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# A NEW INTEGRAL CONTAINING THE JACOBI POLYNOMIAL

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**Abstract.** A new integral containing the Jacobi polynomials is given with an elementary multiplier function.

## 1. Introduction

The Jacobi polynomials are orthogonal polynomials [3, p.285, (5) and (9)] over the interval $(-1,1)$ with respect to the weight function $(1-x)^a(1+x)^b$, if Re $a>-1$, Re $b>-1$. In fact in order to make the weight function non-negative and integrable, we assume $a>-1, b>-1$. However many of the formal results are valid without these restrictions [4, p.274]. This remark of Luke is true in case of the result presented in this paper.

We will present a new integral containing the Jacobi polynomials over the interval $(1,\infty)$ with respect to the multiplier function $(x-1)^{a+1}(x+1)^{b+n-m-2}$, if Re $a>-2$, Re $(a+b)<-2n$. In order to make the multiplier function non-negative and integrable, we have $a>-1, (a+b)<-2n$.

The Jacobi polynomials are defined by the relations [2, p.170, (160)]:

$$P_n^{(a,b)}(x)=\frac{(1+a)_n}{n!}\left(\frac{1+x}{2}\right)^n {}_2F_1\left(-n,-n-b;a+1;\frac{x-1}{x+1}\right), \tag{1.1}$$

provided $a$ is not a negative integer.

The following formulae are required in the proof.

The modified form of the integral [3, p.201,(6)]:

$$\int_1^\infty (x-1)^{w-1}(x+1)^b\,dx=2^{w+b}\frac{\Gamma(w)\Gamma(-b-w)}{\Gamma(-b)},$$

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$$\text{Re } b < 0,\ \text{Re } w > 0,\ \text{Re } (w+b) < 0 \tag{1.2}$$

The Saalschutz's theorem [1, p.188, (3)]:

$$ {}_3F_2 \left[ \begin{matrix} -n, a, b; & 1 \\ c, 1-c+a+b-n & \end{matrix} \right] = \frac{(c-a)_n (c-b)_n}{(c)_n (c-a-b)_n} \tag{1.3}$$

The modified form of the relation [1, p.3, (4)]:

$$\Gamma(1+a-n) = \frac{(-1)^n (1+a)}{(-a)_n} \tag{1.4}$$

The formula [1, p.3, (6)]:

$$\sin \pi z = \frac{\pi}{\Gamma(z)\Gamma(1-z)} \tag{1.5}$$

## 2. The integral

The integral to be established is

$$\int_1^\infty (x-1)^{a+1}(x+1)^{b+n-m-2} P_m^{(a,b)}(x) P_n^{(a,b)}(x)\,dx = 0, \text{ if } m < n-1 \tag{2.1}$$

$$= \frac{2^{a+b+1}\Gamma(1+a+n)\Gamma(b+n)\sin \pi b}{b(n-1)!\Gamma(1+a+b+n)\sin \pi(1+a+b)}, \text{ if } m = n-1 \tag{2.2}$$

$$= -\frac{2^{a+b}\Gamma(1+a+n)\Gamma(1+b+n)}{n!\Gamma(1+a+b+n)} . \frac{\sin\pi(1+b)}{\sin\pi(1+a+b)} . \frac{\Gamma(-1-b)}{(2-b)}$$
$$ .[n(a+n)(1-b) + (n+1)(1+a+n)(1+b)], \quad m = n \tag{2.3}$$

$$= \frac{2^{a+b-1}\Gamma(2+a+n)\Gamma(2+b+n)}{n!\Gamma(1+a+b+n)} . \frac{\sin\pi(2+b)}{\sin\pi(1+a+b)}$$
$$ .\left[ -\frac{n(a+n)\Gamma(-2-b)}{\Gamma(1-b)} + \frac{(n+1)(1+a+n)(-1-b)}{\Gamma(2-b)} - \right.$$
$$ \left. -\frac{(n+2)(2+a+n)\Gamma(-b)}{2\Gamma(3-b)} \right], m = n+1 \tag{2.4}$$

where Re $a > -2$, Re $(a+b) < -2n$, provided $(1+b)$ and $(1+a+b)$ are nonintegrals.

PROOF. In view of (1.1), the integral (2.1) can be written as

$$\frac{(1+a)_m}{m!2^m}\frac{(1+a)_n}{n!2^n}\int_1^\infty (x-1)^{a+1}(x+1)^{b+2n-2}{}_2F_1\left(-m,-m-b;1+a;\frac{x-1}{x+1}\right)$$

$$\cdot {}_2F_1\left(-n,-n-b;1+a;\frac{x-1}{x+1}\right)dx$$

$$\frac{(1+a)_m(1+a)_n}{m!n!2^{m+n}}\sum_{r=0}^{m}\frac{(-m)_r(-m-b)_r}{(1+a)_r r!}\sum_{u=0}^{n}\frac{(-n)_u(-n-b)_u}{(1+a)_u u!}$$

$$\cdot\int_1^\infty (x-1)^{a+r+u+1}(x+1)^{b+2n-2-r-u}dx \tag{2.5}$$

Evaluating the last integral in (2.5) with the help of (1.2), then using (1.4) and simplifying, the right hand side of (2.5) becomes

$$\frac{(1+a)_m(1+a)_n\Gamma(-a-b-2n)}{m!n!2^{m-n-a-b}}\sum_{r=0}^{m}\frac{(-m)_m(-m-b)_r}{(1+a)_r r!}$$

$$\cdot\frac{\Gamma(a+r+2)}{\Gamma(2-b-2n+r)}{}_3F_2\left[\begin{matrix}-n,-n-b,2+a+r; & 1\\ 1+a,2-b-2n+r & \end{matrix}\right] \tag{2.6}$$

Now applying Saalschutz's theorem (1.3) to (2.6), we get

$$\frac{(1+a)_m\Gamma(-a-b-2n)}{m!n!2^{m-n-a-b}}.$$

$$\sum_{r=0}^{m}\frac{(-m)_r(-1-r)_n\Gamma(a+r+2)(-m-b)_r(1+a+b+n)_n}{(1+a)_r r!\Gamma(2-b-2n+r)(b+n-r-1)_n} \tag{2.7}$$

If $r < n-1$, the numerator of (2.7) vanishes and since $r$ runs from 0 to $m$, if follows that (2.7) vanishes when $m < n-1$, which proves (2.1).

When $m = n-1$, using the standard result $(-n)^n = (-1)^n n!$ and (1.4), we obtain

$$\int_1^\infty (x-1)^{a+1}(x+1)^{b-1}P_{n-1}^{(a,b)}(x)P_n^{(a,b)}(x)dx$$

$$=\frac{2^{a+b+1}\Gamma(1+a+n)\Gamma(b+n)\Gamma(1+a+b)\Gamma(-a-b)}{b(n-1)!\Gamma(1+a+b+n)\Gamma(b)\Gamma(1-b)} \tag{2.8}$$

Applying (1.5) to (2.8) we get (2.2).

When $m = n$, using standart results like $(-n)_{n-1} = (-1)^n n!$ and $(-n-1)_n = (-1)^n (n+1)!$ and adding the two resulting terms $(r = n-1, n)$, we have

$$\int_1^\infty (x-1)^{a+1}(x+a)^{b-2}\{P_n^{(a,b)}\}dx$$

$$= -\frac{2^{a+b}\Gamma(1+a+n)\Gamma(1+b+n)\Gamma(-a-b)\Gamma(1+a+b)\Gamma(-1-b)}{n!\Gamma(1+a+b+n)\Gamma(-b)\Gamma(1+b)\Gamma(2-b)}$$
$$.[n(a+n)(1-b)+(n+1)(1+a+n)(1+b)] \tag{2.9}$$

On applying (1.5) to (2.9), we obtain (2.3)

When $m = n+1$, using standard results $(-n)_n = (-1)^n n!, (-n-1)_n = (-1)^1(n+1)!$ and $(-2-n)_n = 1/2(-1)^n(n+2)!$ and adding the resulting terms $(r = n-1, n, n+1)$, we get

$$\int_1^\infty (x-1)^{a+1}(x+1)^{b-3}P_{n+1}^{(a,b)}(x)P_n^{(a,b)}(x)dx$$
$$= \frac{2^{a+b-1}\Gamma(2+a+n)\Gamma(2+b+n)\Gamma(1+a+b)\Gamma(-a-b)}{n!\Gamma(1+a+b+n)\Gamma(-1-b)\Gamma(2+b)}$$
$$\cdot\Big[-\frac{n(a+n)\Gamma(-2-b)}{\Gamma(1-b)}+\frac{(n+1)(1+a+n)\Gamma(-1-b)}{\Gamma(2-b)}-$$
$$\frac{(n+2)(2+a+n)\Gamma(-b)}{2\Gamma(3-b)}\Big] \tag{2.10}$$

By applying (1.5) to (2.10) we get (2.4).

REMARK. On continuing as above, we can find the values of the integral (2.1) for $m = n+2, n+3, n+4, \ldots$, since the value of the integral is not zero for any value of $m < n-1$.

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