ACTA MATHEMATICA VIETNAMICA
Volume 18, Number 2, 1993

A SUFFICIENT CONDITION FOR BIJECTIVITY OF POLYNOMIAL MAPS ON THE REAL PLANE

NGUYEN VAN CHAU

Abstract. It is shown that the non-singular polynomial map f of R^2 into itself is a global diffeomorphism of R^2 if $0 \notin Co\{Df(x)v \mid ||x|| > Const.\}$ for a vector v. This result is a variant of a theorem of Olech and Meister [3,6] for the polynomial case.

1. Introduction

Considering the global diffeomorphism problem for differentiable maps of R^2, Olech and Meister [3,6] have obtained the following interesting result:

The continuously differentiable map f of R without any singularity is injective if

$$0 \notin Co\{Df(x)v \mid x \in R^2\} \tag{1}$$

for two linearly independent vectors v.

Here $Co\{.\}$ indicates the convex hull of the set $\{.\}$.

We will prove the following variant of this result for the polynomial case.

THEOREM. *Suppose f is a non-singular polynomial map of R^2 into itself. If there exist $v \in R^2$ and $C > 0$ such that*

$$0 \notin Co\{Df(x)v \mid ||x|| > C\} \tag{2}$$

then f is a global diffeomorphism of R^2.

A special case of condition (2) is when one of the partial derivatives of f does not vanish outside a bounded set. The maps

$$f(x,y) = (y - x(x^2+y^2-\epsilon), -x - y(x^2+y^2-\epsilon)), \quad \epsilon < \sqrt{3}$$

Received April 2, 1992; in revised form March 15, 1993

give simple examples for this case.

Note that the famous Jacobian Conjecture (cf. [1,4,6]), which remains unsolved even for the two-dimensional case, asserts that every non-singular polynomial map of R^2 into itself is bijective.

2. Proof of Theorem

Let us use the Cartesian coordinate $x = (x_1, x_2)$ in the Euclidean space R^2 and the symbol $< \dots >$ to indicate the scalar product in R^2. Let f be a polynomial map of R^2 satisfying the assumptions of the theorem, $f = (f_1, f_2)$. By condition (2) and in view of the Hahn-Banach Theorem there exist real numbers a and b, $a^2 + b^2 > 0$ such that

$$\langle grad(af_1 + bf_2)(x), v\rangle > 0, \text{ for } \|x\| > C. \tag{3}$$

We put $F = (F_1, F_2)$, $F_1 = af_1 + bf_2$, $F_2 = af_2 - bf_1$. Then F is a polynomial map of R^2 without any singularity and f is a diffeomorphism of R^2 iff so is F. Furthermore, by (3)

$$\langle gradF_1(x), v\rangle > 0 \quad \text{for } \|x\| > C.$$

Next, we observe that the restriction of F_2 on each connected component of level sets of F_1 is injective, because F has no singularity. Then, by the well-known fact that every injective polynomial map of R^n is bijective [2,5], we only need to show that every level set of F_1 is connected. The proof will be completed by the following.

LEMMA. *Let g be a polynomial of n real variables without any singularity. Assume that there exist $v \in R^n$ and $C > 0$ such that*

$$\langle gradg(x), v\rangle > 0, \quad \text{for } \|x\| > C. \tag{4}$$

Then there exists a diffeomorphism Φ of R^n such that

$$g.\Phi(x_1, x_2, \dots, x_n) = x_1. \tag{5}$$

PROOF. Without loss of generality we may assume that $v = (1, 0, \ldots, 0)$ in (4). Since g has no any singularity and because of (4), using unite partition we can construct a differentiable vector field W on R^n such that

$$W(x) = \begin{cases} \text{grad}g(x).||\text{grad}g(x)||^2 & \text{for } ||x|| < C - \epsilon \\ < \text{grad}g(x), v >^{-1} & \text{for } ||x|| > C + \epsilon, \end{cases} \tag{6}$$

and

$$< \text{grad}g(x), W(x) > = 1 \text{ on } R^n, \tag{7}$$

where ϵ is a positive number.

Let $\varphi(x,t)$ be the local flow on R^n induced by the vector field W. Let us fix a point x_0 and consider the trajectory $\varphi(x_0, t)$. By (7) we have

$$g.\varphi(x_0, t) = g(x_0) + t. \tag{8}$$

The trajectory $\varphi(x_0, t)$ must tend to infinity in two directions. Indeed, because of (6) the parts outside the ball $\{x \mid ||x|| < C + \epsilon\}$ of this trajectory are half lines of directions v and $-v$. Hence the restriction of the polynomial g on the trajectory $\varphi(x_0, t)$ obtain all real values. This means that $\varphi(x_0, t)$ is defined for all t. Thus W induces the one-parameter group $\varphi(., t)$ of diffeomorphisms of R^n.

Set $V = \{x \in R^n \mid g(x) = 0\}$, which can be seen as a submanifold of R^n. We define the map $\varphi_1 : V \times R \to R^n$ by $\varphi_1(x,t) = \varphi(x,t)$. Clearly, φ_1 is diffeomorphic. Let us fix a constant $a > C + \epsilon$ and denote by L the surface $x_1 = a$. We define the map $\varphi_2 : L \to V$ by

$$\varphi_2(a, x_2, \ldots, x_n) = \varphi((a, x_2, \ldots, x_n), -g(a, x_2, \ldots, x_n)).$$

Because of (8) this map is well defined. Since $a > C + \epsilon$, from (6) it follows that at each point of L the vector field W is orthogonal to the surface L. Using this fact one can see that φ_2 is also diffeomorphic.

Now, we can construct the map Φ from R^n into itself by setting

$$\Phi(x_1, x_2, \ldots, x_n) = \varphi_1(\varphi_2(a, x_2, \ldots, x_n), x_1).$$

It is easy to verify that this map is a diffeomorphism of R^n and that

$$\Phi(x_1, x_2, \ldots, x_n) = x_1.$$

ACKNOWLEDGEMENTS. The author wishes to thank Prof. C. Olech for stimulating discussions and valuable suggestions.

REFERENCES

[1] H. Bass, E.H Connell and D. Wright, *The Jacobian Conjecture*, Bull Amer. Math. Soc. **7** (1982), 287-330.

[2] A. Bialynicki-Birula and M.Rosenlicht, *Injective morphisms of real algebraic varieties*, Proc. Amer. Math. Soc. **13** (1962), 200-204.

[3] G.H. Meister and C. Olech, *A Jacobian condition for injectivity of differentiable plane maps*, Ann. Polon. Math. **LI** (1990), 249-254.

[4] G.H. Meister and C. Olech, *A poly-flow formulation of the Jacobian Conjecture*, Bull. Polish Acad. Sci. Math. **35** (1987), 725-731.

[5] D. J. Newman, *One-one polynomial maps*, Proc. Amer. Math. Soc. **11** (1960), 867-870.

[6] C. Olech, *Global Diffeomorphism Question and Differential Equations*, Colloquia Mathematica Societatis Janos Bolyai 53, Qualitative theory of differential equations, Szeged (Hungary), 1988, 465-471.

HANOI INSTITUTE OF MATHEMATICS
P.O. BOX 631, 10.000 HANOI, VIETNAM