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SOME NOTES ON SEMIDIFFERENTIABILITY AND GENERALIZED SUBDIFFERENTIALS

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Abstract. The notion of generalized semiderivative is studied. Some necessary or sufficient conditions for $\mathcal{G}$-semidifferentiability, for compactness of $\mathcal{C}$-subdifferentials and the relationship between these notions and the others are established.

1. Introduction

In recent years there has been an increasing interest in nondifferentiable optimization, both from the theory and pratical applications. Efforts to find adequate tools for the purposes of local approximation in differential calculus for nonsmooth problems have resulted in the use of methods and techniques borrowed from convex analysis in which linear functions have been replaced by sublinear functions and single gradients vectors have been subtituted by convex compact sets (subdifferentials)(see, e.g., [14, 10, 11, 8, 12, 13, 5]). This work is concerned with one of these concepts proposed by F. Giannessi [4].

In Section 2 the generalized derivatives are defined in an abstract way and some remarks are given to explain these notions. In Section 3 some neccesary and sufficient conditions for semidifferentiability and the relationships between it and some well-known notions are established.

The subdifferential corresponding to generalized derivatives is a closed, convex set but it might not be compact. This may restrict its applications. In Section 4 of this note we give some neccesary and sufficient conditions for compactness of the subdifferential. In particular, we show that the generalized

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subdifferential for a Lipschitz function of class $\mathcal{S}^-$ coincides with the subdifferential in the sense of Michel - Penot in the case of regularity.

2. Definitions and preliminaries

First of all, let us present a partial list of notations that we employ in this paper. The *inner product* on $\mathbb{R}^n$ is defined as the bi-linear form

$$\langle y, x\rangle := \sum_{i=1}^{n} y_i x_i.$$

We denote a *norm* on $\mathbb{R}^n$ by $\|.\|$ and the associated closed unit ball for this norm by B. The *indicator* and *support functions* for a subset A are given by

$$\phi(x|A) := \begin{cases} 0 & \text{if } x \in A, \\ +\infty & \text{otherwise,} \end{cases}$$

and

$$\psi^*(x|A) := \sup\{\langle x^* x\rangle : x^* \in A\},$$

respectively. Moreover, we write int A for the interior of A, cl A for the closure of A. The *relative interior* of A, denoted ri A, is the interior of A relative to the *affine hull* of A which is given by

$$\text{aff } A := \left\{ \sum_{k=1}^{s} \lambda_k x_k \,\middle|\, \begin{array}{c} s \in \{1, 2, \cdots\}, x_k \in A \text{ and } \lambda_k \in \mathbb{R} \\ \text{for } k = 1, 2, \cdots, s, \text{ with } \sum_{k=1}^{s} \lambda_k = 1 \end{array} \right\}.$$

We denote the Dini lower and upper directional derivatives by

$$f^-(\bar{x}, z) := \liminf_{t \downarrow 0} \frac{f(\bar{x} + tz) - f(\bar{x})}{t},$$

$$f^+(\bar{x}, z) := \limsup_{t \downarrow 0} \frac{f(\bar{x} + tz) - f(\bar{x})}{t},$$

and the Dini-Hadamard lower and upper directional derivatives by

$$f_T^-(\bar{x}, z) := \liminf_{u \to z, t \downarrow 0} \frac{f(\bar{x} + tu) - f(\bar{x})}{t},$$

$$f_T^+(\bar{x}, z) := \limsup_{u \to z, t \downarrow 0} \frac{f(\bar{x} + tu) - f(\bar{x})}{t}.$$

The *Dini subdifferential* is defined (see Ref.[10-11]) by

$$\partial^- f(\bar{x}) := \{u \in \mathbb{R}^n : \langle u, z\rangle \leq f^-(x,z), \quad \forall z \in \mathbb{R}^n\}.$$

The set A is said to be *convex* if the line segment connecting any two points in A is also contained in A. The *convex hull* of A, denoted by co A, is the smallest convex set which contains A, that is, co A is the intersection of all convex sets which contain A. Let X be a closed cone of $\mathbb{R}^n$ with apex at $\bar{x} \in X$ and let $Z := X - \bar{x}$. Let $\mathcal{G}$ denote the set of positively homogeneous of degree one functions $g(\bar{x}, .) : Z \to \mathbb{R} \cup \{+\infty\}$, namely $g \in \mathcal{G}$ if

$$g(\bar{x}, \alpha z) = \alpha g(\bar{x}, z), \forall \alpha \geq 0, \forall z \in Z. \tag{1}$$

The subset $\mathcal{C} \subset \mathcal{G}$ of all sublinear functions plays a special role in our development.

Let f be a real-valued function defined in a neighbourhood of $\bar{x}$ and G a subset of $\mathcal{G}$. For a given function $g \in G$ we consider the function

$$\epsilon_{f,g}(\bar{x}, z) := f(\bar{x} + z) - f(\bar{x}) - g(\bar{x}, z). \tag{2}$$

The sets

$$LGA_f(\bar{x}) := \{g \in G : \liminf_{z \to 0, z \in Z} \frac{\epsilon_{f,g}(\bar{x}, z)}{|z|} \geq 0\}, \tag{3}$$

$$UGA_f(\bar{x}) := \{g \in G : \limsup_{z \to 0, z \in Z} \frac{\epsilon_{f,g}(\bar{x}, z)}{|z|} \leq 0\} \tag{4}$$

are called the sets of lower and upper G-approximations of f at $\bar{x}$, respectively.

Now, let us recall briefly the definitions of semidifferentiabilities [4] and give some remarks on the class of semidifferentiable functions.

DEFINITION 1. The function f is said to be lower (resp. upper) G-semidifferentiable at $\bar{x}$ if the set of lower (resp. upper) G-approximations of f at $\bar{x}$ is nonempty:

$$LGA_f(\bar{x}) \neq \emptyset, \tag{5}$$

$$(\text{ resp.} UGA_f(\bar{x}) \neq \emptyset), \tag{6}$$

and $LGA_f(\bar{x})$ possesses a maximum element

$$\underline{D}_G f(\bar{x}, z) = \max_{g \in LGA_f(\bar{x})} g(\bar{x}, z) \quad \forall z \in Z, \tag{7}$$

(resp. $UGA_f(\bar{x})$ possesses a minimum element

$$\bar{D}_G f(\bar{x}, z) = \min_{g \in UGA_f(\bar{x})} g(\bar{x}, z) \quad \forall z \in Z). \tag{8}$$

$\underline{D}_G f(\bar{x}, .)$ and $\bar{D}_G f(\bar{x}, .)$ are called the lower and upper G-semiderivative of f at $\bar{x}$, respectively.

REMARK 1. Conditions (5) and (6) should be called **the existence conditions** for G-approximations, and (7) and (8) the **uniqueness conditions** (existence of the "best" approximations). When $G = \mathcal{C}$ condition (6) always holds and the set $U\mathcal{C}A_f(\bar{x})$ is a subset of the set of Pschenichnyi's upper sublinear approximations defined as the set of all sublinear functions which majorize the upper Dini derivative (see Ref.[11]). The following example, however, shows that these two notions do not always coincide.

EXAMPLE 1. Let $X = \mathbb{R}^2, G = \mathcal{G}$, and

$$f(x) := \begin{cases} 0 & \text{if } x \in l_2 := \{(0, \alpha), \alpha \geq 0\}, \\ \frac{|x|^2}{|\frac{x}{|x|} - (0,1)|} & \text{otherwise.} \end{cases}$$

It is easy to see that $g(\bar{x}, .) = 0$ is a Pshenichnyi's upper sublinear approximation of f at 0 and the point-wise maximum of the following set

$$U\mathcal{G}A_f(0) = \{g_\alpha(.) : \quad \alpha < 0\},$$

where

$$g_\alpha(z) := \begin{cases} 0 & \text{if } x \in l_2, \\ \frac{\alpha||x||}{||\frac{x}{||x||} - (0,1)||} & \text{otherwise.} \end{cases}$$

However, $g(\bar{x}, .)$ does not belong to this set. So the function f is not upper $\mathcal{G}-$semidifferentiable.

REMARK 2. The following relations are trivial (see e.g. [4])

$$\underline{D}_G f(\bar{x}, z) \leq f^-(\bar{x}, z) \quad \forall z \in Z,$$

$$\bar{D}_G f(\bar{x}, z) \geq f^+(\bar{x}, z) \quad \forall z \in Z.$$

Assume that $f(.)$ is a given lower (resp. upper) K-semidifferentiable function ($K \subset \mathcal{C}$) with its lower (resp. upper) K-semiderivative $g(\bar{x},.)$. The set

$$\partial_K f(\bar{x}) := \{\xi \in \mathrm{R}^n : g(\bar{x}, z) \geq \langle \xi, z\rangle \quad \forall z \in Z\} \tag{9}$$

is called the lower (resp. upper) generalized subdifferential for $f(.)$ at $\bar{x}$.

It is easy to see that if f is lower (res. upper) $G-$semidifferentiable with

$$\epsilon_f(\bar{x},.) = \epsilon_{f,\underline{D}_G f}(\bar{x},.) \qquad (\text{ res. }) = \epsilon_{f,\bar{D}_G f}(\bar{x},.))$$

satisfying

$$\lim_{z\to 0, z\in Z} \frac{\epsilon_f(\bar{x},z)}{|z|} = 0 \tag{10}$$

then f is upper (res. lower) $G-$semidifferentiable. So we consider the following notion.

DEFINITION 2. ([4]) In case (10) the function f is said to be G-differentiable and the lower (or upper) G-semiderivative is called G-derivative.

REMARK 3. From Remark 2 we see that if f is G-differentiable, then f is directional differentiable, (i.e. there exists the directional derivative

$$f'(\bar{x},z) := \lim_{t\downarrow 0} \frac{f(\bar{x}+tz) - f(\bar{x})}{t} \quad \forall z \in Z$$

and $f'(\bar{x},z)$ coincides with the G-derivative of f at $\bar{x}$. The converse assertion is not always true, even in case $G = \mathcal{G}$.

EXAMPLE 2. Let us consider the function f defined in Example 1. It is easy to see that its directional derivative is $f'(\bar{x},.) = 0$, which is neither lower nor upper $\mathcal{G}-$ semidifferentiable, as the uniqueness conditions (7) and (8) are not satisfied for $L\mathcal{G}A_f(0)$, $U\Gamma A_f(0)$.

In Theorem 2.1 of [13] there is an assertion that the continuity (resp. semicontinuity) of the considered function follows from its G-differentiability (resp. G-semidifferentiability). From Remark 3 it is easy to see that this assertion failed, because a function, which is directional differentiable, is not necessarily continuous (resp. semicontinuous)

EXAMPLE 3. Let $X = \mathbb{R}^2, G = \mathcal{G}$, and

$$f(x) := \begin{cases} 0, & \text{if } x \in l_1 := \{(\alpha, 0), \alpha \geq 0\}, \\ 0, & \text{if } x \in l_2 := \{(0, \alpha), \alpha \geq 0\}, \\ \left(\frac{|x|}{|\frac{x}{|x|} - (1,0)|} - \frac{|x|}{|\frac{x}{|x|} - (0,1)|}\right) & \text{otherwise.} \end{cases}$$

It is easy to see that $f'(\bar{x}, .) = f(z)$ is the directional derivative of f at $\bar{x} = 0$ and its $\mathcal{G}$-derivative at the same point, but f is not continuous at 0.

Now let us recall the definition of class $\mathcal{S}^-$ from [4].

DEFINITION 3. A function $f : X \to \mathbb{R}$ is said to be of class $\mathcal{S}^-$ at $\bar{x}$ iff there exist a neighborhood N of $\bar{x}$ and a $\mathcal{C}$-differentiable function $\sigma_f : X \to \mathbb{R}$ whose epigraph is minimal in the sense of inclusion and whose $\mathcal{C}$-derivative is closed such that

$$f(x) \geq \sigma_f(x), \quad \forall x \in X \cap N \text{ and } f(\bar{x}) = \sigma_f(\bar{x}).$$

σ_f is called the lower support of f at $\bar{x}$. The notion of generalized subdifferential of a function of class $\mathcal{S}^-$ is the one of the lower support.

It turns out that the class $\mathcal{S}^-$ is exactly the class of $\mathcal{C}$-differetiable functions.

PROPOSITION 4. *The function f is of class $\mathcal{S}^-$ iff it is $\mathcal{C}$-differentiable.*

PROOF. *Necessity.* Assume that f is of class $\mathcal{S}^-$ and σ_f is its lower support. We will show that $f(x) = \sigma_f(x) \quad \forall x \in X \cap N$. Ab absurdo, suppose that there exists $\hat{x} \in X \cap N \setminus \{\bar{x}\}$ such that $f(\hat{x}) > \sigma_f(\hat{x})$. Then the function

$$\hat{\sigma}_f(x) := \begin{cases} \sigma_f(x) & \text{if } x \neq \hat{x}, \\ f(x) & \text{if } x = \hat{x}, \end{cases}$$

is a $\mathcal{C}$-differentiable function which is a lower support for f and epi $\hat{\sigma}_f$ is stricly contained in epi σ_f. This contradicts the assumption that σ_f is the lower support of f. So $f(x) = \sigma_f(x) \quad \forall x \in X \cap N$ and f is $\mathcal{C}$-differentiable.

Sufficiency. Straightforward.

3. $\mathcal{G}$-semidifferentiability

In this section the necessary and sufficient conditions for $\mathcal{G}$-semidifferentiability are established. From these conditions we can see that the $\mathcal{G}$-semiderivatives are the Dini derivatives for functions whose Dini-Hadamard derivatives are the lower limit of Dini derivatives. The class of functions satisfying this property is larger then the class of Lipschitz functions.

THEOREM 5. *If f is lower $\mathcal{G}-$semidifferentiable, then*

$$\underline{D}_{\mathcal{G}} f(\bar{x}, z) = f^{-}(\bar{x}, z), \tag{11}$$

$$\liminf_{y \longrightarrow z} f^{-}(\bar{x}, y) = f_T^{-}(\bar{x}, z) = \liminf_{y \longrightarrow z} f_T^{-}(\bar{x}, y). \tag{12}$$

Moreover, we have analogous equalities for the upper version of $\mathcal{G}-$semiderivatives.

PROOF. From the definitions we have

$$\begin{aligned} f^{-}(\bar{x}, z) &= \liminf_{t \longrightarrow 0} \frac{f(\bar{x} + tz) - f(\bar{x})}{t} \\ &= \liminf_{t \longrightarrow 0} \frac{\underline{D}_{\mathcal{G}} f(\bar{x}, tz) + \epsilon_{f, \underline{D}_{\mathcal{G}} f}(\bar{x}, tz)}{t} = \\ &= \underline{D}_{\mathcal{G}} f(\bar{x}, z) + \liminf_{t \to 0} \frac{\epsilon_{f, \underline{D}_{\mathcal{G}} f}(\bar{x}, tz)}{t}. \end{aligned}$$

From the uniqueness condition (7) it follows that

$$\mu := \liminf_{t \to 0} \frac{\epsilon_{f, \underline{D}_{\mathcal{G}} f}(\bar{x}, tz)}{t} = 0$$

as, in the other case ($\mu > 0$), by taking

$$\underline{D}^1_{\mathcal{G}} f(\bar{x}, z) := \underline{D}_{\mathcal{G}} f(\bar{x}, z) + \|z\| \mu$$

we can get a contradiction to (7). So (11) follows. Now let $\{z_n\}$ be a sequence converging to $z \neq 0$ such that there exists a sequence $\{t_n\}$ converging to 0 satisfying

$$\lim_{n \to \infty} \frac{f(\bar{x} + t_n z_n) - f(\bar{x})}{t_n} = f_T^{-}(\bar{x}, z).$$

Then

$$f_T^{-}(\bar{x}, z) - \liminf_{y \longrightarrow z} f^{-}(\bar{x}, y) = f_T^{-}(\bar{x}, z) - \liminf_{y \to z} \underline{D}_{\mathcal{G}} f(\bar{x}, y)$$

$$\geq \lim_{n\to\infty} \frac{f(\bar{x}+t_n z_n)-f(\bar{x})}{t_n} - \limsup_{n\to\infty} \underline{D}_G f(\bar{x}, z_n)$$
$$= \liminf_{n\to\infty} \frac{f(\bar{x}+t_n z_n)-f(\bar{x}) - \underline{D}_G f(\bar{x}, t_n z_n)}{t_n}$$
$$= \liminf_{n\to\infty} \frac{\epsilon_{\underline{D}}(\bar{x}, t_n z_n)}{|t_n z_n|}|z_n| \geq 0.$$

So

$$f_T^-(\bar{x}, z) \geq \liminf_{y\to z} f^-(\bar{x}, y). \tag{13}$$

Now we will show that the Dini-Hadamard lower directional derivative is lower semicontinuous. Indeed, let $\{z_n\}$ be a sequence converging to z such that

$$\lim_{n\to\infty} f_T^-(\bar{x}, z_n) = \liminf_{y\to z} f_T^-(\bar{x}, y).$$

If there exists a subsequence $\{z_{n_k}\}$ such that $f_T^-(\bar{x}, z_{n_k}) > -\infty \quad \forall n_k$ then there exist sequences $\{y_{n_k} \in \mathbb{R}^n\}$ and $\{t_{n_k} \in (0, \frac{1}{n_k})\}$ such that

$$|y_{n_k} - z_{n_k}| \leq \frac{1}{n_k},$$
$$\frac{f(\bar{x}+t_{n_k} y_{n_k})-f(\bar{x})}{t_{n_k}} \leq f_T^-(\bar{x}, z_{n_k}) + \frac{1}{n_k}.$$

Obviously, the sequence $\{y_{n_k}\}$ converges to z, so we have

$$\liminf_{y\to z} f_T^-(\bar{x}, y) = \lim_{n\to\infty} f_T^-(\bar{x}, z_n)$$
$$\geq \limsup_{n_k\to\infty} \frac{f(\bar{x}+t_{n_k} y_{n_k})-f(\bar{x})}{t_{n_k}}$$
$$\geq \liminf_{y\to z, t\downarrow 0} \frac{f(\bar{x}+ty)-f(\bar{x})}{t}$$
$$= f_T^-(\bar{x}, z).$$

Hence

$$\liminf_{y\to z} f_T^-(\bar{x}, y) \geq f_T^-(\bar{x}, z). \tag{14}$$

Combining (13) with the last inequality and the following obvious inequality

$$\liminf_{y\to z} f^-(\bar{x}, y) \geq \liminf_{y\to z} f_T^-(\bar{x}, y)$$

we get (12) .

REMARK 4. Theorem 5 can be considered as the necessary conditions for lower $\mathcal{G}-$ semidifferentiability. By combining Theorem 5 with Remark 1 we get that f is lower $\mathcal{G}$-semidifferentiable if and only if

$$f^{-}(\bar{x},.) \in LGA_f(\bar{x}). \tag{15}$$

The following sufficient condition gives us another relationship between the notion of $\mathcal{G}$-semidifferentiability and the Dini directional derivatives.

THEOREM 6. *If for every $z \in Z$,*

$$f_T^{-}(\bar{x},z) \geq \limsup_{y\to z} f^{-}(\bar{x},y), \tag{16}$$

then f is lower $\mathcal{G}$ -semidifferentiable.

PROOF. We will show that $\underline{D}_{\mathcal{G}}f(\bar{x},z) = f^{-}(\bar{x},z)$. From the definition we have

$$\liminf_{z\to 0} \frac{\epsilon_{f,f^-}(\bar{x},z)}{|z|} = \liminf_{z\to 0} \frac{f(\bar{x}+z)-f(\bar{x})-f^{-}(\bar{x},z)}{|z|}.$$

Let $z_n \longrightarrow 0$ be a sequence such that

$$\lim_{n\to\infty} \frac{f(\bar{x}+z_n)-f(\bar{x})-f^{-}(\bar{x},z_n)}{|z_n|} = \liminf_{z\to 0} \frac{\epsilon_{f,f^-}(\bar{x},z)}{|z|}.$$

By taking a subsequence if necessary we may assume that the sequence $\{\frac{z_n}{|z_n|}\}$ converges to $u \neq 0$. Then

$$\lim_{n\to\infty} \frac{f(\bar{x}+z_n)-f(\bar{x})-f^{-}(\bar{x},z_n)}{\|z_n\|}$$

$$\geq \liminf_{n\to\infty} \frac{f(\bar{x}+z_n)-f(\bar{x})}{\|z_n\|} - \limsup_{n\to\infty} f^{-}(\bar{x},\frac{z_n}{\|z_n\|})$$

$$\geq \liminf_{t\downarrow 0, y\to u} \frac{f(\bar{x}+ty)-f(\bar{x})}{t} - \limsup_{y\to u} f^{-}(\bar{x},y).$$

From (16) it follows that the latter difference is positive. So we get $f^{-}(\bar{x},.) \in LGA_f(\bar{x})$ and from Remark 4 we get the lower $\mathcal{G}-$semidifferentiability of f.

REMARK 5. It is easily seen that for locally Lipshitz functions the condition (16) is fulfilled, and, as a consequence of Theorem 4, it follows that Lipshitz functions are lower $\mathcal{G}-$semidifferentiable (see e.g. [4]).

4. $\mathcal{C}$-subdifferentials

In this section we will consider the generalized semisubdifferential (9) of a $K(\subset \mathcal{C})$-semidifferentiable function.

Obviously, the lower semisubdifferential is a closed, convex set. But, from the definition, for the case $Z \neq \mathbf{R}^n$, the generalized subdifferential might be noncompact when the semiderivatives are bounded. It is well-known that the compactness of subdifferential plays an important role in applications. The following natural consequence of the definition of lower subdifferential

$$\partial_K f(\bar{x}) + Z^* \subset \partial_K f(\bar{x}) \tag{17}$$

give us a necessary and sufficient condition for the compactness of the K-subdifferential.

PROPOSITION 7. *The K-subdifferential is bounded if and only if $Z = \mathbf{R}^n$.*

PROOF. It follows from (17).

From now on we only consider the case $Z = \mathbf{R}^n$.

PROPOSITION 8. *If the function f is lower K-semidifferentiable with nonempty K-generalized semisubdifferential, then the Dini subdifferential is nonempty and $\partial_K f(\bar{x}) \subset \partial^- f(\bar{x})$.*

PROOF. Let $u \in \partial_K f(\bar{x})$. From Remark 1 we get

$$\langle u, z\rangle \leq \underline{D}_K f(\bar{x}, z) \leq f^-(\bar{x}, z), \forall z.$$

Hence

$$\langle u, z\rangle \leq f_T^-(\bar{x}, z) \forall z \in \mathbf{R}^n,$$

and the conclusion follows.

From Proposition 8 it follows that the Dini subdifferentiability is necessary for the nonemptiness of $\mathcal{C}$ -semisubdifferential. It turns out that for Lipschitz functions these two notions are equivalent.

THEOREM 9. *Assume that f is a Lipschitz function. If the Dini subdifferential is nonempty then the function f is $\mathcal{C}$-lower semidifferentiable. In this case, the lower $\mathcal{C}$-semiderivative is the support function for the Dini-subdifferential*

$$\bar{D}_{\mathcal{C}} f(\bar{x}, z) = \psi^*(z|\partial^- f(\bar{x})), \tag{18}$$

and hence $\partial_{\mathcal{C}} f(\bar{x}) = \partial^- f(\bar{x})$.

PROOF. As f is Lipschitz, we get

$$\liminf_{z\to 0} \frac{f(\bar{x}+z) - f(\bar{x}) - \sup_{u\in\partial^- f(\bar{x})}\langle u, z\rangle}{|z|}$$

$$\geq \liminf_{z\to 0} \frac{f(\bar{x}+z) - f(\bar{x}) - f^-(\bar{x}, z)}{|z|} \geq 0$$

It follows that the support function of Dini-subdifferential is in the set $LCA_f(\bar{x})$. Since $\mathcal{C}$-semiderivative should be smaller than the lower Dini derivative, so the support function is the maximum element in this set.

CONCLUDING REMARKS. From Remark 3 we can see that if f is $\mathcal{C}-$differentiable then f possesses a convex directional derivative which coincides with its $\mathcal{C}$-derivative. So the function f is *quasidifferentiable* in the Pshenichnyi's definition (see [12]). Recall that f is said to be quasidifferentiable at $\bar{x}$ if there exists a closed convex set K in $\mathbb{R}^n$ such that

$$f'(\bar{x}, .) = \sup_{u\in K}\langle u, .\rangle.$$

Another useful notion is that of ∂-regularity. Assume that $\bar{\partial}$ is a given kind of subdifferential. A function f is said to be $\bar{\partial}$-regular if the usual directional derivative $f'(\bar{x}, z)$ exists and equals $\psi^*(z\|\bar{\partial} f(\bar{x}))$ for all $z \in Z$. We can see that the generalized subdifferential for a function of class $\mathcal{S}^-$ is always regular. Moreover, it is easy to see from Proposition 5 of [2] that for Lipschitz functions the $\mathcal{C}$-differentiability is equivalent to the regularity of the Michel-Penot subdifferential and, by Theorems 8 and 9, to the Dini subdifferentiability.

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