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HOPF BIFURCATION AT A DOUBLE EIGENVALUE*

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1. Introduction

Let R^n be the n-dimensional Euclidean space, $n \geq 4$. It is customary to simplify a notation by dropping the superscript $R^1 = R$. We consider Hopf bifurcation points of the following dynamic system

$$\frac{dx}{dt} = \dot{x} = f(x,\mu),\ (x,\mu) \in R^{n+1}, \tag{1}$$

where f is a smooth mapping from R^{n+1} into R^n, $f(0,\mu) = 0$ for all $\mu \in R$. By $A(\mu)$ we denote the derivative of $f(\cdot,\mu)$ with respect to $x \in R^n$ at zero, i.e. $A(\mu) = f_x(0,\mu)$.

Let $\mu_0 \in R$ be such that the $n \times n$-matrix $A = A(\mu_0)$ has a pair of pure imaginary eigenvalues $\lambda(\mu_0) = \pm i\beta$. Without loss of generality we may assume $\beta = 1$. In the case when $\pm i$ are simple eigenvalues of A, there are many results on the Hopf bifurcation of Eq. (1), (see, for example [1], [3], [4], [6], [7], and the references therein). The purpose of this paper is to study the Hopf bifurcation of (1) in the case when $\pm i$ are double eigenvalues of A. In what follows we only consider the case when i is a double eigenvalue of A. By $R_0 = \ker(A - iI)$ we denote the null space of the mapping $A - iI$, i.e., $R_0 = \{x \in R^n/(A - iI)x = 0\}$. Here, I stands for the identity mapping. Let

$$R_0 = [v^1, v^2]$$

and

$$S_0 = \ker(A - iI)^* = [\gamma^1, \gamma^2].$$

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Setting

$$\phi^j = Re(e^{it}v^j) = \cos t Re(v^j) - \sin t Im(v^j),$$
$$\phi^{j+1} = Im(e^{it}v^j) = \cos t Im(v^j) + \sin t Re(v^j),$$
$$\psi^j = Re(e^{it}\gamma^j) = \cos t Re(\gamma^j) - \sin t Im(\gamma^j),$$
$$\psi^{j+1} = Im(e^{it}\gamma^j) = \cos t Im(\gamma^j) + \sin t Re(\gamma^j),\ j = 1, 2,$$

we can easily see that $\phi^k,\ \psi^k,\ k = 1, ..., 4$ satisfy the equations

$$\dot{x} = Ax,$$
$$\dot{y} = A^* y,$$

respectively.

Further, let $X = C_{2\pi}(R, R^n)$ ($Y = C_0([0, 2\pi], R^n)$) denote the space of continuous 2π-periodic functions from R into R^n (continuous functions $h : [0, 2\pi] \to R^n$ with $h(0) = 0$). The scalar product $<,>$ and the norm $||\cdot||$ in X and Y are given by

$$< x, y >= \int_0^{2\pi} (x(t), y(t))dt,$$
$$||x|| = (< x, x >)^{1/2},$$

where $(,)$ denotes the scalar product in R^n defined by usual ways. We want to find (x, μ) satisfying (1) with x being a nonzero $(1 + \rho)2\pi$-periodic function (ρ is unknown and to be determined). Inserting $t = (1 + \rho)\tau$ into (1) we obtain the equation

$$\dot{x} = \frac{dx}{d\tau} = (1 + \rho)(f(x, \mu),\ (x, \mu) \in R^{n+1}. \tag{2}$$

To find (x, μ) as above, it is enough to find (x, μ, ρ) satisfying (2) with x being a nonzero 2π-periodic function, i.e., $x \in X,\ x \neq 0$.

2. Lyapunov-Schmidt procedure for evolution equations

Using Taylor's expansion, we see that Eq. (2) can be written as

$$\dot{x} = (1 + \rho)\{Ax + (\mu - \mu_0)f_{x\mu}(0, \mu_0)x + \frac{1}{2}f_{xx}(0, \mu_0)x^2 \tag{3}$$
$$\frac{1}{6}f_{xxx}(0, \mu_0)x^3 + \text{ high order term } (HoT)\},$$

where $(HoT) = \mathcal{O}(|\mu - \mu_0|^2 x,\ \|x\|^3)$. Putting $\lambda = \mu - \mu_0,\ L = f_{x\mu}(0, \mu_0)$, the equation (3) becomes

$$\dot{x} = (1+\rho)\{Ax + \lambda Lx + \frac{1}{2} f_{xx}(0,\mu_0)x^2 + \frac{1}{6} f_{xxx}(0,\mu_0)x^3 + HoT\}. \qquad (4)$$

Further, set

$$X_0 = [\phi^1, \phi^2, \phi^3, \phi^4],$$
$$Y_0 = [\psi^1, \psi^2, \psi^3, \psi^4],$$

where $[\varphi^1, ..., \varphi^n]$ denotes the real subspace spanned by $\{\varphi^1, ..., \varphi^n\}$. One can easily verify that

$$X = X_0 \oplus X_1,$$
$$Y = Y_0 \oplus Y_1,$$

with

$$X_1 = \{x \in X \ / \ < x, z > = 0 \text{ for all } z \in X_0\},$$
$$Y_1 = \{y \in Y \ / \ < y, v > = 0 \text{ for all } v \in Y_0\}.$$

We define the projectors $P : Y \to Y_0,\ Q : Y \to Y_1$ by

$$P(y) = \sum_{j=1}^{4} < y, \psi^j > \psi^j, \quad Q(y) = y - P(y).$$

Then Eq. (4) can be reduced to the following equations

$$Q(\dot{x} - (1+\rho)\{Ax + \lambda Lx + \frac{1}{2} f_{xx}(0,\mu_0)x^2 + \frac{1}{6} f_{xxx}(0,\mu_0)x^3 + HoT\}) = 0$$
$$(5)$$
$$P(\dot{x} - (1+\rho)\{Ax + \lambda Lx + \frac{1}{2} f_{xx}(0,\mu_0)x^2 + \frac{1}{6} f_{xxx}(0,\mu_0)x^3 + HoT\}) = 0.$$

But any $x \in X$ can be written as $x = \sum_{j=1}^{4} \epsilon \phi^j + z$, for some $\epsilon_1, ..., \epsilon_4 \in R,\ z \in X_1$. Therefore, to solve the system (5) is equivalent to find $\rho, \lambda \in R^1, \epsilon = (\epsilon_1, ..., \epsilon_4) \in R^4,\ z \in X_1$ satisfying the system

$$\dot{z} - Az - Q(\rho A(\sum_{j=1}^{p} \epsilon_j \phi^j + z) + (1+\rho)\{\lambda L(\sum_{j=1}^{4} \epsilon_j \phi^j + z)$$

$$+ \frac{1}{2} f_{xx}(0,\mu_0)(\sum_{j=1}^{4} \epsilon_j \phi^j + z)^2 + \frac{1}{6} f_{xxx}(0,\mu_0)(\sum_{j=1}^{4} \epsilon_j \phi^j + z)^3\} + HoT) = 0,$$

$$P(\frac{d}{dt}(\sum_{j=1}^{4} \epsilon_j \phi^j + z) - (1+\rho)\{A(\sum_{j=1}^{4} \epsilon_j \phi^j + z) + \lambda L(\sum_{j=1}^{4} \epsilon_j \phi^j + z)$$

$$+ \frac{1}{2} f_{xx}(0,\mu_0)(\sum_{j=1}^{4} \epsilon_j \phi^j + z)^2 + \frac{1}{6} f_{xxx}(0,\mu_0)(\sum_{j=1}^{4} \epsilon_j \phi^j + z)^3\} + HoT) = 0.$$

This is equivalent to

$$\dot{z} - Az - Q(\rho A(\sum_{j=1}^{4} \epsilon_j \phi^j + z) + (1+\rho)\{\lambda L(\sum_{j=1}^{4} \epsilon_j \phi^j + z)$$

$$+ \frac{1}{2} f_{xx}(0,\mu_0)(\sum_{j=1}^{4} \epsilon_j \phi^j + z)^2 + \frac{1}{6} f_{xxx}(0,\mu_0)(\sum_{j=1}^{4} \epsilon_j \phi^j + z)^3 + HoT\}) = 0, \tag{6}$$

$$< \rho A(\sum_{j=1}^{4} \epsilon_j \phi^j + z) + (1+\rho)\{\lambda L(\sum_{j=1}^{4} \epsilon_j \phi^j + z) + \frac{1}{2} f_{xx}(0,\mu_0)(\sum_{j=1}^{4} \epsilon_j \phi^j + z)^2 +$$

$$\frac{1}{6} f_{xxx}(0,\mu_0)(\sum_{j=1}^{4} \epsilon_j \phi^j + z)^3\} + HoT\}, \psi^k > = 0 \quad k = 1,..,4. \tag{7}$$

LEMMA 1. *There exist neighborhoods I_1 of zero in R, U_1 of the origin in R^4, D_1 of the origin in X_1, and a continuous mapping $z : V_1 = I_1 \times I_1 \times U_1 \to D_1$, $z(0,0,0) = 0$, such that for any $(\rho,\lambda,\epsilon) \in V_1$ we have*

$$\dot{z}(\rho,\lambda,\epsilon) - Az(\rho,\lambda,\epsilon) - Q(\rho A(\sum_{j=1}^{4} \epsilon_j \phi^j + z(\rho,\lambda,\epsilon))$$

$$+ (1+\rho)\{\lambda L(\sum_{j=1}^{4} \epsilon_j \phi^j + z(\rho,\lambda,\epsilon)) + \frac{1}{2} f_{xx}(0,\mu_0)(\sum_{j=1}^{4} \epsilon_j \phi^j + z(\rho,\lambda,\epsilon))^2$$

$$+ \frac{1}{6} f_{xxx}(0,\mu_0)(\sum_{j=1}^{4} \epsilon_j \phi^j + z(\rho,\lambda,\epsilon))^3 + HoT\}) = 0. \tag{8}$$

Moreover, for any natural number $m = 1, 2, \ldots$, there exist constants E_m, F_m such that

$$\Gamma(\alpha) = \max\{||z(\pm\alpha^2, \alpha^2, \alpha y)||, ||z(\pm\alpha^2, \pm\alpha^3, \alpha y)||, ||z(\pm\alpha^3, \pm\alpha^2, \alpha y)|| \\ ||z(\pm\alpha^3, \pm\alpha^3, \alpha y)||\} \leq E_m(1 + |y|)\alpha^3 + F_m\alpha^{2n+1}$$

for all $(\alpha, \alpha, \alpha y) \in V_1$. Consequently, we conclude that $\frac{\Gamma(\alpha)}{\alpha^2} = o(|\alpha|)$ as $\alpha \to 0$.

PROOF. We define the mapping $G : R^6 \times X_1 \to Y_1$ by

$$G(\rho, \lambda, \epsilon, z) = \dot{z} - Az - Q(\rho A(\sum_{j=1}^{4} \epsilon_j \phi^j + z) + (1 + \rho)\{\lambda L(\sum_{j=1}^{4} \epsilon_j \phi^j + z)$$

$$+ \frac{1}{2} f_{xx}(0, \mu_0)(\sum_{j=1}^{4} \epsilon_j \phi^j + z)^2 + \frac{1}{6} f_{xxx}(0, \mu_0)(\sum_{j=1}^{4} \epsilon_j \phi^j + z)^3 + HoT\}) = 0.$$

It then follows that $G(0, 0, 0, 0) = 0$ and $G_z(0, 0, 0) = \dot{z} - Az$ which is a one-to-one mapping from X_1 onto Y_1. Using the implicit function theorem we obtain the first assertion. The proof of the second assertion proceeds exactly as the one of Lemma 2 in [8].

3. The main results

For $\sigma = 1$ or $\sigma = -1$ we define the mappings $\mathcal{A}, \mathcal{B}^\sigma, \mathcal{C}^\sigma, \mathcal{D}^\sigma : R^4 \to R^4$, $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_4)$, $\mathcal{B}^\sigma = (\mathcal{B}_1^\sigma, \ldots, \mathcal{B}_4^\sigma)$, $\mathcal{C} = (\mathcal{C}_1^\sigma, \ldots, \mathcal{C}_4^\sigma)$, $\mathcal{D}^\sigma = (\mathcal{D}_1^\sigma, \ldots, \mathcal{D}_4^\sigma)$, by

$$\begin{aligned}
\mathcal{A}_k(y) &= \frac{1}{6} < f_{xxx}(0, \mu_0)(\sum_{j=1}^{4} y_j \phi^j)^3, \psi^k >, \\
\mathcal{B}_k^\sigma(y) &= \sigma < A(\sum_{j=1}^{4} y_j \phi^j), \psi^k > + \mathcal{A}_k(y), \\
\mathcal{C}_k^\sigma(y) &= \sigma < L(\sum_{j=1}^{4} y_j \phi^j), \psi^k > + \mathcal{A}_k(y), \\
\mathcal{D}_k^\sigma(y) &= \sigma < (A + L)(\sum_{j=1}^{4} y_j \phi^j), \psi > + \mathcal{A}_k(y), \\
k &= 1, \ldots, 4, \ y = (y_1, \ldots, y_4) \in R^4.
\end{aligned} \tag{9}$$

Further, we make the following hypotheses on these mappings

HYPOTHESIS 1. There exists a point $\bar{y} = (\bar{y}_1, ..., \bar{y}_4) \in R^4$, $\bar{y} \neq 0$ such that $\mathcal{A}_k(\bar{y}) = 0$ and the 4×4-matrix

$$(\frac{\partial \mathcal{A}_k}{\partial y_j}(\bar{y}))_{k,j=1,...,4}$$

is nonsingular.

HYPOTHESIS 2. Hypothesis 1 with $\bar{y}$ and $\mathcal{A}$ replaced by $\bar{y}^\sigma$ and $\mathcal{B}^\sigma$, respectively.

HYPOTHESIS 3. Hypothesis 1 with $\bar{y}$ and $\mathcal{A}$ replaced by $\bar{y}^\sigma$ and $\mathcal{C}^\sigma$, respectively.

HYPOTHESIS 4. Hypothesis 1 with $\bar{y}$ and $\mathcal{A}$ replaced by $\bar{y}^\sigma$ and $\mathcal{D}^\sigma$, respectively.

THEOREM 1. *Under Hypothesis 1, $(0, \mu_0)$ is a bifurcation point of periodic solutions of Eq. (1). More precisely, there exists a neighborhood I_0 of zero in R and continuous mappings $\mu_\pm, \rho_\pm : I_0 \to R$, $y^\pm : I_0 \to R^4$, $y^\pm = (y_1^\pm, ..., y_4^\pm)$ such that $(x_\pm(\alpha), \mu_\pm(\alpha), \rho_\pm(\alpha))$, $\alpha \in I_0$, with*

$$x_\pm(\alpha) = \sum_{j=1}^{4} \alpha y_j^\pm(\alpha)\phi^j + o(\alpha) \text{ as } \alpha \to 0$$

and

$$\mu_\pm(\alpha) = \mu_0 \pm \alpha^3, \ \rho_\pm(\alpha) = \pm\alpha^3$$

satisfies Eq. (3), $x_\pm(\alpha) \to 0$, $\mu_\pm(\alpha) \to \mu_0$, $\rho_\pm(\alpha) \to 0$ as $\alpha \to 0$, $x_\pm(\alpha) \neq 0$ for $\alpha \neq 0$ and $\tilde{x}(\alpha) = x(\alpha)(\frac{t}{1+\rho(\alpha)})$ is $(1 + \rho_\pm(\alpha))2\pi$-periodic function.

PROOF. Let I_1, U_1, z be as in Lemma 1 and Ω a neighborhood of the point $\bar{y}$ in R^4 such that $0 \notin \overline{\Omega}$ and $\alpha\Omega \subset U_1$ for all $\alpha \in I_1$. We define the mapping $E^\pm : I_1 \times \Omega \to R^4$, $E^\pm = (E_1^\pm, ..., E_4^\pm)$, by

$$E_k^\pm(\alpha, y) = \begin{cases} < \pm A(\sum_{j=1}^4 \alpha y_j \phi^j + z(\alpha)) + (1 \pm \alpha^3)\{\pm L(\sum_{j=1}^4 \alpha y_j \phi^j + z(\alpha)) \\ + \frac{1}{\alpha}\frac{1}{2} f_{xx}(0, \mu_0)(\sum_{j=1}^4 y_j \phi^j + \frac{z(\alpha)}{\alpha})^2 \\ + \frac{1}{6} f_{xxx}(0, \mu_0)(\sum_{j=1}^4 y_j \phi^j + \frac{z(\alpha)}{\alpha})^3 + HoT, \psi^k >, \text{ for } \alpha \neq 0, \\ \mathcal{A}_k(y), \text{ for } \alpha = 0, \ y = (y_1, ..., y_4) \in R^4. \end{cases}$$

where $z(\alpha) = z(\pm\alpha^3, \pm\alpha^3, \alpha y)$.

We have $E^{\pm}(0,\bar{y}) = 0$. Observing that.

$$\frac{1}{\alpha} < \frac{1}{2} f_{xx}(0,\mu_0)(\sum_{j=1}^{4} y_j\phi^j + \frac{z(\alpha)}{\alpha})^2, \psi^k > =$$

$$\frac{1}{\alpha}\{< \frac{1}{2} f_{xx}(0,\mu_0)(\sum_{j=1}^{4} y_j\phi^j)^2, \psi^k > \; + < f_{xx}(0,\mu_0)(\sum_{j=1}^{4} y_j\phi^j)(\frac{z(\alpha)}{\alpha}), \psi^k > +$$

$$< \frac{1}{2} f_{xx}(0,\mu_0)(\frac{z(\alpha)}{\alpha})^2, \psi^k >\}.$$

A simple calculation shows that $< f_{xx}(0,\mu_0)(\sum_{j=1}^{4} y_j\phi^j)^2, \psi^k > = 0$ for all $k = 1,...,4$ and $|\frac{z(\alpha)}{\alpha}| \to 0$. Therefore

$$\frac{1}{\alpha} < \frac{1}{2} f_{xx}(0,\mu_0)(\sum_{j=1}^{4} y_j\phi^j + \frac{z(\alpha)}{\alpha}), \psi^k > \to 0 \text{ as } \alpha \to 0 \text{ for all } k = 1,...,4.$$

One can easily show that $E^{\pm}$ is a continuous mapping and $E^{\pm}_y$ exists. Moreover, it is continuous and

$$(\frac{\partial E^{\pm}_k(0,\bar{y})}{\partial x_j}) = (\frac{\partial A_k}{\partial x_j}(\bar{y}))_{k,j=1,...,4}$$

is a nonsingular mapping.

Applying the implicit function theorem we conclude that there exist neighborhoods I_0 of zero in R, Ω_0 of $\bar{y}$ in $R^4, I_0 \subset I_1, \Omega_0 \subset \Omega$ and a differentiable mapping $y^{\pm} : I_0 \to \Omega_0, y^{\pm}(0) = \bar{y}$, such that $E^{\pm}(\alpha, y^{\pm}(\alpha)) = 0$ for all $\alpha \in I_0$. It then follows that

$$\begin{aligned} &< \pm\alpha^3 A(\sum_{j=1}^{4} \alpha y_j(\alpha)\phi^j + z(\alpha)) + (1 \pm \alpha^3)\{\pm\alpha^3 L(\sum_{j=1}^{4} \alpha y_j^{\pm}\phi^j + z(\alpha)) \\ &+ \frac{1}{2} f_{xx}(0,\mu_0)(\sum_{j=1}^{4} \alpha y_j\phi^j + z(\alpha))^2 \\ &+ \frac{1}{6} f_{xxx}(0,\mu_0)(\sum_{j=1}^{4} \alpha y_j^{\pm}(\alpha)\phi^j + z(\alpha))^3 + HoT\}, \psi^k >= 0 \end{aligned} \tag{10}$$

for all $\alpha \in I_0$. Setting $x^{\pm}(\alpha) = \sum_{j=1}^{4} \alpha y_j^{\pm}(\alpha)\phi^j + z(\alpha)$ and noticing that $< \frac{d(x^{\pm}(\alpha))}{d\tau} - A(x^{\pm}(\alpha)), \psi^k > = 0$ for all $k = 1,...,4$, we deduce from (10) the equation

$$< \frac{d(x^{\pm}(\alpha))}{d\tau} - (1 \pm \alpha^3)\{A(x^{\pm}(\alpha)) \pm \alpha^3 L(x^{\pm}(\alpha) + \frac{1}{2} f_{xx}(0, \mu_0)(x^{\pm}(\alpha))^2 + \frac{1}{6} f_{xxx}(0, \mu_0)(x^{\pm}(\alpha))^3 + HoT\}, \psi^k > = 0, \ k = 1, ..., 4. \tag{11}$$

On the other hand, since $\pm\alpha^3, \pm\alpha^3, \alpha y_j^{\pm}(\alpha)) \in I_1 \times I_1 \times U_1$, it follows from (8) that

$$\dot{z}(\alpha) - A(z(\alpha)) - Q(\pm\alpha^3 A(x^{\pm}(\alpha)) + (1 \pm \alpha^3)\{\pm L(x^{\pm}(\alpha) + \frac{1}{2} f_{xx}(0, \mu_0)(x^{\pm}(\alpha))^2 + \frac{1}{6} f_{xxx}(0, \mu_0)(x^{\pm}(\alpha))^3 + HoT\}) = 0$$

and hence

$$Q(\frac{d(x^{\pm}(\alpha))}{d\tau} - (1 \pm \alpha^3)\{A(x^{\pm}(\alpha)) \pm \alpha^3 L(x^{\pm}(\alpha) + \frac{1}{2} f_{xx}(0, \mu_0)(x^{\pm}(\alpha))^2 + \frac{1}{6} f_{xxx}(0, \mu_0)(x^{\pm}(\alpha))^3 + HoT\}) = 0 \tag{12}$$

A combination of (11) and (12) give

$$\frac{d(v^{\pm}(\alpha))}{d\tau} = (1 \pm \alpha^3) f(v^{\pm}(\alpha), \mu(\alpha))$$

with $\mu_{\pm}(\alpha) = \mu_0 \pm \alpha^3$. Thus $\pm\alpha^3, \mu_{\pm}(\alpha), x^{\pm}(\alpha)$ satisfies Eq. (2) for any $\alpha \in I$. Further, it is clear that $\mu_{\pm}(\alpha) \to \mu_0$ and $x^{\pm}(\alpha) \to 0$ as $\alpha \to 0$.

By choosing $I_0' \subset I_0$ if necessary we can easily see that $x^{\pm}(\alpha) \neq 0$ for all $\alpha \neq 0$. Indeed, assume that for any neighborhood $I_n \subset I_0$, $I_{n+1} \subset I_n, \bigcap I_n = \{0\}$ there exists $\alpha_n \neq 0$ such that $x^{\pm}(\alpha_n) = 0$. Then

$$\sum_{j=1}^{4} y_j^{\pm}(\alpha)\phi^j = \frac{z(\alpha_n)}{\alpha_n} \in X_0 \bigcap X_1 = \{0\}.$$

The linear independence of $\{\phi^1, ..., \phi^4\}$ gives $y^{\pm}(\alpha_n) = (y_1^{\pm}(\alpha_n), ..., y_4^{\pm}(\alpha_n) = 0 \in \Omega_0$. This contradicts to $0 \notin \overline{\Omega}$. The proof of the theorem is now complete.

THEOREM 2. *Under Hypothesis 2, $(0,\mu_0)$ is a bifurcation point of periodic solutions of Eq. (1). More precisely, the same conclusions of Theorem 1 continue to hold with $\mu_\pm, \rho_\pm, y, x$ replaced by $\mu^\sigma_\pm, \rho^\sigma_\pm, y^{\sigma\pm}, x^{\sigma\pm}$, where $y^\sigma(0) = \bar{y}^\sigma$,*

$$\mu^\sigma_\pm(\alpha) = \mu_0 \pm \alpha^3,$$
$$\rho^\sigma_\pm(\alpha) = \sigma\alpha^2,$$
$$x^{\sigma\pm}(\alpha) = \sum_{j=1}^{4} \alpha y_j^{\sigma\pm}(\alpha)\phi^j + o(\alpha) \text{ as } \alpha \to 0.$$

THEOREM 3. *Under Hypothesis 3, $(0,\mu_0)$ is a bifurcation point of periodic solutions of Eq. (1). More precisely, the same conclusion of Theorem 1 continue to hold with $\mu_\pm, \rho_\pm, y^\pm, x^\pm$ replaced by $\mu^\sigma_\pm, \rho^\sigma_\pm, y^{\sigma\pm}, x^{\sigma\pm}$ respectively, where $y^{\sigma\pm}(0) = \bar{y}^\sigma$,*

$$\mu^\sigma_\pm(\alpha) = \mu_0 + \sigma\alpha^2,$$
$$\rho^\sigma_\pm(\alpha) = \pm\alpha^3,$$
$$x^\sigma_\pm(\alpha) = \sum_{j=1}^{4} \alpha y_j^{\sigma\pm}(\alpha)\phi^j + o(\alpha) \text{ as } \alpha \to 0.$$

THEOREM 4. *Under Hypothesis 4, $(0,\mu_0)$ is a bifurcation point of periodic solutions of Eq. (1). More precisely, the same conclusions of Theorem 1 continue to hold with $\mu_\pm, \rho_\pm, y^\pm, x^\pm$ replaced by $\mu^\sigma_\pm, \rho^\sigma_\pm, y^{\sigma\pm}, x^{\sigma\pm}$, where $y^{\sigma\pm}(0) = \bar{y}^\sigma$,*

$$\mu^\sigma_\pm(\alpha) = \mu_0 + \sigma\alpha^2,$$
$$\rho^\sigma_\pm(\alpha) = \pm\alpha^2,$$
$$x^\sigma_\pm(\alpha) = \sum_{j=1}^{4} \alpha y_j^{\sigma\pm}(\alpha)\phi^j + o(\alpha) \text{ as } \alpha \to 0.$$

The proofs of Theorems 2-4 proceed exactly as the one of Theorem 1 by replacing the mapping $E^\pm$ by $F^{\sigma\pm}, G^{\sigma\pm}, J^{\sigma\pm} = (J_1^{\sigma\pm}, \ldots, J_4^{\sigma\pm})$ respectively, where

$$F_k^{\sigma\pm}(\alpha,y)=\begin{cases} <\sigma A(\sum_{j=1}^{4}y_j\phi^j+\frac{z(\alpha)}{\alpha})+(1\pm\sigma\alpha^2)\{\pm L(\sum_{j=1}^{4}\alpha y_j\phi^j+z(\alpha)) \\ +\frac{1}{2\alpha}f_{xx}(0,\mu_0)(\sum_{j=1}^{4}y_j\phi^j+\frac{z(\alpha)}{\alpha}))^2 \\ +\frac{1}{6}f_{xxx}(0,\mu_0)(\sum_{j=1}^{4}y_j\phi^j+\frac{z(\alpha)}{\alpha}))^3+HoT\},\psi^k, \ for\ \alpha\neq 0, \\ \mathcal{B}_k(y) \quad for \quad \alpha=0, \end{cases}$$

$$G_k^{\sigma\pm}(\alpha,y)=\begin{cases} <\mp A(\sum_{j=1}^{4}\alpha y_j\phi^j+z(\alpha)+(1\pm\alpha^2)\{\sigma L(\sum_{j=1}^{4}y_j\phi^j+\frac{z(\alpha)}{\alpha}) \\ +\frac{1}{2\alpha}f_{xx}(0,\mu_0)(\sum_{j=1}^{4}y_j\phi^j+\frac{z(\alpha)}{\alpha}))^2 \\ +\frac{1}{6}f_{xxx}(0,\mu_0)(\sum_{j=1}^{4}y_j\phi^j+\frac{z(\alpha)}{\alpha}))^3+HoT\},\psi^k>, \ for\ \alpha\neq 0, \\ \mathcal{C}_k(y) \quad for \quad \alpha=0, \end{cases}$$

$$J_k^{\sigma\pm}(\alpha,y)=\begin{cases} <\mp A(\sum_{j=1}^{4}y_j\phi^j+\frac{z(\alpha)}{\alpha})+(1\pm\alpha^2)\{\sigma L(\sum_{j=1}^{4}y_j\phi^j+\frac{z(\alpha)}{\alpha}) \\ +\frac{1}{2\alpha}f_{xx}(0,\mu_0)(\sum_{j=1}^{4}y_j\phi^j+\frac{z(\alpha)}{\alpha}))^2 \\ +\frac{1}{6}f_{xxx}(0,\mu_0)(\sum_{j=1}^{4}y_j\phi^j+\frac{z(\alpha)}{\alpha}))^3+HoT\},\psi^k>, \ for\ \alpha\neq 0, \\ \mathcal{D}_k(y) \text{ for } \alpha=0, \end{cases}$$

REMARK 1. The above results show that there always exist at least two different parameter families of nontrivial periodic solutions in a neighborhood of $(0,\mu_0)$.

REMARK 2. If there are two different points $\bar{y}^{1\sigma},\bar{y}^{2\sigma}$ satisfying the above hypotheses, then we can prove that there also exist at least four different parameter families $(\lambda_{\pm}^{1\sigma}(\alpha),x_{\pm}^{1\sigma}(\alpha))\neq(\lambda_{\pm}^{2\sigma}(\alpha),x_{\pm}^{2\sigma}(\alpha))$ of nontrivial periodic solutions in a neighborhood of $(0,\mu_0)$.

REMARK 3. In the case when the mapping f is c^k-mapping, we conclude that $x^{\sigma}_{\pm}$ are also c^k-mappings.

To illustrate the above results, we give the following example:

EXAMPLE 1. We consider the equation in R^4

$$\frac{du}{dt} = f(\lambda, u), \qquad (\lambda, u) \in R \times C_{2\pi}(R, R^4), \tag{13}$$

where $f(\lambda, u) = T(u) + \lambda L(u) + H(\lambda, u) + K(\lambda, u)$, with

$$T = \begin{pmatrix} -1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$H(\lambda, u) = \lambda(\sum_{j=1}^{4} u_j)^3 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \qquad u = (u_1, u_2, u_3, u_4),$$

and K is a differentiable mapping, $||K(\lambda, u)|| = o(||u||^3)$ as $||u|| \to 0$ uniformly to λ from any bounded subset of R. A simple calculation shows that i is an eigenvalue of the mapping $f_u(1, 0) = T + L$ with multiplicity 2. We have

$$v^1 = \gamma^1 = \frac{1}{2\sqrt{2\pi}} \begin{pmatrix} 1 \\ -i \\ 1 \\ i \end{pmatrix} = \frac{1}{2\sqrt{2\pi}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \frac{i}{2\sqrt{2\pi}} \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix},$$

$$v^2 = \gamma^2 = \frac{1}{2\sqrt{2\pi}} \begin{pmatrix} i \\ 1 \\ -i \\ 1 \end{pmatrix} = \frac{1}{2\sqrt{2\pi}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \frac{i}{2\sqrt{2\pi}} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

and hence

$$\phi^1 = \psi^1 = \frac{1}{2\sqrt{2\pi}} \begin{pmatrix} \cos t \\ -\sin t \\ \cos t \\ \sin t \end{pmatrix}; \phi^2 = \psi^2 = \frac{1}{2\sqrt{2\pi}} \begin{pmatrix} -\sin t \\ -\cos t \\ -\sin t \\ \cos t \end{pmatrix},$$

$$\phi^3 = \psi^3 = \frac{1}{2\sqrt{2\pi}} \begin{pmatrix} \sin t \\ \cos t \\ -\sin t \\ \cos t \end{pmatrix}; \phi^4 = \psi^4 = \frac{1}{2\sqrt{2\pi}} \begin{pmatrix} \cos t \\ -\sin t \\ -\cos t \\ -\sin t \end{pmatrix}.$$

One can easily see that

$$\begin{aligned}
\mathcal{D}_1^-(x) &= -x_1 + \frac{3}{8\pi}(\frac{1}{2}(x_1 + x_3)^3 + (x_1 + x_3)(x_2 + x_4)^2), \\
\mathcal{D}_2^-(x) &= -x_2 + \frac{3}{8\pi}(\frac{1}{2}(x_2 + x_4)^3 + (x_2 + x_4)(x_1 + x_3)^2), \\
\mathcal{D}_3^-(x) &= -x_3 + \frac{3}{8\pi}(\frac{1}{2}(x_1 + x_3)^3 + (x_1 + x_3)(x_2 + x_4)^2), \\
\mathcal{D}_4^-(x) &= -x_4 + \frac{3}{8\pi}(\frac{1}{2}(x_2 + x_4)^3 + (x_2 + x_4)(x_1 + x_3)^2).
\end{aligned}$$

Taking $\bar{x}_2^{\pm} = \bar{x}_4^{\pm} = 0,\ \bar{x}_1^{\pm} = \bar{x}_3^{\pm} = \pm(\frac{2\pi}{3})^{1/2}$, we can see that $\mathcal{D}^-(\bar{x}^{\pm}) = 0$ with $\bar{x}^{\pm} = (\bar{x}_1^{\pm}, \bar{x}_2^{\pm}, \bar{x}_3^{\pm}, \bar{x}_4^{\pm})$ and

$$\frac{\partial \mathcal{D}_k^-}{\partial x_j}(\bar{x}^{\pm}) = \begin{pmatrix} 2 & 0 & 3 & 0 \\ 0 & 2 & 0 & 3 \\ 3 & 0 & 2 & 0 \\ 0 & 3 & 0 & 2 \end{pmatrix}$$

is a nonsingular matrix. Therefore, applying Theorem 4, we conclude that (1,0) is a bifurcation point of periodic solutions of Eq. (13). Analogously, if we choose $\bar{x}_2^{\pm} = \bar{x}_4^{\pm} = \pm(\frac{2\pi}{3})^2$ and $\bar{x}_1^{\pm} = \bar{x}_3^{\pm} = 0$, we can prove that the assumptions of this theorem are satisfied. Using Remark 2 we deduce that there exist at least 8 distinct parameter families of nontrivial periodic solutions of Eq. (13).

4. The degenerate cases

In this section we consider the cases when there exist points $\bar{y}, \bar{y}^{\sigma} \neq 0$ satisfying one of the following equations

$$\mathcal{A}(y) = 0,$$
$$\mathcal{B}^\sigma(y) = 0,$$
$$\mathcal{C}^\sigma(y) = 0,$$
$$\mathcal{D}^\sigma(y) = 0,$$

which correspond to the singular matrices

$$\frac{\partial \mathcal{A}_k}{\partial y_j}(\bar{y}),\ \frac{\partial \mathcal{B}_k^\sigma}{\partial y_j}(\bar{y}^\sigma),\ \frac{\partial \mathcal{C}_k^\sigma}{\partial y_j}(\bar{y}^\sigma),\ \frac{\partial \mathcal{D}_k^\sigma}{\partial y_j}(\bar{y}^\sigma),$$

respectively. This means that none of Hypotheses 1-4 is satisfied. These cases are called degenerate cases.

Now, let $\bar{y} \in R^4$ be such that $\mathcal{A}(\bar{y}) = 0$ and the matrix $\overline{\mathcal{A}} = (\frac{\partial \mathcal{A}_k}{\partial x_j})_{i,j=1,\dots,4}$ is a singular 4×4-matrix. We put

$$R_0 = \{x \in R^4 \;/\; \overline{\mathcal{A}}x = 0\},$$
$$R_0^* = \{x \in R^4 \;/\; \overline{\mathcal{A}}^* x = 0\},$$

and assume that $R_0 = [\xi^1, \dots, \xi^r]$ $R_0^* = [\xi^{*1}, \dots, \xi^{*r}]$, $1 \le r \le 4$, and R_1, R_1^* are such that $R_0 \oplus R_1 = R_0^* \oplus R_1^* = R^4$. By P, Q we denote the projectors of R^4 into R_0^*, R_1^*, respectively. Further, by $D^j\mathcal{A}(\bar{y}) = \mathcal{A}_{x\dots x}(\bar{y})$ (j-times) we denote the j-th derivative of the mapping $\mathcal{A}$ at the point $\bar{y}$, $j = 1, 2, \dots$. Let c be the smallest number such that $D^c\mathcal{A}(\bar{y}) \not\equiv 0$ on R. We define the mapping $f : R^r \to R^r$, $f = (f_1, \dots, f_r)$, by

$$f_k(t) = \frac{1}{c} < D^c\mathcal{A}(\bar{y})(\sum_{j=1}^{r} t_j \xi^j)\xi^{*k} >, k = 1, \dots, r,\ t = (t_1, \dots, t_r) \in R^n. \quad (15)$$

Analogously we define the mappings $g^\sigma, h^\sigma, q^\sigma : R^r \to R^n$ with $\mathcal{A}$ replaced by $\mathcal{B}^\sigma, \mathcal{C}^\sigma, \mathcal{D}^\sigma$, respectively (of course, the spaces R_0, R_1, R_0^*, R_1^* are also defined by replacing $\overline{\mathcal{A}}$ by $\overline{\mathcal{B}}^\sigma = (\frac{\partial \mathcal{B}_k^\sigma}{\partial x_j}(\bar{y}^\sigma))$, $\overline{\mathcal{C}}^\sigma = (\frac{\partial \mathcal{C}_k^\sigma}{\partial x_j}(\bar{y}^\sigma))$, $\overline{\mathcal{D}}^\sigma = (\frac{\partial \mathcal{D}_k^\sigma}{\partial x_j}(\bar{y}^\sigma))$, respectively).

We make the following hypotheses on these mappings:

HYPOTHESIS 5. There exists a point $t^* \in R^r$, $t^* = (t_1^*, ..., t_r^*)$, and a neighborhood U^* of t^* in R^r such that the topological degree, deg $(f, U^*, 0)$ of f with respect to U^* and the origin in R^r is defined and different from zero.

HYPOTHESIS 6. Hypothesis 5 with t^*, U^* and f replaced by $t^{*\sigma}, V^{*\sigma}$ and g^σ, respectively.

HYPOTHESIS 7. Hypothesis 5 with t^*, U^* and f replaced by $t^{*\sigma}, V^{*\sigma}$ and h^σ, respectively.

HYPOTHESIS 8. Hypothesis 5 with t^*, U^* and f replaced by $t^{*\sigma}, V^{*\sigma}$ and q^σ, respectively.

THEOREM 5. *Let $\mathcal{A}, \bar{y}, f$ ect. be as above. Under Hypothesis 5, $(0, \mu_0)$ is a bifurcation point of periodic solutions of Eq. (1). More precisely, to given d with $dc < 2$, there exists a neighborhood I_0 of zero in R such that for each $\alpha \in I_0$, $\alpha \neq 0$, one can find $t^\pm(\alpha) = (t_1^\pm(\alpha), ..., t_r^\pm(\alpha)) \in U^*$ for which $x_\pm^\sigma(\alpha), \mu_\pm(\alpha), \rho_\pm(\alpha))$ with*

$$x_\pm^\sigma(\alpha) = \sum_{k=1}^{4} \alpha(\bar{y}_k + \sum_{j=1}^{r} |\alpha|^d t_j^\pm(\alpha) \xi_k^j) \phi^k + o(|\alpha|) \text{ as } |\alpha| \to 0$$

and

$$\mu_\pm(\alpha) = \mu_0 \pm \alpha^3; \quad \rho(\alpha) = \pm\alpha^3$$

satisfies Eq. (3), $x_\pm^\sigma(\alpha) \to 0$, $\mu_\pm(\alpha) \to 0$, $\rho_\pm(\alpha) \to 0$ as $\alpha \to 0$, $x_\pm^\sigma(\alpha) \neq 0$ for $\alpha \neq 0$, $\tilde{x}(\alpha)(t) = x(\alpha)(\frac{t}{1+\rho_\pm(\alpha)})$ is $(1 + \rho_\pm(\alpha))2\pi$ -periodic function.

PROOF. Let I_2, U_2 and z be as in Lemma 1. Without loss of generality we may assume that $\bar{y} \notin U_2$. The proof of this theorem proceeds exactly as the one of Theorem 1 in [9] with A replaced by $\mathcal{A}$ and C replaced by $\mathcal{N}^\pm = (\mathcal{N}_1^\pm, ..., \mathcal{N}_4^\pm)$, where

$$\mathcal{N}_k^\pm(y, \alpha) = \begin{cases} < \pm A(\sum_{j=1}^4 \alpha y_j \phi^j + z(\alpha)) + (1 \pm \sigma\alpha^3)\{\pm L(\sum_{j=1}^4 \alpha y_j \phi^j \\ +z(\alpha)) + \frac{1}{2\alpha} f_{xx}(0, \mu_0)(\sum_{j=1}^4 y_j \phi^j + \frac{z(\alpha)}{\alpha}))^2 \\ +\frac{1}{6} f_{xxx}(0, \mu_0)(\sum_{j=1}^4 y_j \phi^j + \frac{z(\alpha)}{\alpha}))^3 \\ +HoT\}, \psi^k > -\mathcal{A}_k^\sigma(y), \text{ for } \alpha \neq 0, \\ 0 \qquad \text{for} \quad \alpha = 0. \end{cases}$$

COROLLARY 6. *Let $\mathcal{A}, \bar{y}, f$ be as above. In addition, assume that c is an odd number and $f(t) \neq 0$ for all $t \in R^r$, $|t| = 1$. Then the conclusions of Theorem 5 continue to hold for $Y^* = \{\xi \in R^r \ / \ |\xi| < 1\}$.*

PROOF. Since c is an odd number, it follows that the mapping f is an odd mapping. The condition of the corollary implies $f(t) \neq 0$ for all $t \in \partial U^*$. So, by the Borsuk Theorem (see, for example, [2, Theorem 4.1]), the topological degree deg $(f, U^*, 0)$ of f with respect to U^* and the origin in R^r is defined and different from zero. Consequently, Hypothesis 5 is satisfied. Therefore, the conclusion follows from Theorem 5.

COROLLARY 7. *Let $\mathcal{A}, \bar{y}, f$ be as above. In addition, assume that $r = 1$ and c is an odd number. Then the conclusions of Theorem 5 continue to hold for $U^* = (-1, 1)$.*

PROOF. Since $r = 1$, we conclude that $f(t) = < D^c\mathcal{A}(\bar{y})(t\xi^1), \xi^{*1} >= t^c < D^c\mathcal{A}(\bar{y})(\xi^1), \xi^{*1} > \neq 0$ for all $t \in R$, $|t| = 1$. Therefore, the corollary immediately follows from Corollary 6.

THEOREM 8. *Let $\mathcal{B}^\sigma, \bar{y}^\sigma, g^\sigma$ be as above. Under Hypothesis 6, $(0, \mu_0)$ is a bifurcation point of periodic solutions of Eq. (1). More precisely, the same conclusions of Theorem 5 continue to hold with $t^\pm$ replaced by $t^{\sigma\pm}$ and $(1 \pm \alpha^3)2\pi$ replaced by $(1 + \sigma\alpha^2)2\pi$.*

PROOF. The proof of this theorem proceeds exactly as the one of Theorem 1 in [9] with A replaced by $\mathcal{B}$ and C replaced by $\mathcal{M}^{\sigma\pm} = (\mathcal{M}_1^{\sigma\pm}, \ldots, \mathcal{M}_4^{\sigma\pm})$, where

$$\mathcal{M}_k^{\pm}(y, \alpha) = \begin{cases} < \sigma A(\sum_{j=1}^{4} y_j\phi^j + \frac{z(\alpha)}{\alpha}) + (1 \pm \sigma\alpha^2)\{\pm L(\sum_{j=1}^{4} \alpha y_j\phi^j + z(\alpha))^2 \\ + \frac{1}{2\alpha} f_{xx}(0, \mu_0)(\sum_{j=1}^{4} y_j\phi^j + \frac{z(\alpha)}{\alpha})) \\ + \frac{1}{6} f_{xxx}(0, \mu_0)(\sum_{j=1}^{4} y_j\phi^j + \frac{z(\alpha)}{\alpha}))^3 \\ + HoT\}, \psi^k > \; -\mathcal{B}_k^\sigma(y), \text{ for } \alpha \neq 0, \\ 0 \quad \text{for} \quad \alpha = 0. \end{cases}$$

The proofs of the following corollaries proceed exactly as the ones of Corollaries 6,7.

COROLLARY 9. *Let $\mathcal{B}^\sigma, \bar{y}^\sigma, g^\sigma$ be as above. In addition, assume that c is an odd number and $g^\sigma(t) \neq 0$ for all $t \in R^r$, $|t| = 1$. Then the conclusions of Theorem 8 continue to hold for $U^{*\sigma} = \{\xi \in R^r \ / \ |\xi| < 1\}$.*

COROLLARY 10. *Let $\mathcal{B}^\sigma, \bar{y}^\sigma, g^\sigma$ be as above. In addition, assume that $r = 1$ and c is an odd number. Then the conclusions of Theorem 8 continue to hold for $U = (-1, 1)$.*

THEOREM 11. *Let $\mathcal{C}^\sigma, \bar{y}^\sigma, h^\sigma$ be as above. Under Hypothesis 7, $(0, \mu_0)$ is a bifurcation point of periodic solutions of Eq. (1). More precisely, the same conclusions of Theorem 5 continue to hold with $t^\pm$ replaced by $t^{\sigma\pm}, \mu_\pm, (1 \pm \alpha^3)2\pi$ replaced by $\mu_\pm^\sigma, (1 \pm \alpha^2)2\pi$, where*

$$\mu_\pm^\sigma(\alpha) = \mu_0 + \sigma\alpha^2.$$

PROOF. The proof of this theorem proceeds exactly as the one of Theorem 1 in [9] with A replaced by $\mathcal{C}$ and C replaced by $\mathcal{R}^{\sigma\pm} = (\mathcal{R}_1^{\sigma\pm}, ..., \mathcal{R}_4^{\sigma\pm})$, where

$$\mathcal{R}_k^\pm(y, \alpha) = \begin{cases} < \pm A(\sum_{j=1}^4 \alpha y_j \phi^j + z(\alpha)) + (1 \pm \alpha^2)\{\sigma L(\sum_{j=1}^4 y_j \phi^j + \frac{z(\alpha)}{\alpha}) \\ + \frac{1}{2\alpha} f_{xx}(0, \mu_0)(\sum_{j=1}^4 y_j \phi^j + \frac{z(\alpha)}{\alpha}))^2 \\ + \frac{1}{6} f_{xxx}(0, \mu_0)(\sum_{j=1}^4 y_j \phi^j + \frac{z(\alpha)}{\alpha}))^3 \\ + HoT\}, \psi^k > -\mathcal{C}_k^\sigma(y), \text{ for } \alpha \neq 0; \\ 0 \quad \text{for} \quad \alpha = 0. \end{cases}$$

The proofs of the following corollaries proceed exactly as the ones of Corollaries 6,7, respectively.

COROLLARY 12. *Let $\mathcal{C}^\sigma, \bar{y}^\sigma, h^\sigma$ be as above. In addition, assume that c is an odd number and $h^\sigma(t) \neq 0$ for all $t \in R^r$, $|t| = 1$. Then the conclusions of Theorem 11 continue to hold for $U^{*\sigma} = \{\xi \in R^r \ / \ |\xi| < 1\}$.*

COROLLARY 13. *Let $\mathcal{C}^\sigma, \bar{y}^\sigma, h^\sigma$ be as above. In addition, assume that $r = 1$ and c is an odd number. Then the conclusions of Theorem 11 continue to hold for $U^{*\sigma} = (-1,$*

THEOREM 14. *Let $\mathcal{D}^\sigma, \bar{y}^\sigma, h^\sigma$ be as above. Under Hypothesis 8, $(0, \mu_0)$ is a bifurcation point of periodic solutions of Eq. (1). More precisely, the same conclusions of Theorem 5 continue to hold with $t^\pm$ replaced by $t^{\sigma\pm}$ and $\mu_\pm, (1 \pm \alpha^3)2\pi$ replaced by $\mu^\sigma_\pm, (1 \pm \alpha^2)2\pi$, where*

$$\mu^\sigma_\pm(\alpha) = \mu_0 + \sigma\alpha^2.$$

PROOF. The proof of this theorem proceeds exactly as the one of Theorem 1 in [9] with A replaced by $\mathcal{D}$ and C replaced by $\mathcal{P}^{\sigma\pm} = (\mathcal{P}^{\sigma\pm}_1, ..., \mathcal{P}^{\sigma\pm}_4)$, where

$$\mathcal{P}^\pm_k(y, \alpha) = \begin{cases} < \pm A(\sum_{j=1}^4 \alpha y_j \phi^j + z(\alpha)) + (1 \pm \alpha^2)\{\sigma L(\sum_{j=1}^4 \alpha y_j \phi^j + z(\alpha)) \\ +\frac{1}{2\alpha} f_{xx}(0, \mu_0)(\sum_{j=1}^4 y_j \phi^j + \frac{z(\alpha)}{\alpha}))^2 \\ +\frac{1}{6} f_{xxx}(0, \mu_0)(\sum_{j=1}^4 y_j \phi^j + \frac{z(\alpha)}{\alpha}))^3 \\ +HoT\}, \psi^k > \ -\mathcal{D}^\sigma_k(y), \text{ for } \alpha \neq 0, \\ 0 \quad \text{for} \quad \alpha = 0. \end{cases}$$

The proofs of the following corollaries proceed exactly as the ones of Corollaries 6,7, respectively.

COROLLARY 15. *Let $\mathcal{D}^\sigma, \bar{y}^\sigma, q^\sigma$ be as above. In addition, assume that c is an odd number and $q^\sigma(t) \neq 0$ for all $t \in R^r$, $|t| = 1$. Then the conclusions of Theorem 14 continue to hold.*

COROLLARY 16. *Let $\mathcal{D}^\sigma, \bar{y}^\sigma, q^\sigma$ be as above. In addition, assume that $r = 1$ and c is an odd number. Then the conclusions of Theorem 14 continue to hold.*

To conclude the paper we give the following example to illustrate the results concerning the degenerate cases.

EXAMPLE 2. Let T, L, K, $\phi^j = \psi^j$, $j = 1, ..., 4$, $\bar{\lambda} = 1$ be the same as in Example 1 and $H(\lambda, u) = \lambda(u, u)$. We have

$$\begin{aligned} \mathcal{D}^\sigma_k(x) &= -\sigma < L(\bar{\lambda}, \sum_{j=1}^4 x_j \phi^j) + H(\bar{\lambda}, \sum_{j=1}^4 x_j \phi^j), \psi^k > \\ &= -\sigma x_k + \frac{6}{64}(x_1^2 + x_2^2 + x_3^2 + x_4^2) x_k \end{aligned}$$

Choosing $\sigma = 1$, $\bar{x}^{\pm} = (\bar{x}_1^{\pm}, ..., \bar{x}_4^{\pm})$ with $\bar{x}_1^{\pm} = \pm\frac{8}{\sqrt{6}}$, $\bar{x}_2^{\pm} = \bar{x}_3^{\pm} = \bar{x}_4^{\pm} = 0$, we can see that

$$D = (\frac{\partial \mathcal{D}_k}{\partial x_j}(\bar{x}^{\pm})) = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and hence

$$R_0 = R_0^* = \{x \in R^4 \ / \ Dx = 0\} = [\xi^1, \xi^2, \xi^3]$$

with $\xi^1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $\xi^2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, $\xi^3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$. It follows that

$$q_k^1(t) = \frac{1}{6} < \mathcal{D}_{xxx}(\bar{x})(\sum_{j=1}^{3} t_j \xi^j), \xi^k > = \frac{1}{32} t_k (3t_k^2 + \sum_{j \neq k}^{3} t_j^2).$$

A simple calculation shows that $q^1(t) \neq 0$ for all $t \in R^3$, $|t| = 1$. Applying Corollary 15 we conclude that $(0, 1)$ is a bifurcation point of periodic solutions of the equation

$$\frac{du}{dt} = f(\lambda, u), \quad (\lambda, u) \in R \times C_{2\pi}(R, R^4)$$

with

$$f(\lambda, u) = -(Tu + \lambda Lu + \lambda(u, u)u + K(\lambda, u)).$$

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