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CONVEX-CONCAVE PROGRAMMING AS A DECOMPOSITION APPROACH TO GLOBAL OPTIMIZATION

LE DUNG MUU

Abstract. We show that many problems of global optimization can be converted into convex-concave programs which can be solved by a decomposition procedure combining branch-and-bound techniques with cutting plane methods. The algorithms use adaptive rectangular and simplicial subdivisions which are not necessarily exhaustive but sufficient to guarantee convergence.

Key words. Convex-concave programming, decomposition, branch and bound, adaptive subdivision, cutting plane, exhaustiveness, d.c. programming.

1. Introduction

Numerical experiences indicate that except certain special cases, the most global optimization problems of realistic sizes cannot yet be solved (see e.g. [4, 15, 16]). Fortunately, in many practical problems the number of "nonconvex variables" is relatively small as compared to the total number of variables of the problem. This suggests applying decomposition techniques for solving global optimization problems. In fact some decomposition methods have been successfully used for solving a lot of global optimization problems. Rosen and Pardalos [16] solved large-scale linear constrained concave quadratic minimization problems by using the eigenstructure of the quadratic form to reduce the objective function to a separable quadratic form. Tuy [21,22] used Bender's decomposition approach for minimizing a concave and a d.c. function. Tuy's method was further investigated by Thach [17] and Thieu [19] for solving concave minimization problems over networks. Horst and Tuy [4] developed another decomposition algorithm for minimizing the sum of a linear and a concave function with linear constraints. Muu and Oettli [8, 9] proposed methods for solving indefinite quadratic problems over convex set and for minimizing a convex-concave

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function with convex constraints. These methods are specialized in [11] to obtain efficient algorithms for solving certain nonconvex optimization problems dealing with the product of two affine fractional functions. A fairly general decomposition scheme for solving a broad class of global optimization problems can be found in [10].

The purpose of this paper is to show that a lot of problems of global optimization can be viewed in form of convex-concave programming problems. Then we propose a decomposition approach by using a combination of branch- and-bound and cutting plane methods for solving a broad class of convex-concave global optimization problems. For branching operation we use a separation function which takes into account iteration points and/or objective function as well as constraints. These subdivisions are, in general, not necessarily exhaustive but sufficient to guarantee the convergence. They take place in the "concave space" only, and therefore it allows us to decompose the problem into convex subprograms, and concave minimization problems whose number of variables is often much less than that of the original problem. The cutting plane, as usually, is used for approximating convex constraints by linear ones, but here it is performed parallel with the branching and bounding operations. The method is described by a similar way as the one in our earlier paper [10]. The main difference lies in the rules for determining the branching operations, and therefore new methods for solving convex-concave programming problems are developed.

The paper is organized as follows. In the next section we state the convex-concave problem to be considered and collect some global optimization problems which can be converted into convex-concave programming problems. The third section is devoted to the description of a unified branch-and-bound and cutting plane algorithm and its convergence. In the last section we shall give some branching operations among which a new simplicial bisection is presented.

2. Convex-concave mathematical programming problems

In what follows we consider the following global optimization problem to which we shall refer as a convex-concave problem

$$\min\{f(x,y) : (x,y) \in S,\ g_j(x,y) \leq 0,\ j = 1,...,\ell\} \tag{CC}$$

where f and each g_j are real continuous functions defined on $R^n \times R^m$, and S is a closed convex set given by

$$S := \{(x,y) \in R^n \times R^m : \varphi(x,y) \leq 0\}.$$

We assume additionally that f and each g_j are convex-concave functions on some convex set S_0 containing S. Problem (CC) where the convex-concave constraints $g_j(x,y) \leq 0$ are missing is considered in our earlier paper [9].

The convex-concave programming problem (CC) contains a broad class of mathematical programming problems as special cases, examples being jointly constrained bilinear programs [1, 8], certain d.c. programming problems [4, 22]. Furthermore many nonconvex optimization problems can be converted into a convex-concave problem of the form (CC). Here we mention some important examples.

1. Affine multiplicative programming problem [7, 11]

$$\min\{\ell_1(x)\ell_2(x) : x \in D\} \tag{AM}$$

where D is a compact convex set in R^n, and ℓ_1, ℓ_2 are affine functions on D. It is easy to see that the product of two affine functions, in general, is neither convex nor concave. Thus the above problem is a multiextremal optimization problem. By setting $y = \ell_1(x)$ Problem (AM) can be rewritten as

$$\min\{y\ell_2(x) : x \in D,\ y = \ell_1(x)\}$$

which is a convex-concave program of the form (CC). Note that in this case y is one-dimensional variable.

2. Indefinite quadratic programming problems [8, 16]

$$\min\{f(z) := c^T z + \frac{1}{2} z^T Q z : z \in D\} \tag{IQ}$$

where D is a closed convex set in R^n, and Q is an $(n \times n)-$ symmetric indefinite matrix.

It is well known that by using only linear transformations one can partition the quadratic form f into $f(z) = f_1(x) + f_2(y)$ with

$$f_1(x) = \sum (c_i x_i + \frac{1}{2}\lambda_i x_i^2) \quad (\lambda_i > 0,\ i = 1, ..., k),$$
$$f_2(y) = \sum (c_i y_i + \frac{1}{2}\lambda_i y_i^2) \quad (\lambda_i \leq 0,\ i = k+1, ..., n).$$

Thus the above indefinite quadratic programming problem is equivalent to

$$\min\{f_1(x) + f_2(y) : (x, y) \in S\}$$

where S is a closed convex set. This problem is again of a form of (CC). We know from [14] that Problem (IQ) even with one negative eigenvalue is NP-hard. It is clear that in the latter case y is one-dimensional variable.

3. Minimization of a linear function over the efficient set [2, 13]

Let C be a $(p \times n)$-matrix and D be a nonempty polytope in R^n. Then the multiple objective linear programming problem, given by

$$V \min\{Cx : x \in D\} \tag{ML}$$

can be viewed as the problem of finding all efficient points of Cx over D.

We recall that a point x^0 is said to be an efficient solution of (ML) when $x^0 \in D$, and whenever $Cx \leq Cx^0$ for some $x \in D$, then $Cx = Cx^0$. By X_E we shall denote the set of all efficient points of (ML).

The problem of optimization of a linear function over the efficient set of Problem (ML) is then given by

$$\min\{d^T x : x \in X_E\}. \tag{LE}$$

Since the efficient set X_E is, in general, nonconvex, Problem (LE) can be classified as a global optimization problem. From [2] we know that one can find a simplex $\Lambda \subset R^p$ such that a point $x^0 \in X_E$ if and only if there exists a parameter $\lambda^0 \in \Lambda$ satisfying $\lambda^0 Cx^0 \leq \lambda^0 Cx$ for all $x \in X$. Thus Problem (LE) is equivalent to

$$\min\{d^T x : x \in X, \lambda \in \Lambda, \lambda Cx \leq \lambda Cz, \forall z \in X\}.$$

By setting $g(\lambda, x) := \lambda Cx - \min_{z \in X} \lambda Cz$ this problem can be rewritten as

$$\min\{d^T x : x \in X, \lambda \in \Lambda, g(\lambda, x) \leq 0\}$$

which, since the function g is convex in λ and linear in x, is of the form (CC).

4. Rank two bilinear programming problems [23]

We know from [23] that the bilinear programming problem

$$\min\{c^T x + d^T y + x^T Qy : x \in X, y \in Y\}, \tag{BL}$$

where Q is a rank two $(n \times m)$-matrix, and X, Y are polytopes in R^n, R^m respectively, can be converted into the form

$$\min\{a^T x + b^T y + p_1^T x p_2^T y + q_1^T x q_2^T y : x \in A, y \in B\}$$

where A and B again are polytopes. Setting $\xi = p_1^T x$ and $\eta = q_2^T y$ the above problem leads to

$$\min\{a^T x + b^T y + \xi p_2^T y + \eta q_1^T x : x \in A, p_1^T x = \xi, y \in B, q_2^T y = \eta\}.$$

It is clear that for each fixed ξ and η this problem is a linear program. Denote by $t(\xi, \eta)$ the optimal value of this program, then

$$t(\xi, \eta) = t_x(\xi, \eta) + t_y(\xi, \eta),$$

where

$$t_x(\xi, \eta) := \min\{a^T x + \eta q_1^T x : x \in A, p_1^T x = \xi\}$$

and

$$t_y(\xi, \eta) := \min\{b^T y + \xi p_2^T y : y \in B, q_2^T y = \eta\}.$$

Let

$$\xi_{\min} = \min\{p_1^T x : x \in A\},$$
$$\xi_{\max} = \max\{p_1^T x : x \in A\},$$
$$\eta_{\min} = \min\{q_2^T y : y \in B\}$$

and

$$\eta_{\max} = \max\{q_2^T y : y \in B\}.$$

Then from the linear programming it follows that the function $t_x(\xi, \eta)$ is convex-concave and $t_y(\xi, \eta)$ is concave-convex on the rectangle

$$R := [\xi_{\min}, \xi_{\max}] \times [\eta_{\min}, \eta_{\max}].$$

5. A problem of optimal design of a water distribution network [3]

Consider a water distribution network which is mathematically represented by a connected directed graph G with n nodes, t arcs (or links) and m independent loops. Suppose that there is only one source node. Let q_i and p_i respectively denote the flow and headloss of link i. An important problem in the optimal design of water distribution networks is to minimize the cost function

$$F(p, q) := \sum a_i q_i^{\lambda\beta/\alpha} p_i^{-\beta/\alpha}$$

under the following constraints:

- Demand on water discharge

$$\sum_{i \in I(k)} q_i - \sum_{j \in O(k)} q_j = b_k \ (k = 1, ..., n);$$

- Loop balance:

$$\sum_{i \in L(l)} p_i = 0 \ (l = 1, ..., m);$$

- Hydraulic head requirement at each node:

$$\sum_{i \in r(k)} p_i \leq h_0 - h_{k\min} \ (k = 1, ..., n);$$

- Physical limit of network:

$$0 < p_{i\min} \le p_i \le p_{i\max}$$

$$0 < q_{i\min} \le q_i \le q_{i\max} \ (i = 1, ..., t),$$

where $I(k)$ and $O(k)$ are respectively the sets of the links connected toward and out of the node k and $L(l)$ denotes the number of the links in the loop l, and $r(k)$ stands for a path connecting the source node with the node k.

We note that the well known Hazen-William equation is already substituted into the objective function F. From the physical conditions it follows that $1 < -\beta/\alpha < 2.5$, $0 < \lambda\beta/\alpha < 1$ and $a_i > 0$ for every i. Thus the objective function is convex in p and concave in q while the constraints are linear. It is worth to note that the "concave variables" q can be expressed via flow change along the orientation of the loops. Namely, let y_i denote the flow change along the loop l, and let $\bar{q}$ be a feasible flow. Then from [3] we have

$$q_i = \bar{q}_i + \sum_{j=1}^{m} \xi_{ij} y_j,$$

where $\xi_{ij} = 1$ if link i lies on loop j and the flow direction coincides with the orientation of loop j in link i; $\xi_{ij} = -1$ if link i lies on loop j and the direction is opposite to the orientation of loop j in link i, and $\xi_{ij} = 0$ otherwise. Thus the problem can be carried out by resorting to new variables $y_1, ..., y_m$ instead of $q_1, ..., q_t$, which may significantly reduce the number of the "concave variables" since in practice m is usually much less than t.

3. A unified branch-and-bound and cutting plane method

In the sequel we shall restrict ourselves to the cases when the variable y is not absent in Problem (CC). We assume that we have fixed two compact polyhedra $X \subset R^n$ and $Y \subset R^m$ such that $X \times Y$ contains the feasible set of (CC). This can be done by standard methods of convex programming if S, in addition, is compact (very often in practical problems) (see e.g. [4]). Let f_* denote the optimal value of (CC) (as usually, we alwavs adopt the convention that the

minimum of a function over an empty set is equal $+\infty$), and let $g := \begin{pmatrix} g_1 \\ \vdots \\ g_\ell \end{pmatrix}$.
Given two polyhedra $B \subset Y$ and $T \supseteq S$ we define Problem C(B,T) as

$$\min\{f(x,y) : x \in X, y \in B, g(x,y) \leq 0, (x,y) \in T\}. \tag{C(B,T)}$$

and the relaxed Problem R(B,T) as

$$\min\{f(x,y) : x \in X, y \in B, g(x,y) \leq 0, (x,u) \in T, u \in B\}. \tag{R(B,T)}$$

By $\beta(B,T)$ we denote the optimal value of $R(B,T)$. Due to our compactness assumption, whenever $\beta(B,T) < \infty$ then $R(B,T)$ has an optimal solution which we denote by (x^{BT}, y^{BT}, u^{BT}) or sometimes by (x^B, y^B, u^B). By α_{k-1} we shall denote the least upper bound for f_* known at the beginning of iteration k.

ALGORITHM.

Initialization. With the two given convex polyhedra $B_0 = Y$ and $T_0 \supseteq S$ solve Problem $R(B_0, T_0)$.

If $\beta(B_0, T_0) = \infty$, terminate; Problem (CC) has no feasible point.

If $\beta(B_0, T_0) < \infty$, let $(x^{B_0}, y^{B_0}, u^{B_0})$ be the obtained optimal solution of $R(B_0, T_0)$. Let $\alpha_{-1} = \infty$ and $\Gamma_0 = \{B_0\}$.

Iteration k (k=0,1...). At the beginning of iteration k we have a collection Γ_k of convex polyhedral subsets $B \subset B_0$ such that one solution of (CC) is contained in $X \times \cup\{B : B \in \Gamma_k\}$, and we have a polyhedron $T_k \supseteq S$. For each $B \in \Gamma_k$ we know an optimal solution (x^B, y^B, u^B) of $R(B, T_k)$. Furthermore, $\alpha_{k-1} \geq f_*$ is at hand.

Let α_k be the currently known smallest upper bound for f_* and, if $\alpha_k < \infty$, let (ξ^k, η^k) be the best feasible point known so far so that $f(\xi^k, \eta^k) = \alpha_k$. Let

$$\Delta_k := \{B \in \Gamma_k : \beta(B, T_k) \leq \alpha_k\}.$$

Select $B_k^* \in \Delta_k$ such that

$$\beta_k := \beta(B_k^*, T_k) = \min\{\beta(B, T_k) : B \in \Delta_k\}.$$

1) If $\beta_k \geq \alpha_k$, terminate; $f_* = \alpha_k$ and, if $\alpha_k < \infty$, then (ξ^k, η^k) is an optimal solution of (CC). Otherwise, if $\alpha_k = \infty$, then (CC) has no feasible point.

2) If $\beta_k < \alpha_k$, then select $B_k \in \Delta_k$ such that

$$d_{B_k}(u^{B_k}) \geq d_{B_k^*}(u^{B_k^*}), \tag{1}$$

where

$$d_B(u) := [[y^B - u]] \tag{2}$$

or

$$d_B(u) := \max\{f(x^B, u) - f(x^B, y^B), \max_j \; g_j(x^B, u)\}. \tag{3}$$

($[[\cdot]]$ stands for a norm in R^m).

2a) If $d_{B_k}(u^{B_k}) \leq 0$, then set $\Gamma_{k+1} = \Delta_{k+1} = \Delta_k$ and go to 3). (Note that in this case $(x^{B_k^*}, u^{B_k^*})$ does not belong to S because otherwise $\beta_k \geq \alpha_k$, and therefore the algorithm already terminates before turning to the case 2a).

2b) If $d_{B_k}(u^{B_k}) > 0$, then select an affine function ℓ_k such that $\| \bigtriangledown \ell_k\| \leq c$ and

$$td_{B_k}(u^{B_k}) \leq \min\{\ell_k(u^{B_k}), -\ell_k(y^{B_k})\}, \tag{*}$$

where $c, t > 0$ are independent of k.

Set

$$B_k^- := \{y \in B_k : \ell_k(y) \leq 0\}, \quad B_k^+ := \{y \in B_k : \ell_k(y) \geq 0\}.$$

(By (*) $y^{B_k} \in B_k^-$, $u^{B_k} \in B_k^+$). Let

$$\Gamma_{k+1} := \Delta_k \setminus \{B_k\} \cup \{B_k^-, B_k^+\}.$$

3) Set

$$T_{k+1} := \begin{cases} T_k & \text{if} \quad (x^{B_k^*}, u^{B_k^*}) \in S; \\ \{(x,u) \in T_k : \varphi(x^{B_k^*}, u^{B_k^*}) & \\ \qquad + t_1(x - x^{B_k^*}) + t_2(u - u^{B_k^*}) \leq 0\}, & \text{otherwise} \end{cases}$$

where (t_1, t_2) is a subdifferential of φ at $(x^{B_k^*}, u^{B_k^*})$.

For each $B \in \Gamma_{k+1}$ solve $R(B, T_{k+1})$ (if $T_{k+1} = T_k$ we solve $R(B_k^-, T_{k+1})$ and $R(B_k^+, T_{k+1})$ only since for the other $B \in \Gamma_{k+1}$ the program $R(B, T_{k+1})$ has been solved.

Increase k by 1 and go to iteration k

This completes the description of the algorithm.

COMMENTS.

1. The just described algorithm can be considered as a new version of the one developed in [10]. The main difference lies in the use of the distance function d. The algorithm is a combination of branch-and-bound and cutting plane methods. It is clear that $S \subseteq T_{k+1} \subseteq T_k$ for all k. Hence if $T_0 = S$, then the algorithm becomes a pure branch-and-bound procedure; if instead $d_{B_k}(u^{B_k}) \leq 0$ (case 2a) for all k, then the algorithm becomes a pure cutting plane method.

2. A crucial operation in the above algorithm is the solution of the relaxed problem $R(B, T_k)$. This question will be discussed at the end of this section.

CONVERGENCE OF THE METHOD.

For simplicity we shall denote by (x^k, y^k, u^k) and (x^{*k}, y^{*k}, u^{*k}) the obtained solutions of Problem $R(B_k, T_k)$ and $R(B_k^*, T_k)$ respectively. Also we shall write d_k for d_{B_k} and d_{*k} for $d_{B_k^*}$. The $< .,. >$ stands for the inner product corresponding to the Euclidian norm $\|\cdot\|$.

It is clear from the construction of T_k that $S \subseteq T_{k+1} \subset T_k$ for every k. This implies $\beta_k \leq \beta_{k+1} \leq f_*$ for all k. Hence $\beta_* := \lim \beta_k$ exists and $\beta_* \leq f_*$. If the algorithm terminates at iteration k, i.e. $\beta_k \geq \alpha_k$, then from $\alpha_k \geq f_*$ it follows that $\alpha_k = \beta_k = f_*$. If the algorithm does not terminate, then we have the following convergence result:

THEOREM. *(a)* $\beta_k \nearrow f_*$, *and* $\{(x^{*k}, u^{*k})\}_0^\infty$ *has a limit point which solves (CC).*

(b) If (x^{*k}, u^{*k}) *is feasible for infinitely many* k, *then* $\alpha_k \searrow f_*$, *and every limit point of the sequence* $\{(\xi^k, \eta^k)\}$ *solves (CC).*

PROOF. (a) If $\beta_k = \infty$ for some k, then the algorithm terminates at iteration k and $f_* = \infty$ (Problem (CC) has no feasible points). Thus if the algorithm does not terminate, then $\beta_k < \infty$ for all k. Since

$$\beta_k = \min\{\beta(B, T_k) : B \in \Delta_k\} = \beta(B_k^*, T_k),$$

Problem $R(B_k^*, T_k)$ has an optimal solution (x^{*k}, y^{*k}, u^{*k}). Note that $(x^{*k}, u^{*k}) \in T_k$ for every k. This and the rule for constructing T_k imply that any limit point of the sequence $\{(x^{*k}, u^{*k})\}$ belongs to S. We distinguish two cases.

Case 1: Case 2b occurs only finitely often. In this case we may disregard the finitely many iterations of case 2b, and therefore we may assume that case 2a occurs for all k. From case 2a and (1) it follows that

$$d_{*k}(u^{*k}) \le d_k(u^k) \le 0 \quad \forall k.$$

This and the definition of d_{*k} by (2) or (3) imply

$$f(x^{*k}, u^{*k}) \le \beta_k, \ g(x^{*k}, u^{*k}) \le 0 \quad \forall k,$$

from which we obtain in the limit

$$f(x^*, u^*) \le \beta_* \le f_*, \ g(x^*, u^*) \le 0$$

for any limit point (x^*, u^*) of $\{(x^{*k}, u^{*k})\}_0^\infty$. Observing that $(x^*, u^*) \in S$ we see that (x^*, u^*) is an optimal solution of (CC) and therefore $\beta_* = f_*$.

Case 2: Case 2b occurs infinitely many times. In this case there exists a decreasing subsequence of $\{B_k\}_0^\infty$. Thus, by extracting a subsequence if necessary we may assume that $B_{k+1} \subset B_k^-$ for all k or $B_{k+1} \subset B_k^+$ for all k. In the first case we have $u^{k+1} \in B_k^-$ which means $\ell_k(u^{k+1}) \le 0$ for all k. We then obtain from (*) that

$$td_k(u^k) \le \ell_k(u^k) \le \ell_k(u^k) - \ell_k(u^{k+1}) \le c||u^k - u^{k+1}|| \to 0.$$

Likewise, if $B_{k+1} \subset B_k^+$ for all k, we use $y^{k+1} \in B_{k+1}^+$ to obtain, by a similar way,

$$td_k(u^k) \le c||y^k - y^{k+1}|| \to 0.$$

Hence always $d_k(u^k) \to 0$. We consider two possibilities.

α) $d_{*k}(u^{*k} > 0$ for infinitely many times. Then by taking again a subsequence if necessary we may assume that $d_{*k}(u^{*k}) > 0$ for all k and that $(x^{*k}, u^{*k}) \to (x^*, u^*)$. From (1) we get $d_{*k}(u^{*k}) \leq d_k(u^k)$. This and $d_k(u^k) \to 0$ imply $d_{*k}(u^{*k}) \to 0$ which together with each of (2) and (3) leads to

$$f(x^*, u^*) \leq \beta_* \leq f_*, \; g(x^*, u^*) \leq 0.$$

This and $(x^*, u^*) \in S$ show that (x^*, u^*) solves (CC), and therefore $\beta_* = f_*$.

β) $d_{*k}(u^{*k}) > 0$ for only finite k. In this case we may assume that $d_{*k}(u^{*k}) \leq 0$ for all k. From (3) it follows that

$$f(x^{*k}, u^{*k}) \leq \beta_k, \quad g(x^{*k}, u^{*k}) \leq 0.$$

Letting $k \to \infty$ and remember that $(x^*, u^*) \in S$ we obtain the result.

(b) Assume now that (x^{*k}, u^{*k}) is feasible for infinitely many k. From part (a) we see that $\{(x^{*k}, u^{*k})\}_0^\infty$ has a limit point (x^*, u^*) that solves (CC). Let (ξ^*, η^*) be any limit point of $\{(\xi^k, \eta^k)\}_0^\infty$. Without loss of generality we may assume that $(x^k, u^k) \to (x^*, u^*)$ and that $(\xi^k, \eta^k) \to (\xi^*, \eta^*)$. From the definition of α_k and (ξ^k, η^k) and the feasibility of (x^{*k}, u^{*k}) it follows that

$$f_* \leq \alpha_k := f(\xi^k, \eta^k) \leq f(x^{*k}, u^{*k}),$$

which together with $f(x^{*k}, u^{*k}) \to f(x^*, u^*)$, and the monotonicity of the sequence $\{\alpha_k\}$ imply the result. The theorem is proved.

REMARK. If the convex-concave constraints $g_j(x, y) \leq 0, \; j = 1, ..., \ell$ are missing and $T_0 = S$, then (x^{*k}, u^{*k}) is feasible for every k. In fact, in this case the feasible region is S, and Problem $R(B, S)$ then reads

$$\min\{f(x, y) : x \in X, \; u \in B, \; y \in B, \; (x, u) \in S\}.$$

As mentioned above the solution of the relaxed problem $R(B, T_k)$ is crucial for implementing the algorithm. Here we give some special cases where Problem $R(B, T_k)$ can be solved implementably.

1. Assume that the convex-concave constraints $g_j(x,y) \leq 0,\ j = 1,...,\ell$, are missing. Then Problem $R(B,T_k)$ becomes

$$\min\{f(x,y) : x \in X,\ y \in B,\ (x,u) \in T_k,\ u \in B\}.$$

Since the minimum of a concave function over a convex set is attained at an extremal point, we have

$$\beta(B,T_k) = \min_i\{\min\{f(x,v^i) : x \in X,\ u \in B,\ (x,u) \in T_k\}\},$$

where v^i are vertices of B. Hence Problem $R(B,T_k)$ is reduced to convex programs, one for each v^i. Note that if $f(x,y) = f_1(x) + f_2(y)$ (D.C. function), then the number of these convex programs just equals one. We observe that in the above algorithm even B is generated from some predecessor B' by adding an affine function, the calculation of the vertices of B, which could be done by some available methods [6, 18], is very costly, since the number of the vertices grows very quickly as the dimension of y-space gets larger. However, for the simplicial bisection the number of the vertices of B is $m+1$. Furthermore the vertex searching can be avoided if $f(x,y)$ for each fixed x, in addition, is separable, i.e.,

$$f(x,y) := \sum_{i=1}^{m} f_i(x,y_i),$$

and B is a rectangle given by

$$B = \{y \in R^m : a_i \leq y_i \leq b_i \quad i = 1,...,m\}.$$

In this case we have

$$\beta(B,T_k) = \min\{\sum_{i=1}^{m} \min\{f_i(x,a_i), f_i(x,b_i)\} : x \in X\}.$$

Note that the objective function of the problem of optimal design of a water distribution network described in Section 2 is separable.

2. Assume now that $f(x,y) = f(x)$ and that $\ell = 1,\quad g_1(x,y) = g(x,y)$ (example 4 in Section 2). Then from the concavity of $g(x,.)$ we get

$$\min\{g(x,y) : y \in B\} = \min_i\ g(x,v^i)$$

and therefore

$$\beta(B,T_k) = \min\{f(x) : x \in X,\ \min_i\ g(x,v^i) \le 0,\ u \in B,\ (x,u) \in T_k\}.$$

Hence for each i it requires minimizing the convex function f over a convex set. This is simplified further if $g(x,y) = g_1(x)+g_2(y)$ (this case $g(x,y) \le 0$ appears as a reverse convex constraint). Then

$$\beta(B,T_k) = \min\{f(x) : x \in X,\ g_1(x)+\xi \le 0,\ u \in B,\ (x,u) \in T_k\}$$

with $\xi := \min_i g_2(v^i) = \min\{g_2(y) : y \in B\}$.

4. Examples for the separation function

Since $[[\cdot]]$ and $||\cdot||$ are two norms in the R^m, there must exist two positive numbers c and C such that $c[[u]] \le ||u|| \le C[[u]]$ for all $u \in R^m$.

1. *Polyhedral bisection.* The first example is designed to the functions defined by (2). In case 2b we have $d_k(u^k) > 0$. Let $r^k := (u^k - y^k)/d_k(u^k)$ and define

$$\ell_k(y) := < r^k, y > - < r^k, (y^k + u^k)/2 > .$$

Then $||\nabla \ell_k|| \le C$. Moreover $\ell_k(u^k - y^k) \ge c^2 d_k(u^k)$ and $\ell_k(u^k) = -\ell_k(y^k)$. Hence (*) is satisfied with $t \le c^2/2$.

2. *Simplicial bisection.* The second example is designed to a simplicial bisection. Suppose that B_k is a fully dimensional simplex in R^m. Let u^k, y^k be two distinct vertices of B_k, and $v^k := (u^k + y^k)/2$ the midpoint of the edge determined by u^k and y^k. Let H_k be the hyperplane through the points obtained from the vertex set of B_k by replacing y^k and u^k by v^k. Denote by h^k the point of H_k such that $u^k - h^k$ is the normal vector of H_k. Let φ_k be the angle between $u^k - h^k$ and $u^k - y^k$. Assume that $||u^k - y^k|| = O(\cos\varphi_k)$. Set

$$d_{B_k}(u) := \cos\varphi_k ||y^{B_k} - u||^2$$

and

$$\ell_k(y) := \cos\varphi_k < u^k - v^k,\ y - v^k > .$$

A simple computation shows that

$$\ell_k(u^k) = \frac{1}{4} d_{B_k}(u^k) = -\ell_k(y^k).$$

Hence (*) is satisfied with $t \leq \frac{1}{4}$.

It is clear from the definition that ℓ_k is the affine function corresponding to the hyperplane H_k, and therefore the sets

$$B_k^- = \{y \in B_k : \ell(y) \leq 0\}$$

and

$$B_k^+ = \{y \in B_k : \ell(y) \geq 0\}$$

are the simplices whose vertex sets are obtained from that of B_k by replacing y^k and u^k by v^k respectively. If u^k and y^k are the vertices of the longest edge of B_k, then this bisection becomes the one introduced first by Horst in [5] (see also [12], [20]).

Note that the edge determined by u^k and y^k is not necessarily the longest, and therefore this simplicial bisection is, in general, not exhaustive (see e.g. [21] for the definition of exhaustiveness).

3. *Subgradiential bisection.* This example corresponds to the function d_k defined by (3) and to the case when f and each g_j are convex in the second argument. Let $r^k \in \partial d_k(u^k)$ (the subgradient of d_k at u^k) such that $r^k \neq 0$. Such a vector exists because $d_k(u^k) > 0 = d_k(y^k)$ and d_k is convex. Assume $||r^k|| \leq L$. Define

$$\ell_k(y) := L^{-1}(< r^k, y - u^k > + d_k(u^k)/2).$$

Then $|| \nabla \ell_k|| \leq 1$. From $r^k \in \partial d_k(u^k)$ it follows that

$$\ell_k(y^k) \leq L^{-1}(d_k(y^k) - d_k(u^k)/2).$$

Since $d_k(y^k) = 0$, $\ell_k(y^k) \leq -(2L)^{-1} d_k(u^k)$. This and $\ell_k(u^k) = (2L)^{-1} d_k(u^k)$ imply (*) with $0 < t \leq (2L)^{-1}$.

4. *Rectangular bisection.* This last example is designed to the case when d_k is defined by the maximal norm. Let j_k be an index such that $\max_i |(u^k - y^k)_j| = |(u^k - y^k)_{j_k}|$. Take $d_k(u^k) := |(u^k - y^k)_{j_k}|$ and define the vector r^k as

$$r_j^k := \begin{cases} (u^k - y^k)_{j_k} \text{ if } j = j_k, \\ 0 \quad \text{otherwise.} \end{cases}$$

Since $d_k(u^k) > 0$ in the case 2b, we have $||r^k|| > 0$. Let

$$\ell_k(y) := ||r^k||^{-1} < r^k, y - (y^k + u^k)/2 > .$$

Then $|| \nabla \ell_k|| = 1$ and

$$\ell_k(y^k) = ||r^k||^{-1} < r^k, (y^k - u^k)/2 > = -|(u^k - y^k)_{j_k}|/2 = -d_k(u^k)/2.$$

Likewise,

$$\ell_k(u^k) = ||r^k||^{-1} < r^k, (u^k - y^k)/2 > = |(u^k - y^k)_{j_k}|/2 = d_k(u^k)/2.$$

Hence (*) is satisfied with $0 < t \leq 1/2$.

Note that with this separation function the set B_k, for every k, is a rectangle provided B_0 is so. In fact, in this case

$$B_k^- = \{y \in B_k : (u^k - y^k)_{j_k} y_{j_k} \leq (u^k - y^k)_{j_k} (u^k + y^k)_{j_k}/2\},$$
$$B_k^+ = \{y \in B_k : (u^k - y^k)_{j_k} y_{j_k} \geq (u^k - y^k)_{j_k} (u^k + y^k)_{j_k}/2\}.$$

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INSTITUTE OF MATHEMATICS
P.O. BOX 631, BOHO, 10 000 HANOI, VIETNAM