph of F.

THEOREM 2.1. *Let φ_F be lower semicontinuous at $x_0 \in X$. Assume that M_F possesses a selection $\bar{y}$ which is bounded in a neighbourhood of x_0. Suppose, in addition, that f is finite at $(x_0, \bar{y}_0)$ and Lipschitz near $(x_0, \bar{y}_0)$ for all $\bar{y}_0 \in \overline{Y}_0$, where $\overline{Y}_0$ is the set of all accumulation points of $\bar{y}(x)$ as $x \to x_0$. Then,*

$$\varphi_F^{\uparrow}(x_0;h) \le \sup_{\bar{y}_0 \in \overline{Y}_0} f^0_{(x,y)}(x_0, \bar{y}_0; h, k) \quad (\forall (h,k) \in \bigcap_{\bar{y}_0 \in \overline{Y}_0} T_{\mathcal{F}}(x_0, \bar{y}_0)), \tag{2.6}$$

where $T_{\mathcal{F}}(x_0, \bar{y}_0)$ stands for the Clarke tangent cone to $\mathcal{F}$ at $(x_0, \bar{y}_0)$.

PROOF. Take any $(h,k) \in \bigcap_{\bar{y}_0 \in \overline{Y}_0} T_{\mathcal{F}}(x_0, \bar{y}_0)$ and any $\varepsilon > 0$. Since φ_F is lower semicontinuous at $x_0, \varphi_F^{\uparrow}$ can be expressed by (2.3). We set

$$\varphi_F^{\varepsilon}(x_0;h) = \limsup_{\substack{x \downarrow_{\varphi_F} x_0 \\ \lambda \downarrow 0}} \inf_{\omega \in h + \varepsilon B} \frac{\varphi_F(x + \lambda\omega) - \varphi_F(x)}{\lambda},$$

and choose sequences $x_n \downarrow_{\varphi_F} x_0$, $\lambda_n \downarrow 0$ satisfying

$$\varphi_F^{\varepsilon}(x_0;h) = \lim_{n \to \infty} \inf_{\omega \in h + \varepsilon B} \frac{\varphi_F(x_n + \lambda_n\omega) - \varphi_F(x_n)}{\lambda_n}. \tag{2.7}$$

For the bounded selection $\bar{y}$, denote $\bar{y}_n = \bar{y}(x_n)$. Then it can be extracted a subsequence $\bar{y}_{n_i} \to \hat{y}_0 \in \overline{Y}_0$. For simplicity of notation we write $\{\bar{y}_n\}$ instead of $\{\bar{y}_{n_i}\}$. Thus,

$$\lim_{n \to \infty} \bar{y}_n = \hat{y}_0 \qquad \text{and} \qquad \varphi_F(x_n) = f(x_n, \bar{y}_n). \tag{2.8}$$

Due to Theorem 2.4.5 of [4], for $\{(x_n, \bar{y}_n)\}$ and $\{\lambda_n\}$ above there exists a sequence $\{(h_n, k_n)\}$ converging to (h, k) such that

$$(x_n, \bar{y}_n) + \lambda_n(h_n, k_n) \in \mathcal{F} ,$$

which implies that

$$\varphi_F(x_n + \lambda_n h_n) \leq f(x_n + \lambda_n h_n,\ \bar{y}_n + \lambda_n k_n). \tag{2.9}$$

Since f is Lipschitz near $(x_0, \hat{y}_0)$, it follows from (2.8), (2.9) that

$$\begin{aligned}
\varphi_F(x_n + \lambda_n h_n) - \varphi_F(x_n) &= \varphi_F(x_n + \lambda_n h_n) - f(x_n, \bar{y}_n) \\
&= \varphi_F(x_n + \lambda_n h_n) - f(x_n + \lambda_n h_n,\ \bar{y}_n + \lambda_n k_n) + \\
&+ f(x_n + \lambda_n h_n,\ \bar{y}_n + \lambda_n k_n) - f(x_n + \lambda_n h, \bar{y}_n + \lambda_n k) + \\
&+ f(x_n + \lambda_n h,\ \bar{y}_n + \lambda_n k) - f(x_n, \bar{y}_n) \\
&\leq f(x_n + \lambda_n h,\ \bar{y}_n + \lambda_n k) - f(x_n, \bar{y}_n) + \lambda_n \varepsilon_n^1,
\end{aligned}$$

where $\varepsilon_n^1 \to 0$ as $n \to \infty$, whence,

$$\frac{\varphi_F(x_n + \lambda_n h_n) - \varphi_F(x_n)}{\lambda_n} \leq \frac{f(x_n + \lambda_n h,\ \bar{y}_n + \lambda_n k) - f(x_n, \bar{y}_n)}{\lambda_n} + \varepsilon_n^1. \tag{2.10}$$

Moreover, for all n sufficiently large we have

$$\inf_{\omega \in h + \varepsilon B} \frac{\varphi_F(x_n + \lambda_n \omega) - \varphi_F(x_n)}{\lambda_n} \leq \frac{\varphi_F(x_n + \lambda_n h_n) - \varphi_F(x_n)}{\lambda_n},$$

which together with (2.7) and (2.10) implies that

$$\varphi_F^\varepsilon(x_0; h) \leq \sup_{\bar{y}_0 \in \overline{Y}_0} f^0_{(x,y)}(x_0, \bar{y}_0) + \lim_{n \to \infty} \varepsilon_n^1.$$

Therefore,

$$\lim_{\varepsilon \downarrow 0} \varphi_F^\varepsilon(x_0; h) \leq \sup_{\bar{y}_0 \in \overline{Y}_0} f^0_{(x,y)}(x_0, \bar{y}_0),$$

which completes the proof.

When M_F possesses a selection $\bar{y}$ which is continuous at $x_0 \in X$, the set $\overline{Y}_0$ of all accumulation points of $\bar{y}(x)$ as $x \to x_0$ shrinks to a singleton $\{y_0\}$, where $y_0 = \lim\limits_{x \to x_0} \bar{y}(x)$. As an immediate consequence of Theorem 2.1 we have the following

COROLLARY 2.1. *Let φ_F be lower semicontinuous at $x_0 \in X$. Assume that M_F possesses a selection $\bar{y}$ which is continuous at x_0. Suppose, furthermore, that f is finite at (x_0, y_0) and Lipschitz near (x_0, y_0). Then, for all $(h, k) \in T_{\mathcal{F}}(x_0, y_0)$,*

$$\varphi_F^{\uparrow}(x_0; h) \leq f^0_{(x,y)}(x_0, y_0; h, k).$$

REMARK 2.1. Theorem 2.1 includes as a special case Theorem 7 of [7] in which φ_F is assumed to be locally Lipschitz.

We close this section with a sufficient condition for the existence of a bounded selection of the set-valued map M_F.

PROPOSITION 2.1. *Let X be a finite dimensional space, the set-valued map F possess compact values and the map $y \mapsto f(x, y)$ be lower semicontinuous on Y. Assume that F is upper semicontinuous in a neighbourhood of x_0. Then the map M_F possesses a selection which is bounded in a neighbourhood of x_0.*

PROOF. Since F possesses compact values and the map $y \mapsto f(x, y)$ is lower semicontinuous, for every $x \in X$ there is an element $\bar{y}(x) \in F(x)$ such that $\varphi_F(x) = f(x, \bar{y}(x))$, which means that $\bar{y}(x) \in M_F(x)$. Since F is upper semicontinuous in a neighbourhood of x_0, there exists a compact neighbourhood $\mathcal{N}(x_0)$ of x_0 such that $\bigcup_{x \in \mathcal{N}(x_0)} F(x)$ is compact (see e.g. [2]). Hence $\bar{y}(\cdot)$ is bounded in $\mathcal{N}(x_0)$.

3. The case of f being directionally Lipschitz

We recall some notions from [4] which will be needed in this section.

DEFINITION 3.1. Let φ be an extended-real-valued function on X. The function φ is said to be directionally Lipschitz at x_0 with respect to d if $|\varphi(x_0)| < +\infty$ and

$$\varphi^+(x_0; d) = \limsup_{\substack{(x,\alpha) \downarrow_\varphi x_0,\ \omega \to d \\ \lambda \downarrow 0}} \frac{\varphi(x + \lambda\omega) - \alpha}{\lambda} < +\infty. \tag{3.1}$$

Denoting by $D_\varphi(x_0)$ the set of all vectors d such that φ is directionally Lipschitz at x_0 with respect to d, we remark that if $D_\varphi(x_0) \neq \emptyset$, then

$$\varphi^{\uparrow}(x_0;d) = \varphi^{+}(x_0;d) \qquad (\forall d \in D_{\varphi}(x_0)).$$

DEFINITION 3.2. Let $Q \subset X$ and $x_0 \in X$. Then a vector $d \in X$ is said to be hypertangent to Q at x_0, if there is a number $\varepsilon > 0$ such that $x + t\omega \in Q$ for all $x \in (x_0 + \varepsilon B) \cap Q$, $\omega \in d + \varepsilon B$, $t \in (0, \varepsilon)$.

Let F_1, F_2 be set-valued maps from X into Y and $\mathcal{F}_1, \mathcal{F}_2$ their graphs, respectively. We define the map $F = F_1 \cap F_2$ by the formula

$$F(x) = F_1(x) \cap F_2(x) \tag{3.2}$$

In this section we shall investigate the marginal function defined by (1.1) and (3.2), i.e.

$$\varphi_F(x) = \inf \{f(x,y) \mid y \in F_1(x) \cap F_2(x)\}.$$

A reason for our interest in this function is a natural connection with optimization problems which will be studied in Section 4.

In accordance with the notations of the previous section M_F is defined by

$$M_F(x) = \{y \in F_1(x) \cap F_2(x) \mid \varphi_F(x) = f(x,y)\}.$$

It is easily seen that

$$\mathcal{F} = \mathcal{F}_1 \cap \mathcal{F}_2,$$

where $\mathcal{F}$ is the graph of the map F defined by (3.2).

Under the directionally Lipschitz assumption an evaluation of the Rockafellar derivative of φ_F can be stated as follows.

THEOREM 3.1. *Assume that M_F possesses a selection $\bar{y}$ which is bounded in a neighbourhood of $x_0 \in X$ and f is directionally Lipschitz at $(x_0, \bar{y}_0)$ $(\forall \bar{y}_0 \in \overline{Y}_0)$ with respect to all $(h,k) \in \bigcap_{\bar{y}_0 \in \overline{Y}_0} T_{\mathcal{F}}(x_0, \bar{y}_0)$, where $\overline{Y}_0$ is the set of all accumulation points of $\bar{y}(x)$ as $x \to x_0$. Suppose, in addition, that $\overline{Y}_0 \subset M_F(x_0)$ and the following hypotheses are fulfilled :*

(i) $T_{\mathcal{F}_1}(x_0, \bar{y}_0) \cap \operatorname{int} T_{\mathcal{F}_2}(x_0, \bar{y}_0) \neq \emptyset \qquad (\forall \bar{y}_0 \in \overline{Y}_0)$,

(ii) There exists at least one hypertangent to $\mathcal{F}_2$ at $(x_0, \bar{y}_0) \qquad (\forall \bar{y}_0 \in \overline{Y}_0)$.

Then, for all $(h,k) \in \bigcap_{\bar{y}_0 \in \overline{Y}_0} \{T_{\mathcal{F}_1}(x_0, \bar{y}_0) \cap T_{\mathcal{F}_2}(x_0, \bar{y}_0)\}$,

$$\varphi_F^{\uparrow}(x_0;h) \leq \sup_{\bar{y}_0 \in \overline{Y}_0} f^{\uparrow}_{(x,y)}(x_0,\bar{y}_0;h,k) \tag{3.3}$$

PROOF. Take any $(h,k) \in \bigcap_{\bar{y}_0 \in \overline{Y}_0} T_{\mathcal{F}}(x_0,\bar{y}_0)$ and any $\varepsilon > 0$. We set

$$\varphi_F^{\varepsilon}(x_0;h) = \limsup_{\substack{(x,\alpha)\downarrow_{\varphi_F} x_0 \\ \lambda \downarrow 0}} \inf_{\omega \in h+\varepsilon B} \frac{\varphi_F(x+\lambda\omega)-\alpha}{\lambda} .$$

and choose sequences $(x_n,\alpha_n) \downarrow_{\varphi_F} x_0$, $\lambda_n \downarrow 0$ such that

$$\varphi_F^{\varepsilon}(x_0;h) = \lim_{n\to\infty} \inf_{\omega \in h+\varepsilon B} \frac{\varphi_F(x_n+\lambda_n\omega)-\alpha_n}{\lambda_n} . \tag{3.4}$$

Then, we remark that

$$\varphi_F(x_n) \leq \alpha_n,\ x_n \to x_0,\ \alpha_n \to \varphi_F(x_0) \tag{3.5}$$

For the bounded selection $\bar{y}$, denote $\bar{y}_n = \bar{y}(x_n)$. Arguing as in the proof of Theorem 2.1 we have

$$\lim_{n\to\infty} \bar{y}_n = \hat{y}_0 \qquad (\hat{y}_0 \in \overline{Y}_0) \qquad \text{and} \qquad \varphi_F(x_n) = f(x_n,\bar{y}_n). \tag{3.6}$$

Note that we have written here $\{\bar{y}_n\}$ instead of its subsequence.

It follows from (3.5) and (3.6) that $((x_n,\bar{y}_n),\alpha_n) \in \operatorname{epi} f$. Hence, observing that $\hat{y}_0 \in M_F(x_0)$, we get

$$((x_n,\bar{y}_n),\alpha_n) \downarrow_f (x_0,\hat{y}_0)$$

In view of Theorem 2.4.5 [4], there exists a sequence $(h_n,k_n) \to (h,k)$ such that

$$(x_n,\bar{y}_n) + \lambda_n(h_n,k_n) \in \mathcal{F} ,$$

whence,

$$\varphi_F(x_n+\lambda_n h_n) - f(x_n+\lambda_n h_n,\ \bar{y}_n+\lambda_n k_n) \leq 0.$$

Consequently,

$$\begin{aligned}\varphi_F(x_n+\lambda_n h_n) - \alpha_n &= \varphi_F(x_n+\lambda_n h_n) - f(x_n+\lambda_n h_n,\ \bar{y}_n+\lambda_n k_n) + \\ &\quad + f(x_n+\lambda_n h_n,\ \bar{y}_n+\lambda_n k_n) - \alpha_n \\ &\leq f(x_n+\lambda_n h_n,\ \bar{y}_n+\lambda_n k_n) - \alpha_n,\end{aligned}$$

which implies that

$$\frac{\varphi_F(x_n+\lambda_n h_n)-\alpha_n}{\lambda_n} \leq \frac{f(x_n+\lambda_n h_n,\ \bar{y}_n+\lambda_n k_n)-\alpha_n}{\lambda_n} \tag{3.7}$$

Moreover, for all n sufficiently large

$$\inf_{\omega\in h+\varepsilon B} \frac{\varphi_F(x_n+\lambda_n\omega)-\alpha_n}{\lambda_n} \leq \frac{\varphi_F(x_n+\lambda_n h_n)-\alpha_n}{\lambda_n},$$

which together with (3.4) and (3.7) implies that

$$\varphi_F^{\varepsilon}(x_0;h) \leq \sup_{\bar{y}_0\in\overline{Y}_0} f^{+}_{(x,y)}(x_0,\bar{y}_0;h,k)\ .$$

Therefore,

$$\varphi_F^{\uparrow}(x_0;h)=\lim_{\varepsilon\downarrow 0}\varphi_F^{\varepsilon}(x_0;h) \leq \sup_{\bar{y}_0\in\overline{Y}_0} f^{+}_{(x,y)}(x_0,\bar{y}_0;h,k). \tag{3.8}$$

Making use of Theorem 2.9.5 [4] we get

$$f^{+}_{(x,y)}(x_0,\bar{y}_0;h,k)=f^{\uparrow}_{(x,y)}(x_0,\bar{y}_0;h,k) \quad (\forall\bar{y}_0\in\overline{Y}_0,\forall(h,k)\in \bigcap_{\bar{y}_0\in\overline{Y}_0} T_{\mathcal{F}}(x_0,\bar{y}_0)). \tag{3.9}$$

Substituting (3.9) in (3.8) yields that

$$\varphi_F^{\uparrow}(x_0;h) \leq \sup_{\bar{y}_0\in\overline{Y}_0} f^{\uparrow}_{(x,y)}(x_0,\bar{y}_0;h,k) \quad (\forall(h,k)\in \bigcap_{\bar{y}_0\in\overline{Y}_0} T_{\mathcal{F}}(x_0,\bar{y}_0)), \tag{3.10}$$

Taking into account Corollary 2 of Theorem 2.9.8 [4] we get

$$T_{\mathcal{F}}(x_0,\bar{y}_0) \supset T_{\mathcal{F}_1}(x_0,\bar{y}_0)\cap T_{\mathcal{F}_2}(x_0,\bar{y}_0) \qquad (\forall\bar{y}_0\in\overline{Y}_0) \tag{3.11}$$

Hence, a combination of (3.10) and (3.11) yields the conclusion of the theorem.

REMARK 3.1. After the proof of Theorem 3.1 we find out that if in its statement we drop assumptions (i) and (ii), then the inequality (3.3) is fulfilled for all $(h,k)\in \bigcap_{\bar{y}_0\in\overline{Y}_0} T_{\mathcal{F}_1\cap\mathcal{F}_2}(x_0,\bar{y}_0)$.

REMARK 3.2. If M_F is upper semicontinuous at x_0, then $\overline{Y}_0\subset M_F(x_0)$. In the case where M_F possesses a continuous selection, by Theorem 3.1 and Remark 3.1 we get the following

COROLLARY 3.1. *Assume that M_F possesses a selection $\bar{y}$ which is continuous at $x_0 \in X$ and f is directionally Lipschitz at (x_0, y_0) with respect to all $(h,k) \in T_{\mathcal{F}}(x_0, y_0)$, where $y_0 = \lim_{x \to x_0} \bar{y}(x) = \bar{y}(x_0)$. Then, for all $(h,k) \in T_{\mathcal{F}_1 \cap \mathcal{F}_2}(x_0, y_0)$,*

$$\varphi_F^{\uparrow}(x_0; h) \leq f^{\uparrow}_{(x,y)}(x_0, y_0; h, k).$$

REMARK 3.3. When X is finite-dimensional, assumption (ii) in Theorem 3.1 can be dropped. Indeed, since $\mathcal{F}_2$ is a closed subset of the finite-dimensional space $X \times Y$, by virtue of Corollary 1 of Theorem 2.5.8 of [4], it follows that for all $\bar{y}_0 \in \overline{Y}_0$,

$$\text{int } T_{\mathcal{F}_2}(x_0, \bar{y}_0) = H_{\mathcal{F}_2}(x_0, \bar{y}_0),$$

where $H_{\mathcal{F}_2}(x_0, \bar{y}_0)$ is the set of all hypertangents to $\mathcal{F}_2$ at $(x_0, \bar{y}_0)$.

In view of assumption (i), condition (ii) of Theorem 3.1 is fulfilled.

4. Applications

In this section we shall be concerned with the following problem

$$(P) \qquad \begin{aligned} & J(x,u) \to \inf\,, \\ & \text{subject to} \\ & u \in F(x)\,, \\ & u \in \Omega\,. \end{aligned}$$

This is a generalization of optimal control problems governed by hemivariational inequalities studied in [8] by Haslinger and Panagiotopoulos. Here, $x \in X$ is the state variable, $u \in Y$ is the control variable, J is an extended-real-valued function on $X \times Y$, F is a set-valued map from X into Y, Ω is a closed subset of Y.

It should be noted here that in [8] the authors have only investigated the existence of solutions for the Lipschitz case. Here we consider the case where f is directionally Lipschitz.

In accordance with notations of the previous section we denote

$$\varphi_F(x) = \inf\,\{J(x,u) \mid u \in F(x) \cap \Omega\},$$

$$M_F(x) = \{u \in F(x) \cap \Omega \mid \varphi_F(x) = J(x,u)\}.$$

A necessary optimality condition for (P) can be stated as follows

THEOREM 4.1. *Let (x_0, u_0) be a local minimum of (P). Assume that the following hypotheses are fulfilled*

(i) M_F posseses a selection $\bar{u}$, which is continuous at $x_0 \in X$, such that $u_0 = \bar{u}(x_0) = \lim_{x \to x_0} \bar{u}(x)$;

(ii) J is directionally Lipschitz at (x_0, u_0) with respect to all $(h, k) \in T_{\mathcal{F} \cap (X \times \Omega)}(x_0, u_0)$, where $\mathcal{F}$ stands for the graph of the map F;

(iii) $T_{\mathcal{F}}(x_0, u_0) \cap (X \times \operatorname{int} T_\Omega(u_0)) \neq \emptyset$. Then,

$$(0,0) \in \overline{\partial}_{(x,u)} J(x_0, u_0) + N_{\mathcal{F}}(x_0, u_0) + \{0\} \times N_\Omega(u_0), \tag{4.1}$$

where $\overline{\partial}_{(x,u)} J$ is the set of subgradients of J, defined by (2.5).

PROOF. It is easy to see that the graph $\mathcal{F}_1$ of the map $F_1 : x \mapsto F(x) \cap \Omega$ is of the form

$$\mathcal{F}_1 = \mathcal{F} \cap (X \times \Omega)$$

Since $\varphi_F(x)$ attains a minimum at $x_0 \in X$, it follows from assumption (i) that

$$\inf_{x \in X} \varphi_F(x) = \inf_{x \in X} J(x, \bar{u}(x)).$$

From this, taking into account Proposition 2.4.1 of [4] one gets

$$0 \in \overline{\partial} \varphi_F(x_0) \tag{4.2}$$

In view of assumption (i), the set of all accumulation points of $\bar{u}(x)$ as $x \to x_0$ shrinks to a singleton $\{u_0\}$, where $u_0 = \lim_{x \to x_0} \bar{u}(x)$. According to Theorem 3.1 and Remark 3.1 we obtain

$$\varphi_F^{\uparrow}(x_0; h) \leq J^{\uparrow}_{(x,u)}(x_0, u_0; h, k) \ , (\forall (h,k) \in T_{\mathcal{F} \cap (X \times \Omega)}(x_0, u_0)) \tag{4.3}$$

Combining (4.2) and (4.3) yields that

$$J^{\uparrow}_{(x,u)}(x_0, u_0; h, k) \geq 0 \ , (\forall (h,k) \in T_{\mathcal{F} \cap (X \times \Omega)}(x_0, u_0)) \tag{4.4}$$

which may be rewritten in the form

$$J^{\uparrow}_{(x,u)}(x_0, u_0; h, k) + \delta_{\mathcal{F},\Omega}(h, k) \geq 0, \tag{4.5}$$

where $\delta_{\mathcal{F},\Omega} := \delta_{T_{\mathcal{F}\cap(X\times\Omega)}}$, the indicator function of the convex cone $T_{\mathcal{F}\cap(X\times\Omega)}(x_0,u_0)$.

Observing that J is directionally Lipschitz at (x_0,u_0) with respect to all $(h,k)\in T_{\mathcal{F}\cap(X\times\Omega)}(x_0,u_0)$, in view of Theorem 2.9.5 of [4] we can see that $J^{\uparrow}_{(x,u)}(x_0,u_0;\cdot)$ is continuous in $T_{\mathcal{F}\cap(X\times\Omega)}(x_0,u_0)$ and $J^{\uparrow}_{(x,u)}(x_0,u_0;h,k)<+\infty$ for all $(h,k)\in T_{\mathcal{F}\cap(X\times\Omega)}(x_0,u_0)$. Moreover, $J^{\uparrow}_{(x,u)}(x_0,u_0;\cdot)$ is positively homogeneous and $T_{\mathcal{F}\cap(X\times\Omega)}(x_0,u_0)$ is closed. Hence,

$$J^{\uparrow}_{(x,u)}(x_0,u_0;0,0)=0\ . \tag{4.6}$$

Since the function $J^{\uparrow}_{(x,u)}(x_0,u_0;\cdot)+\delta_{\mathcal{F},\Omega}(\cdot)$ is convex proper, it follows from (4.5), (4.6) that

$$(0,0)\in\partial_{(h,k)}(J^{\uparrow}_{(x,u)}+\delta_{\mathcal{F},\Omega})(0,0)\ , \tag{4.7}$$

where ∂ stands for the subdifferential of a convex function.

Applying the Moreau-Rockafellar Theorem 0.3.3 of [10] yields

$$\begin{gathered}\partial_{(h,k)}(J^{\uparrow}_{(x,u)}+\delta_{\mathcal{F},\Omega})(0,0)=\\ \partial_{(h,k)}(J^{\uparrow}_{(x,u)}(x_0,u_0;0,0))+\partial_{(h,k)}\delta_{\mathcal{F},\Omega}(0,0)\end{gathered} \tag{4.8}$$

From the convex analysis we have

$$\begin{aligned}\partial\delta_{\mathcal{F},\Omega}(0,0)&=\{(x^*,u^*):<(x^*,u^*),(h,k)>\le 0\ ,\\ &\qquad\forall(h,k)\in T_{\mathcal{F}\cap(X\times\Omega)}(x_0,u_0)\}\\ &=N_{\mathcal{F}\cap(X\times\Omega)}(x_0,u_0)\end{aligned} \tag{4.9}$$

It follows from assumption (iii) and Corollary 1 of Theorem 2.5.8 of [4] that there exists at least one hypertangent to Ω at u_0. By virtue of assumption (iii) and making use of Corollary 2 of Theorem 2.9.8 and Corollary of Theorem 2.4.5 of [4] we get

$$N_{\mathcal{F}\cap(X\times\Omega)}(x_0,u_0)\subset N_{\mathcal{F}}(x_0,u_0)+\{0\}\times N_{\Omega}(u_0) \tag{4.10}$$

Moreover, in view of (4.6) and taking into account Corollary of Theorem 2.9.1 of [4], $\overline{\partial}_{(x,u)}J$ can be expressed by (2.5). Consequently,

$$\partial_{(h,k)}J^{\uparrow}_{(x,u)}(x_0,u_0;0,0)=\overline{\partial}_{(x,u)}J(x_0,u_0)\ . \tag{4.11}$$

Finally, combinining (4.7)–(4.11) we obtain (4.1).

The proof is complete.

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