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ON A CLASS OF RESOLVABLE LIE ALGEBRAS

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1. Introduction

Let $\mathcal{G}$ be a real finite dimensional Lie algebra , G the corresponding connected simply-connected Lie group and $Ad : G \longrightarrow Aut(\mathcal{G})$ its adjoint representation in $\mathcal{G}$. Then G acts on the dual space $\mathcal{G}^*$ of $\mathcal{G}$ by : $Ad^* : G \longrightarrow Aut(\mathcal{G}^*)$ with

$$< Ad^*_g(f), x >=< f, Ad_{g^{-1}}(x) > \quad (f \in \mathcal{G}^*, x \in \mathcal{G}, g \in G).$$

This is the K-representation of the Lie group G and its orbits in $\mathcal{G}^*$ are called the K-orbits [1].

DEFINITION 1. *We say that the Lie algebra $\mathcal{G}$ satisfies the property MD (or $\overline{MD}$) if the dimension of every K-orbit in $\mathcal{G}^*$ is 0 or maximal (i.e., 0 or equal to dim $\mathcal{G}$).*

If the Lie algebra $\mathcal{G}$ is resolvable and satisfies the property MD ($\overline{MD}$), then it is said belonging to the class MD ($\overline{MD}$) or MD's ($\overline{MD}$'s) Lie algebra .

This is an interesting and imporant class of Lie algebras because the C^* -algebra of their corresponding Lie groups can be studied by methods of homological K-theory [2], [3], [5], [6], [7].

It should be noted that the class $\overline{MD}$ is a subclass of the class MD and it has been completely studied in [4] by H.H. Viet. It is surprising that there are only two non-commutative Lie algebras in this class : real affine Lie algebra $Aff(R)$ and complex affine Lie algebra $Aff(C)$. On the other hand V.M. Son and H.H. Viet have proved in [5] that if $\mathcal{G}$ is a Lie algebra of the class MD then $\mathcal{G}^3 = (0)$.

The aim of this paper is to explore some important properties of the Lie algebras of the class MD and classify all the Lie algebras of the class MD with

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dimension not greater than 5. Preliminary results on dimension 3 and 4 have been developed in [8]. Our method is very elementary. Roughly speaking, we only use technics of Lie algebra and Linear algebra.

2. Some fundamental properties of the Lie algebras of the class MD

Let f be an element of $\mathcal{G}^*$, Ω_f the K-orbit containing f . Let G_f be the stable subgroup of f and $\mathcal{G}_f$ the Lie algebra of G_f . It is obvious that $\Omega_f \cong G/G_f$. Suppose that B_f is the bilinear antisymmetric form defined by $B_f(x,y) = < f, [x,y] > (x, y \in \mathcal{G})$.

PROPOSITION 1. *The Lie algebra $\mathcal{G}_f$ coincides with Ker B_f .*

PROOF. Let $ad^* : \mathcal{G} \longrightarrow \mathrm{End}(\mathcal{G}^*)$ be the differential of the K-representation : $Ad^* : G \longrightarrow \mathrm{Aut}(\mathcal{G}^*)$. It is easy to see that $x \in \mathcal{G}_f \Longleftrightarrow ad^*_x(f) = 0 \Longleftrightarrow < ad^*_x(f), y >= 0, \forall y \in \mathcal{G} \Longleftrightarrow < f, [x,y] >= B_f(x,y) = 0, \forall y \in \mathcal{G} \Longleftrightarrow x \in \mathrm{Ker} B_f$.

This means that $\mathcal{G}_f = \mathrm{Ker} B_f$.

COROLLARY 1. *$\dim \Omega_f$ is even.*

PROOF. We obviously have

$\dim \Omega_f = \dim G/G_f = \dim G - \dim G_f =$

$= \dim \mathcal{G} - \dim \mathcal{G}_f = = \dim \mathcal{G} - \dim \mathrm{Ker} B_f = rank B_f$.

Since the form B_f is antisymmetric, rank B_f is even and so is $\dim \Omega_f$.

PROPOSITION 2. *$\dim \Omega_f$ is positive if and only if $f|\mathcal{G}^1 \neq 0$.*

PROOF. It is easy to see that $\dim \Omega_f = 0 \Longleftrightarrow \mathrm{Ker} B_f = \mathcal{G} \Longleftrightarrow f|\mathcal{G}^1 = 0$.

DEFINITION 2. *A Lie algebra $\mathcal{G}$ is called irreducible if it can not be decomposed into the direct sum of two non-trivial ideals.*

REMARK. It should be noted that irreducible Lie algebras in this sense have been first defined in [8] and are later called "indecomposable" by L.A.Vu in [7].

It is clear that every commutative Lie algebra belongs to the class MD. We shall denote the n-dimensional commutative Lie algebra by R^n .

THEOREM 1. *Let $\mathcal{G}$ be a Lie algebra of the class MD. Then $\mathcal{G}$ is decomposed into a direct sum : $\mathcal{G} \cong R^n \oplus \tilde{\mathcal{G}}$, where $\dim \mathcal{G} \geq n \geq 0$ and $\tilde{\mathcal{G}}$ is an irreducible ideal of $\mathcal{G}$.*

PROOF. Suppose that $\mathcal{G} = \mathcal{G}_1 \oplus \mathcal{G}_2$, where $\mathcal{G}_1$ and $\mathcal{G}_2$ are both non-trivial and non-commutative ideals of $\mathcal{G}$. Then $\mathcal{G}_1 = \mathcal{G}_1^1 \oplus \mathcal{G}_2^1$. Let f_1 and f_2 be the linear forms of $\mathcal{G}$ such that

$$f_1|\mathcal{G}_1^1 \neq 0, \quad f_2|\mathcal{G}_2^1 \neq 0,$$

$$f_1|\mathcal{G}_2 = 0, \quad f_2|\mathcal{G}_1 = 0.$$

We shall denote the restriction of the form B_f on $\mathcal{G}_i$ by $B^i_{f_i}(i = 1,2)$. It is easy to verify that

$$\mathrm{Ker}B_{f_1} = \mathrm{Ker}B^1_{f_1} \oplus \mathcal{G}_2,$$

$$\mathrm{Ker}B_{(f_1+f_2)} = \mathrm{Ker}B^1_{f_1} \oplus \mathrm{Ker}B^2_{f_2}.$$

From the properties of f_1 and f_2 we obtain

$$\mathrm{Ker}B^1_{f_1} \neq \mathcal{G}_1 \text{ and } \mathrm{Ker}B^2_{f_2} \neq \mathcal{G}_2.$$

Hence $\mathcal{G} \not\supseteq \mathrm{Ker}B_{f_1} \not\supseteq \mathrm{Ker}B_{(f_1+f_2)}$ and $\dim \Omega_{(f_1+f_2)} > \dim \Omega_{f_1} > 0$. But this is impossible since $\mathcal{G}$ is a MD's Lie algebra .

Let us now consider a Lie algebra $\mathcal{G}$ and the Lie factor algebra $\tilde{\mathcal{G}} = \mathcal{G}/\mathcal{G}^2$. Suppose $\pi : \mathcal{G} \longrightarrow \tilde{\mathcal{G}}$ is the natural homomorphism and $\pi^* : \tilde{\mathcal{G}}^* \longrightarrow \mathcal{G}^*$ is the induced homomorphism defined by

$$< \pi^*(\tilde{f}), x > = < \tilde{f}, \pi(x) > \quad (\tilde{f} \in \tilde{\mathcal{G}}^*, x \in \mathcal{G}).$$

It is clear that π^* is a monomorphism from $\tilde{\mathcal{G}}^*$ into $\mathcal{G}^*$ and $\mathcal{G}^{2*}$ can be identified with a subspace of $\mathcal{G}^*$.

PROPOSITION 3. *The space $\mathcal{G}^*$ is the direct sum of $\pi^*(\tilde{\mathcal{G}}^*)$ and $\mathcal{G}^{2*}$*

PROOF. The equality $\dim \mathcal{G}^* = \dim \pi^*(\tilde{\mathcal{G}}^*) + \dim \mathcal{G}^{2*}$ is obvious. Thus it remains to prove $\pi^*(\tilde{\mathcal{G}}^*) \cap \mathcal{G}^{2*} = (0)$. Assume that f is an element of $\pi^*(\tilde{\mathcal{G}}^*) \cap \mathcal{G}^{2*}$ and $f = \pi^*(\tilde{f})$ for some element $\tilde{f}$ of $\tilde{\mathcal{G}}^*$. It is obvious that $\pi(\mathrm{Ker} f) = \tilde{\mathcal{G}}$, and if

$x \in \mathrm{Ker}f$, then $< \tilde{f}, \pi(x) >=< f, x >= 0$. Hence $\pi(\mathrm{Ker}f) \subset \mathrm{Ker}\tilde{f}$ and therefore $f = 0$.

PROPOSITION 4. *Let $\tilde{f}$ be an element of $\tilde{\mathcal{G}}^*$ and $f = \pi^*(\tilde{f})$ its image in $\mathcal{G}^*$. If $\tilde{f}|_{\tilde{\mathcal{G}}_1} \neq 0$, then $f|^1_{\mathcal{G}} \neq 0$.*

PROOF. This statement is obvious because $\pi(\mathcal{G}^1) = \tilde{\mathcal{G}}^1$.

Let $\tilde{\Omega}_{\tilde{f}}$ denote the K-orbit in $\tilde{\mathcal{G}}^*$ containing the element $\tilde{f}$ and let $\tilde{B}_{\tilde{f}}$ be the bilinear form corresponding to $\tilde{f}$ in $\tilde{\mathcal{G}}$.

THEOREM 2. *For every element $\tilde{f}$ of $\tilde{\mathcal{G}}^*$ we have*

$$\dim \tilde{\Omega}_{\tilde{f}} = \dim \Omega_{\pi^*}(\tilde{f}).$$

PROOF. Let x and y be two arbitrary elements of $\mathcal{G}$. Then

$$B_{\pi^*(\tilde{f})}(x, y) =< \pi^*(\tilde{f}), [x, y] >=< \tilde{f}, [\pi(x), \pi(y)] > .$$

Therefore,

$$\mathrm{Ker}B_{\pi^*(\tilde{f})} = \pi^{-1}(\mathrm{Ker}\tilde{B}_{\tilde{f}}),$$

$$\mathrm{Ker}\tilde{B}_{\tilde{f}} \cong \mathrm{Ker}B_{\pi^*(\tilde{f})}/\mathcal{G}^2.$$

Now the equality $\dim \tilde{\Omega}_{\tilde{f}} = \dim \Omega_{\pi^*(\tilde{f})}$ is obvious.

COROLLARY 2. *If $\mathcal{G}$ is a MD's Lie algebra , then so is $\tilde{\mathcal{G}}$. Moreover, the maximal dimension of the K-orbits in $\mathcal{G}^*$ and the maximal dimension of the K-orbits in $\tilde{\mathcal{G}}^*$ are equal to each other.*

PROOF. This corollary is obvious by Propositions 2, 4 and Theorem 2.

COROLLARY 3. *Suppose that $\mathcal{G}$ is a $\overline{MD}'s$ Lie algebra . Then $\mathcal{G}^2 = (0)$.*

PROOF. This corollary is a trivial consequence of Theorem 2 and the formula $\dim \tilde{\mathcal{G}} + \dim \mathcal{G}^2 = \dim \mathcal{G}$.

REMARK. This result has been proved by H.H. Viet in [4] (Proposition 1) in another way.

In the sequel we shall denote by ad^1_v the restriction of ad_v on $\mathcal{G}^1$ for each element V of $\mathcal{G}$. If $\{V, U, ...\}$ is a basis of $\mathcal{G}$, then $\{V^*, U^*, ...\}$ will denote its dual basis in $\mathcal{G}^*$.

3. Classification of 3-dimensional Lie algebras of the class MD.

First notice that from Corollary 1 it follows that every Lie algebra $\mathcal{G}$ whose dimension is not greater than 3 always satisfies the property MD. By Theorem 1 we can restrict our consideration on irreducible Lie algebras . It is clear that R^1 and $Aff(R)$ are the unique MD's Lie algebras with dimensions 1 and 2, respectively. Thus, it remains to classify irreducible 3-dimensional resolvable Lie algebras .

THEOREM 3. *Let $\mathcal{G}$ be an irreducible 3-dimensional Lie algebra , $\{X, Y, Z\}$ a basis of $\mathcal{G}$. Then $\mathcal{G}$ belongs to the class MD if and only if $\mathcal{G}$ is isomorphic to one of the following Lie algebras :*

1) $\dim \mathcal{G}^1 = 1, \mathcal{G}^1 = gen(Z)$ *with*

$$\mathcal{G}_{3.1} : [X, Y] = Z, [X, Z] = 0, [Y, Z] = 0$$

(This is the 3-dimensional Heisenberg Lie algebra usually denoted by $\mathcal{H}_3$).

2) $\dim \mathcal{G}^1 = 2, \mathcal{G}^1 = gen(X, Y) \cong R^2$ *with*

2.1) $\mathcal{G}_{3.2.1} : ad^1_X = \begin{Vmatrix} \lambda & 0 \\ 0 & 1 \end{Vmatrix}, \quad \lambda \neq 0,$

2.2) $\mathcal{G}_{3.2.2} : ad^1_X = \begin{Vmatrix} 1 & 1 \\ 0 & 1 \end{Vmatrix},$

2.3) $\mathcal{G}_{3.2.3} : ad^1_X = \begin{Vmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{Vmatrix}, \quad \varphi \in (0, \pi).$

PROOF. It is obvious that every Lie algebra written above is resolvable and irreducible. Moreover, they are not isomorphic to each other.

Now let $\mathcal{G}$ be an irreducible resolvable 3-dimensional Lie algebra . Since $\mathcal{G}^1$ is a nilpotent Lie algebra and $\dim \mathcal{G}^1 < \dim \mathcal{G} = 3, \mathcal{G}^1$ is commutative. Let us consider the following cases:

1) $\dim \mathcal{G}^1 = 1$. Suppose that $\mathcal{G}^1 = gen(Z)$ and $[X, Y] = \alpha Z$; $[Y, Z] = \beta Z; [Z, X] = \gamma Z$, where α, β, γ are not simultaneously equal to zero.

If $\beta \neq 0$, we put

$$\tilde{X} = X + \frac{\gamma}{\beta} Y + \frac{\alpha}{\beta} Z, \tilde{Y} = \frac{1}{\beta} Y, \tilde{Z} = Z.$$

One easily verifies that $[\tilde{X},\tilde{Y}] = [\tilde{X},\tilde{Z}] = 0$ and $[\tilde{Y},\tilde{Z}] = \tilde{Z}$. Therefore, $\mathcal{G} = gen(\tilde{X}) \oplus gen(\tilde{Y},\tilde{Z})$ and $\mathcal{G}$ is not irreducible.

By analogous arguments we prove that if $\gamma \neq 0$, then $\mathcal{G}$ is not irreducible. Thus β and γ must be both equal to 0, but α is not. Now put $\tilde{X} = X, \tilde{Y} = Y, \tilde{Z} = \alpha Z$. Then we have $[\tilde{X},\tilde{Y}] = \tilde{Z}, [\tilde{X},\tilde{Z}] = [\tilde{Y},\tilde{Z}] = 0$. Therefore, $\mathcal{G} \cong \mathcal{G}_{3.1} = \mathcal{H}_3$.

2) $\dim \mathcal{G}^1 = 2$. Suppose that $\mathcal{G}^1 = gen(Y,Z) \cong R^2$. Then $ad^1_X : \mathcal{G}^1 \longrightarrow \mathcal{G}^1$ is an isomorphism. Without loss of generality we may assume that in the basis $\{Y, Z\}$ the transformation ad^1_X has one of the following real Jordan matrices:

2.1) $\begin{Vmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{Vmatrix}, \quad \lambda_1\lambda_2 \neq 0,$

2.2) $\begin{Vmatrix} \lambda & 1 \\ 0 & \lambda \end{Vmatrix}, \quad \lambda \neq 0,$

2.3) $\begin{Vmatrix} a & b \\ -b & a \end{Vmatrix}, \quad b > 0.$

Putting

$$\tilde{X} = \frac{X}{\lambda_2}, \tilde{Y} = Y, \tilde{Z} = Z, \lambda = \frac{\lambda_1}{\lambda_2}$$

in case 2.1) we have $\mathcal{G} \cong \mathcal{G}_{3.2.1}$. In case 2.2) we make the following changes

$$\tilde{X} = \frac{X}{\lambda}, \tilde{Y} = \frac{Y}{\lambda}, \tilde{Z} = Z.$$

It is obvious that $\mathcal{G} \cong \mathcal{G}_{3.2.2}$. Finally, in case 2.3) we put

$$\tilde{X} = \frac{X}{\sqrt{a^2+b^2}}, \tilde{Y} = Y, \tilde{Z} = Z,$$

$$\cos\varphi = \frac{a}{\sqrt{a^2+b^2}}, \sin\varphi = \frac{b}{\sqrt{a^2+b^2}} (\varphi \in (0.\pi)).$$

Then we obtain $\mathcal{G} \cong \mathcal{G}_{3.2.3}$.

4. Classification of 4-dimensional Lie Algrbras of the class MD

In this section we shall prove the following result.

Theorem 4. *Let $\mathcal{G}$ be an irreducible 4-dimensional Lie algebra and $\{T, X, Y, Z\}$ a basis of $\mathcal{G}$. Then $\mathcal{G}$ belongs to the class MD if and only if it is isomorphic to one of the following Lie algebras:*

1/ dim $\mathcal{G}^1 = 2, \mathcal{G}^1 = gen(Y, Z) = R^2$ *with*

1.1) $\mathcal{G}_{4.2.1} : [T, X] = 0, \quad ad^1_T = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}, \quad ad^1_X = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}.$

(This is the complex affine Lie algebra usually denoted by $Aff(C)$).

1.2) $\mathcal{G}_{4.2.2} : [T, X] = Y, ad^1_X = 0, \quad ad^1_T = \begin{Vmatrix} 0 & 0 \\ 0 & 1 \end{Vmatrix}.$

1.3) $\mathcal{G}_{4.2.3} : [T, X] = Z, ad^1_X = 0, \quad ad^1_T = \begin{Vmatrix} 0 & 1 \\ 0 & 0 \end{Vmatrix}.$

2/ dim $\mathcal{G}^1 = 3, \mathcal{G}^1 = gen(X, Y, Z) = R^3$ *with*

$$2.1)\ \mathcal{G}_{4.3.1} : \quad ad^1_T = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{Vmatrix}, \quad \lambda_1\lambda_2 \neq 0,$$

$$2.2)\ \mathcal{G}_{4.3.2} : \quad ad^1_T = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{Vmatrix}, \quad \lambda \neq 0,$$

$$2.3)\ \mathcal{G}_{4.3.3} : \quad ad^1_T = \begin{Vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{Vmatrix},$$

$$2.4)\ \mathcal{G}_{4.3.4} : \quad ad^1_T = \begin{Vmatrix} \lambda & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{Vmatrix}, \quad \lambda \neq 0, \varphi \in (0, \pi).$$

3) dim $\mathcal{G}^1 = 3, \mathcal{G}^1 = gen(X, Y, Z) \cong \mathcal{H}_3$ *with*

$$3.1)\ \mathcal{G}_{4.3.5} : \quad ad^1_T = \begin{Vmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{Vmatrix}.$$

(This is a real diamond Lie algebra).

$$3.2)\ \mathcal{G}_{4.3.6} : \quad ad^1_T = \begin{Vmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix}.$$

Proof. One immediately checks that all the above Lie algebras are irreducible and belong to the class MD. Moreover, they are not isomorphic to each other.

Now let $\mathcal{G}$ be an irreducible 4-dimensional MD's Lie algebra. Then $\mathcal{G}^1$ is a nilpotent Lie-algebra with $dim\mathcal{G}^1 < 4$.

We shall distinguish the following cases according to the values of $dim\mathcal{G}^1$.

Suppose that we have $dim\mathcal{G}^1 = 1$; $\mathcal{G}^1 = gen(Z)$ and

$$[T, Z] = \tau Z, \qquad [X, Z] = \kappa Z, \qquad [Y, Z] = \gamma Z,$$
$$[T, X] = aZ, \qquad [X, Y] = bZ, \qquad [Y, T] = cZ,$$

where $\tau, \kappa, \gamma, a, b, c$ do not simultaneously vanish.

If $\tau \neq \emptyset$, we make the following transformation:

$$\tilde{X} = X - \frac{\kappa}{\tau}T - \frac{a}{\tau}Z, \qquad \tilde{T} = \frac{1}{\tau}T,$$
$$\tilde{Y} = Y - \frac{\gamma}{\tau}T + \frac{c}{\tau}Z, \qquad \tilde{Z} = Z.$$

It is clear that this transformation is regular and we have

$$[\tilde{T}, \tilde{Z}] = \tilde{Z}, \ [\tilde{X}, \tilde{Z}] = [\tilde{Y}, \tilde{Z}] = [\tilde{T}, \tilde{X}] = [\tilde{X}, \tilde{Y}] = [\tilde{Y}, \tilde{T}] = 0.$$

Therefore, $\mathcal{G} = gen(\tilde{T}, \tilde{Z}) \oplus gen(\tilde{X}, \tilde{Y}) \cong R^2 \oplus Aff(R)$. Since $\mathcal{G}$ is irreducible, this case is excluded.

In a similar manner we can prove that the case $\kappa \neq \emptyset$ (or $\gamma \neq 0$) is excluded, too. Let us consider the case $\tau = \gamma = 0$ and $a \neq 0$. Put

$$\tilde{T} = \frac{1}{a}T, \tilde{X} = X, \tilde{Y} = Y + \frac{b}{a}T\frac{c}{a}X, \tilde{Z} = Z.$$

Then $\{\tilde{T}, \tilde{X}, \tilde{Y}, \tilde{Z}\}$ is a basis of $\mathcal{G}$ which satisfies the following equalities:

$$[\tilde{T}, \tilde{X}] = \tilde{Z}, [\tilde{T}, \tilde{Y}] = [\tilde{T}, \tilde{Z}] = [\tilde{X}, \tilde{Y}] = [\tilde{X}, \tilde{Z}] = [\tilde{Y}, \tilde{Z}] = 0.$$

This means

$$\mathcal{G} = gen(\tilde{T}, \tilde{X}, \tilde{Z}) \oplus gen(\tilde{Y}) \approx R^1 + \mathcal{H}_3.$$

So this case is excluded.

By similar arguments one can show that the case $\tau = \kappa = \gamma = 0$ and $b \neq 0$ (or $\tau = \kappa = \gamma = 0$ and $c \neq 0$) is excluded, too. Thus there does not exist any irreducible 4-dimensional Lie algebra $\mathcal{G}$ with $\dim \mathcal{G}^1 = 1$.

REMARK. The above transformations of basis have been used by L.A. Vu in [7]. But he wrongly claimed that these Lie algebras are irreducible.

Let us pass to the cases where $dim\mathcal{G}^1 = 2$ or 3.

1) Suppose that $dim\mathcal{G}^1 = 2$ and $\mathcal{G}^1 = gen(Y, Z) \cong R^2$. Since $ad^1_{[T,X]} = [ad^1_T, ad^1_X]$ and $[T, X] \in \mathcal{G}^1$, the transformations ad^1_T and ad^1_X commute between themselves.

LEMMA 1. *There exists in $\mathcal{G}^1$ a basis for which the matrices of ad^1_T and ad^1_X have one of the following forms: a) $ad^1_T = \begin{Vmatrix} a & b \\ -b & a \end{Vmatrix}, ad^1_X = \begin{Vmatrix} c & d \\ -d & c \end{Vmatrix}, (b^2 + d^2 \neq 0)$.*

b) $ad^1_T = \begin{Vmatrix} \lambda_1 & \nu \\ 0 & \lambda_2 \end{Vmatrix}, ad^1_X = \begin{Vmatrix} \mu_1 & \gamma \\ 0 & \mu_2 \end{Vmatrix}$.

PROOF. If a^1_T has a complex eigenvalue $a + b\sqrt{-1}, (b > 0)$ or ad^1_X has a complex eigenvalue $c + d\sqrt{-1}, (d > 0)$, then there exists in $\mathcal{G}^1$ a basis for which the matrices of ad^1_T and ad^1_X have the forms in a).

If ad^1_T and ad^1_X only have real eigenvalues, then since $ad^1_T ad^1_X = ad^1_X ad^1_T$, they are simultaneously triangulated. Therefore, there exists in $\mathcal{G}^1$ a basis for which the matrices of ad^1_T and ad^1_X have the matrices as in b).

1.1) Assume that for the chosen basis $\{Y, Z\}$ of $\mathcal{G}^1$ we have

$$ad^1_T = \begin{Vmatrix} a & b \\ -b & a \end{Vmatrix}, ad^1_X = \begin{Vmatrix} c & d \\ -d & c \end{Vmatrix}, (b^2 + d^2 \neq 0).$$

Without loss of generality we may assume $b > 0$. Since the transformation ad^1_T is regular, there exists two coefficients a_1, a_2 such that

$$[T, X] = a_1 ad^1_T(Y) + a_2 ad^1_T(Z) = [T, a_1 Y, a_2 Z].$$

Therefore, $[T, X - a_1 Y - a_2 Z] = 0$. Changing X by $(X - a_1 Y - a_2 Z - \frac{d}{b}T)$, we may assume that

$$ad^1_X = \begin{Vmatrix} c & 0 \\ 0 & c \end{Vmatrix}, \text{ and } [T, X] = 0.$$

Here $c \neq 0$. In fact, if $c = 0$, then $[X, \mathcal{G}] = (0)$ and so we have

$$\mathcal{G} = gen(X) \oplus gen(T, Y, Z).$$

This contradicts the irreducibility of $\mathcal{G}$. Let us now make the following transformation

$$\tilde{T} = \frac{1}{b}(T - \frac{a}{c}X), \tilde{X} = \frac{1}{c}X, \tilde{Y} = Y, \tilde{Z} = Z.$$

This transformation is obviously regular and for the basis $\{\tilde{T}, \tilde{X}, \tilde{Y}, \tilde{Z}\}$ we have

$$[\tilde{T}, \tilde{X}] = 0, ad^1_{\tilde{T}} = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}, ad^1_{\tilde{X}} = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}.$$

It follows that $\mathcal{G} \cong \mathcal{G}_{4.2.1} = Aff(c)$.

1.2) Suppose now that in the chosen basis $\{Y, Z\}$ of $\mathcal{G}^1$ we have

$$ad^1_T = \begin{Vmatrix} \lambda_1 & \nu \\ 0 & \lambda_2 \end{Vmatrix}, ad^1_X = \begin{Vmatrix} \mu_1 & \gamma \\ 0 & \mu_2 \end{Vmatrix}.$$

Let $f = yY^* + zZ^*$ be a linear form of $\mathcal{G}^{1*}$ (by Proposition 2 it is sufficient to consider such forms). Let $\varphi(y, z)$ be the matrix of the form B_f for the basis $\{T, X, Y, Z\}$. Then we have

$$\varphi(y, z) = \begin{Vmatrix} 0 & \langle f, [T, X] \rangle & \lambda_1 y & (\nu y + \lambda_2 z) \\ \langle f, [X, T] \rangle & 0 & \mu_1 y & (\gamma y + \mu_2 z) \\ -\lambda_1 y & -\mu_1 y & 0 & 0 \\ -(\nu y + \lambda_2 z) & (-\gamma y + \mu_2 z) & 0 & 0 \end{Vmatrix}.$$

Since $rank\ \varphi(0, z) < 4$ anf $\mathcal{G}$ is a MD's Lie algebra, $rank\ \varphi(y, z) < 4, \forall (y, z)$. From this it follows that $\lambda_1 \gamma = \nu \mu_1$ and $\lambda_1 \mu_2 = \lambda_2 \mu_1$. That means ad^1_T and ad^1_X must be proportional. Moreover, $[T, X], [T, \mathcal{G}^1]$ and $[X, \mathcal{G}^1]$ generate $\mathcal{G}^1$. Hence one of ad^1_T and ad^1_X must be different from zero. Assume that $ad^1_T \neq 0$ and $ad^1_X = t ad^1_T (t \in R)$. Changing X by $(X - tT)$ we may assume $ad^1_X = 0$. Let us now consider the following cases:

Case 1. $\lambda_1 \neq 0, \lambda_2 \neq 0$. In this case ad^1_T is regular and there exist two coefficients a_1, a_2 such that

$$[T, X] = a_1 ad^1_T(Y) + a_2 ad^1_T(Z),$$

hence $[T, X - a_1 Y - a_2 Z] = 0$. Since $ad^1_{(X - a_1 Y - a_2 Z)} = 0$, we have

$$\mathcal{G} = gen(X - a_1 Y - a_2 Z) \oplus gen(T, Y, Z).$$

So $\mathcal{G}$ is not irreducible and the case is excluded.

Case 2. $\lambda_1 = 0, \lambda_2 \neq 0$. In this case, ad^1_T is diagonal. Without loss of generality we may assume $\{Y, Z\}$ to be a proper basis of ad^1_T, and so we have

$$ad^1_T = \begin{Vmatrix} 0 & 0 \\ 0 & \lambda_2 \end{Vmatrix}, ad^1_X = 0.$$

If $[T, X] = a_1 Y + a_2 Z$, then $a_1 \neq 0$. Make the following transformation

$$\tilde{T} = \frac{T}{\lambda_2}, \tilde{X} = X - \frac{a_2}{\lambda_2} Z, \tilde{Y} = \frac{a_1}{\lambda_2} Y, \tilde{Z} = Z.$$

It is obvious that $\{\tilde{T}, \tilde{X}, \tilde{Y}, \tilde{Z}\}$ is a basis of $\mathcal{G}$ and for this basis we have

$$[\tilde{T}, \tilde{X}] = \tilde{Y}, ad^1_{\tilde{T}} = \begin{Vmatrix} 0 & 0 \\ 0 & 1 \end{Vmatrix}, ad^1_{\tilde{X}} = 0.$$

That means $\mathcal{G} \cong \mathcal{G}_{4.2.2}$.

Case 3. $\lambda_1 = \lambda_2 = 0$. Since $ad^1_T \neq 0$, we may assume $\nu = 1$ and so

$$ad^1_T = \begin{Vmatrix} 0 & 1 \\ 0 & 0 \end{Vmatrix}, ad^1_X = 0.$$

If $[T, X] = a_1 Y + a_2 Z$, then $a_2 \neq 0$. Now let us make the following changes

$$\tilde{T} = T, \tilde{X} = X - a_1 Z, \tilde{Y} = a_2 Y, \tilde{Z} = a_2 Z.$$

This is obviously a new basis of $\mathcal{G}$ which satisfies the equalities

$$[\tilde{T}, \tilde{X}] = \tilde{Z}, ad^1_{\tilde{T}} = \begin{Vmatrix} 0 & 1 \\ 0 & 0 \end{Vmatrix}, ad^1_{\tilde{X}} = 0.$$

This means $\mathcal{G} \cong \mathcal{G}_{4.2.3}$.

REMARK. In [7] Vu has missed these two important cases.

2) Suppose that $dim\ \mathcal{G}^1 = 3$ and $\mathcal{G}^1 = gen(X, Y, Z) \approx R^3$. Then ad^1_T is a regular transformation of $\mathcal{G}^1$. It is easy to show that every 4-dimensional Lie algebra with $\mathcal{G}^1 \approx R^3$ is irreducible and belongs to the class MD. They can be classified by the real normal Jordan matrices of ad^1_T.

Let us now consider the following possibilities

2.1) Assume that for the chosen basis $\{X, Y, Z\}$ of $\mathcal{G}^1$ we have

$$ad_T^1 = \begin{Vmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{Vmatrix}, \lambda_1\lambda_2\lambda_3 \neq 0.$$

By putting $\tilde{T} = \frac{1}{\lambda_1}T, \tilde{X} = X, \tilde{Y} = Y, \tilde{Z} = Z, \tilde{\lambda_1} = \frac{\lambda_2}{\lambda_1}, \tilde{\lambda_2} = \frac{\lambda_3}{\lambda_1}$, we obtain $\mathcal{G} \cong \mathcal{G}_{4.3.1}$.

2.2) Suppose now that for the basis $\{X, Y, Z\}$ of $\mathcal{G}^1$ we have

$$ad_T^1 = \begin{Vmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{Vmatrix}; \lambda_1\lambda_2 \neq 0.$$

By changing $\tilde{T} = \frac{1}{\lambda_1}T, \tilde{X} = X, \tilde{Y} = \frac{Y}{\lambda_1}, \tilde{Z} = Z, \tilde{\lambda} = \frac{\lambda_2}{\lambda_1}$, we get $\mathcal{G} \cong \mathcal{G}_{4.3.2}$.

2.3) If for the basis $\{X, Y, Z\}$ the matrix of ad_T^1 has the form

$$ad_T^1 = \begin{Vmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{Vmatrix}, \lambda \neq 0,$$

then we can use the following transformation

$$\tilde{T} = \frac{T}{\lambda}, \tilde{X} = \frac{X}{\lambda^2}; \tilde{Y} = \frac{Y}{\lambda}, \tilde{Z} = Z.$$

Now, it is obvious that $\mathcal{G} \cong \mathcal{G}_{4.3.3}$.

2.4) Finally, assume that

$$ad_T^1 = \begin{Vmatrix} \lambda & 0 & 0 \\ 0 & a & b \\ 0 & -b & a \end{Vmatrix}, \lambda \neq 0, b > 0$$

for the chosen basis $\{X, Y, Z\}$. Put

$$\tilde{T} = \frac{T}{\sqrt{a^2 + b^2}}, \tilde{X} = X, \tilde{Y} = Y, \tilde{Z} = Z,$$

$$\cos\varphi = \frac{a}{\sqrt{a^2 + b^2}}, \sin\varphi = \frac{b}{\sqrt{a^2 + b^2}}, \tilde{\lambda} = \frac{\lambda}{\sqrt{a^2 + b^2}}.$$

Since $b > 0$, we have $\varphi \in (0, \pi)$. Therefore $\mathcal{G} \cong \mathcal{G}_{4.3.4}$.

Let us pass to the last case.

3) $dim\ \mathcal{G}^1 = 3$ and $\mathcal{G}^1 = gen(X, Y, Z)$ is not commutative. Since $\mathcal{G}^1$ is a nilpotent 3-dimensional Lie algebra, from Theorem 3 we get $\mathcal{G}^1 \approx \mathcal{H}_3$ (Heisenberg Lie algebra). Suppose that

$$[X, Y] = Z, [X, Z] = [Y, Z] = 0.$$

Since $\mathcal{G}^2 = gen(Z)$ is an ideal of $\mathcal{G}$, the matrix of ad^1_T for the basis $\{X, Y, Z\}$ has the following form

$$ad^1_T = \begin{Vmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{Vmatrix}.$$

From Jacobi identities written for the triplets $(T, X, Y), (T, X, Z), (T, Y, Z)$ we obtain $a_{33} = a_{11} + a_{22}$.

Suppose that $f = xX^* + yY^* + zZ^*$ is an arbitrary linear form of $\mathcal{G}^{1*}$ and $\varphi(x, y, z)$ is the matrix of the form B_f in the basis $\{T, X, Y, Z\}$. Then

$$\varphi(x, y, z) = \begin{Vmatrix} 0 & \langle f, [T, X]\rangle & \langle f, [T, Y]\rangle & a_{33}z \\ \langle f, [X, T]\rangle & 0 & z & 0 \\ \langle f, [Y, T]\rangle & -z & 0 & 0 \\ -a_{33}z & 0 & 0 & 0 \end{Vmatrix}.$$

Since $rank\ \varphi(x, y, 0) < 4$ and $\mathcal{G}$ is a MD's Lie algebra, $rank\ \varphi(x, y, z) < 4$, $\forall (x, y, z)$. Therefore $a_{33} = 0$ and

$$ad^1_T = \begin{Vmatrix} a_{11} & a_{12} & 0 \\ a_{21} & -a_{11} & 0 \\ a_{31} & a_{32} & 0 \end{Vmatrix}.$$

Changing T by $\tilde{T} = T - a_{32}X + a_{31}Y$ we get

$$ad^1_{\tilde{T}} = \begin{Vmatrix} a_{11} & a_{12} & 0 \\ a_{21} & -a_{11} & 0 \\ 0 & 0 & 0 \end{Vmatrix}.$$

On the other hand, $\mathcal{G}^2 + [T, \mathcal{G}^1] = \mathcal{G}^1$ implies $a_{11}^2 + a_{21}a_{12} \neq 0$.

Let $L = L(X,Y)$ be the subspace of $\mathcal{G}^1$ generated by X and Y. Then L is an invariant subspace of $ad^1_{\tilde{T}}$ and the matrix of $ad^1_{\tilde{T}}\Big|_L$ for the basis $\{X,Y\}$ is

$$ad^1_{\tilde{T}}\Big|_L = \begin{Vmatrix} a_{11} & a_{12} \\ a_{21} & -a_{11} \end{Vmatrix}.$$

This matrix has two eigenvalues

$$\lambda_{1,2} = \pm\sqrt{a_{11}^2 + a_{12}a_{21}}.$$

There may be two possibilities:

3.1) $a_{11}^2 + a_{12}a_{21} = \partial^2 > 0$. In this case, $ad^1_{\tilde{T}}$ is diagonal. Then there exists a basis $\{\tilde{X},\tilde{Y}\}$ of L such that

$$[\tilde{T},\tilde{X}] = -\delta\tilde{X}, [\tilde{T},\tilde{Y}] = \delta\tilde{Y}.$$

Now put

$$T' = \frac{1}{\delta}\tilde{T}, X' = \tilde{X}, Y' = \tilde{Y}, Z' = [\tilde{X},\tilde{Y}].$$

Since $[L,L] = \mathcal{G}^2$, Z' does not vanish. This means that $\{T',X',Y',Z'\}$ is a basis of $\mathcal{G}$ and we obtain

$$[X',Y'] = Z', [X',Z'] = [Y',Z'] = 0,$$

$$ad^1_{T'} = \begin{Vmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{Vmatrix}.$$

Therefore $\mathcal{G} \cong \mathcal{G}_{4.3.5}$.

3.2)$a_{11}^2 + a_{11}a_{21} = -\delta^2 < 0$. In this case, the eigenvalues $\lambda_{1,2} = \pm\delta\sqrt{-1}$ are purely imaginary. Then there exists a basis $\{\tilde{X},\tilde{Y}\}$ of L such that $[\tilde{T},\tilde{X}] = -\delta\tilde{T}, [\tilde{T},\tilde{Y}] = \delta\tilde{X}$. Put

$$T' = \frac{1}{\delta}\tilde{T}, X' = \tilde{X}, Y' = \tilde{Y}, Z' = [\tilde{X},\tilde{Y}].$$

It is clear that $\{T',X',Y',Z'\}$ is a basis of $\mathcal{G}$ and we have

$$[X',Y'] = Z', [X',Z'] = [Y',Z'] = 0,$$

$$ad_{T'} = \begin{Vmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix}.$$

Thus $\mathcal{G} \cong \mathcal{G}_{4.3.6}$. The proof of Theorem 4 is now completed.

REMARK. In [7] Vu has given explicit formulas of $\tilde{X}$ and $\tilde{Y}$.

5. Classification of 5-dimensional Lie Algebras of the class MD

In this section we shall use the method developed in Section 4 to prove the following theorem

THEOREM 5. *Suppose that $\mathcal{G}$ is an irreducible 5-dimensional Lie algebra and $\{D,T,X,Y,Z\}$ is a basis of $\mathcal{G}$. Then $\mathcal{G}$ belongs to the class MD if and only if $\mathcal{G}$ is isomorphic to one of the following Lie algebras:*

1/ $dim \mathcal{G}^1 = 1, \mathcal{G}^1 = gen(Z)$ with

$$\mathcal{G}_{5.1}: \quad \begin{aligned} &[D,T] = Z, [X,Y] = Z, \\ &[D,Z] = [T,Z] = [X,Z] = [Y,Z] = 0, \\ &[T,X] = [T,Y] = [D,X] = [D,Y] = 0. \end{aligned}$$

(This is the 5-dimensional Heisenberg Lie Algebra usually denoted by $\mathcal{H}_5$).

2/ $dim\ \mathcal{G}^1 = 2, \mathcal{G}^1 = gen(Y,Z) \approx R^2$ with

$$\mathcal{G}_{5.2}: \quad \begin{aligned} &[D,T] = Y, [D,X] = 0, [X,T] = Z, \\ &ad^1_D = ad^1_T = ad^1_X = 0. \end{aligned}$$

3/ $dim\ \mathcal{G}^1 = 3, \mathcal{G}^1 = gen(X,Y,Z) \approx R^3$ with

$$[D,T] = Z, ad^1_T = 0,$$

$$\mathcal{G}_{5.3.1}: \quad ad^1_D = \begin{Vmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 \end{Vmatrix}, \varphi \in (0,\pi),$$

$$[D,T] = Z, ad^1_T = 0,$$

$$\mathcal{G}_{5.3.2}: \quad ad_D^1 = \begin{Vmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{Vmatrix}, (\lambda \neq 0),$$

$$[D,T] = Y, ad_T^1 = 0,$$

$$\mathcal{G}_{5.3.3}: \quad ad_D^1 = \begin{Vmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda \end{Vmatrix}, (\lambda \neq 0),$$

$$[D,T] = Z, ad_T^1 = 0,$$

$$\mathcal{G}_{5.3.4}: \quad ad_D^1 = \begin{Vmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{Vmatrix},$$

$$[D,T] = Z,$$

$$\mathcal{G}_{5.3.5}: \quad ad_D^1 = \begin{Vmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix}, ad_T^1 = \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{Vmatrix},$$

$$[D,T] = Z, ad_T^1 = 0,$$

$$\mathcal{G}_{5.3.6}: \quad ad_D^1 = \begin{Vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{Vmatrix}.$$

4/ $dim\ \mathcal{G}^1 = 3, \mathcal{G}^1 = gen(X,Y,Z) \approx \mathcal{H}_3$ *with*

$$\mathcal{G}_{5.3.7}: \quad ad_D^1 = \begin{Vmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix}, ad_T^1 = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{Vmatrix},$$

$$[D,T] = 0.$$

5/ $dim\ \mathcal{G}^1 = 4, \mathcal{G}^1 = gen(T,X,Y,Z) \approx R^4$ *with*

$$\mathcal{G}_{5.4.1}: \quad ad_D^1 = \begin{Vmatrix} \cos\varphi & \sin\varphi & 0 & 0 \\ -\sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\beta & \alpha \end{Vmatrix}, \varphi \in (0,\pi), \beta > 0,$$

$$\mathcal{G}_{5.4.2}: \qquad ad_D^1 = \begin{Vmatrix} \cos\varphi & \sin\varphi & 1 & 0 \\ -\sin\varphi & \cos\varphi & 0 & 1 \\ 0 & 0 & \cos\varphi & \sin\varphi \\ 0 & 0 & -\sin\varphi & \cos\varphi \end{Vmatrix}, \varphi \in (0,\pi),$$

$$\mathcal{G}_{5.4.3}: \qquad ad_D^1 = \begin{Vmatrix} \cos\varphi & \sin\varphi & 0 & 0 \\ -\sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{Vmatrix}, \varphi \in (0,\pi), \lambda_1\lambda_2 \neq 0,$$

$$\mathcal{G}_{5.4.4}: \qquad ad_D^1 = \begin{Vmatrix} \cos\varphi & \sin\varphi & 0 & 0 \\ -\sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{Vmatrix}, \varphi \in (0,\pi), \lambda \neq 0,$$

$$\mathcal{G}_{5.4.5}: \qquad ad_D^1 = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{Vmatrix}, \lambda_1\lambda_2\lambda_3 \neq 0,$$

$$\mathcal{G}_{5.4.6}: \qquad ad_D^1 = \begin{Vmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{Vmatrix}, \lambda_1\lambda_2 \neq 0,$$

$$\mathcal{G}_{5.4.7}: \qquad ad_D^1 = \begin{Vmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \lambda \end{Vmatrix}, \lambda \neq 0,$$

$$\mathcal{G}_{5.4.8}: \qquad ad_D^1 = \begin{Vmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{Vmatrix}, \varphi, \lambda \neq 0,$$

$$\mathcal{G}_{5.4.9}: \qquad ad_D^1 = \begin{Vmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{Vmatrix}.$$

PROOF. One can immediately show that the above Lie algebras are irreducible MD's Lie algebras and they are not isomorphic to each other.

Now let $\mathcal{G}$ be an irreducible 5-dimensional Lie algebra of the class MD. Then $\mathcal{G}^1$ is a nilpotent Lie algebra with $dim\ \mathcal{G}^1 \leq 4$. We will consider the following cases of $\mathcal{G}^1$:

1/ $dim\ \mathcal{G}^1 = 1$. Suppose that $\mathcal{G}^1 = gen(Z)$ and

$$[D, Z] = \delta Z, [T, Z] = \tau Z, [X, Z] = \kappa Z, [Y, Z] = \gamma Z,$$
$$[D, T] = aZ, [T, X] = bZ, [X, Y] = cZ, [Y, D] = dZ,$$
$$[X, D] = eZ, [Y, T] = gZ,$$

where the coefficients $\delta, \tau, \kappa, \ldots, e, g$ are not simultaneously equal to 0.

1.1)If $\delta \neq 0$, then without loss of generality we may assume that $\delta = 1$. Make the following regular transformation:

$$\tilde{T} = T - \tau D - aZ,$$
$$\tilde{X} = X - \kappa D + eZ, \tilde{D} = D,$$
$$\tilde{Y} = Y - \gamma D + dZ, \tilde{Z} = Z.$$

It is easy to verify that

$$[\tilde{D}, \tilde{Z}] = \tilde{Z}, [\tilde{T}, \tilde{Z}] = [\tilde{X}, \tilde{Z}] = [\tilde{Y}, \tilde{Z}] = 0, [\tilde{D}, \tilde{T}] = [\tilde{D}, \tilde{X}] = [\tilde{D}, \tilde{Y}] = 0.$$

Therefore $[\tilde{T}, \tilde{X}] = [\tilde{X}, \tilde{Y}] = [\tilde{Y}, \tilde{T}] = 0$. It follows that

$$\mathcal{G} = gen(\tilde{D}, \tilde{Z}) \oplus gen(\tilde{T}, \tilde{X}, \tilde{Y}) \approx Aff(R) \oplus R^3.$$

Analogously we can show that if $\tau \neq 0$ or $\kappa \neq 0$ or $\gamma \neq 0$, then $\mathcal{G}$ is not irreducible. So these cases are excluded.

1.2) Suppose that $\delta = \tau = \kappa = \gamma = 0$. Then one of the coefficients a, b, c, d, e, g is different from 0. Without loss of generality we may assume $a = 1$ so that we have

$$[D, Z] = [T, Z] = [X, Z] = [Y, Z] = 0, [D, T] = Z,$$
$$[T, X] = bZ, [X, Y] = cZ, [Y, D] = dZ,$$
$$[X, D] = eZ, [Y, T] = gZ.$$

Now put

$$\tilde{X} = X + bD + eT, \tilde{D} = D, \tilde{T} = T, \tilde{Z} = Z, \tilde{Y} = Y - gD + dT.$$

It is trivial to show that this transformation is regular and

$$[\tilde{D},\tilde{Z}] = [\tilde{T},\tilde{Z}] = [\tilde{X},\tilde{Z}] = [\tilde{Y},\tilde{Z}] = 0,$$
$$[\tilde{T},\tilde{X}] = [\tilde{T},\tilde{Y}] = [\tilde{D},\tilde{X}] = [\tilde{D},\tilde{Y}] = 0,$$
$$[\tilde{D},\tilde{T}] = \tilde{Z}, \quad [\tilde{X},\tilde{Y}] = \tilde{c}\tilde{Z} \text{ (for some coefficient } \tilde{c}).$$

If $\tilde{c} = 0$, then

$$\mathcal{G} = gen(\tilde{D},\tilde{T},\tilde{Z}) \oplus gen(\tilde{X},\tilde{Y}) \cong \mathcal{H}_3 \oplus R^2.$$

So $\mathcal{G}$ is not irreducible and this case is excluded. In the case $\tilde{c} \neq 0$, changing $\tilde{X}$ by $\frac{\tilde{X}}{\tilde{c}}$ we can easily show that $\mathcal{G} \cong \mathcal{G}_{5.1} = \mathcal{H}_5$.

2/ Suppose that dim $\mathcal{G}^1 = 2$ and $\mathcal{G}^1 = gen(Y,Z) \approx R^2$. First we note that since $[D,T],[T,X],[X,D] \in \mathcal{G}^1$,

$$[ad^1_T, ad^1_D] = [ad^1_T, ad^1_X] = [ad^1_X, ad^1_D] = 0.$$

This means that the transformations ad^1_D, ad^1_T, ad^1_X are pairwise commutative.

Lemma 2. *There exists in $\mathcal{G}^1$ a basis for which the matrices of the transformations ad^1_D, ad^1_T, ad^1_X have one of the following forms:*

a/ $ad^1_D = \begin{Vmatrix} \alpha & \beta \\ -\beta & \alpha \end{Vmatrix}, ad^1_T = \begin{Vmatrix} \delta & \gamma \\ -\gamma & \delta \end{Vmatrix}, ad^1_X = \begin{Vmatrix} \vartheta & \tau \\ -\tau & \vartheta \end{Vmatrix}, \beta^2+\gamma^2+\tau^2 \neq 0.$

b/ $ad^1_D = \begin{Vmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{Vmatrix}, ad^1_T = \begin{Vmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{Vmatrix}, ad^1_X = \begin{Vmatrix} \vartheta_1 & 0 \\ 0 & \vartheta_2 \end{Vmatrix},$
$(\alpha_1-\alpha_2)^2 + (\delta_1-\delta_2)^2 + (\vartheta_1-\vartheta_2)^2 \neq 0.$

c/ $ad^1_D = \begin{Vmatrix} \alpha & \beta \\ 0 & \alpha \end{Vmatrix}, ad^1_T = \begin{Vmatrix} \delta & \gamma \\ 0 & \delta \end{Vmatrix}, ad^1_X = \begin{Vmatrix} \vartheta & \tau \\ 0 & \vartheta \end{Vmatrix}.$

Proof.

a/ If ad^1_D or ad^1_T or ad^1_X has a complex eigenvalue $\alpha+\beta\sqrt{-1}$ $(\beta>0)$ or $\delta+\gamma\sqrt{-1}$ $(\gamma>0)$ or $\vartheta+\tau\sqrt{-1}$ $(\tau>0)$, respectively, then there exists in $\mathcal{G}^1$ a basis for which the matrices of ad^1_D, ad^1_T and ad^1_X have the forms in a/, respectively. The condition $\beta^2+\gamma^2+\tau^2 \neq 0$ means that β,γ and τ are not simultaneously equal to 0.

b/ Suppose that each of the transformations has only real eigenvalues and one of them has two different ones. Then we can find in $\mathcal{G}^1$ a basis such that the matrices of ad^1_D, ad^1_T and ad^1_X have the forms in b/.

c/ Finally assume that every transformation in our consideration has only multiple eigenvalue. Since ad^1_D, ad^1_T and ad^1_X are pairwise commutative, they are simultaneously triangulated. Therefore, we can find a basis for which the matrices of ad^1_D, ad^1_T and ad^1_X have the forms in c/.

Thus Lemma 2 is proved.

Let us now consider the possibilities listed in Lemma 2.

2.1) Suppose that for the chosen basis $\{Y, Z\}$ of $\mathcal{G}^1$ we have

$$ad^1_D = \begin{Vmatrix} \alpha & \beta \\ -\beta & \alpha \end{Vmatrix}, ad^1_T = \begin{Vmatrix} \delta & \gamma \\ -\gamma & \delta \end{Vmatrix}, ad^1_X = \begin{Vmatrix} \vartheta & \tau \\ -\tau & \vartheta \end{Vmatrix},$$

where β, γ, τ are not simultaneously equal to 0. Without loss of generality we may assume that $\beta > 0$. Then the transformation ad^1_D is regular and there exist coefficients a_1, b_1, a_2, b_2 such that

$$[D,T] = a_1[D,Y] + b_1[D,Z] = [D, a_1Y + b_1Z],$$
$$[D,X] = a_2[D,Y] + b_2[D,Z] = [D, a_2Y + b_2Z].$$

Changing T by $(T - a_1Y - b_1Z)$ and X by $(X - a_2Y - b_2Z)$ we may assume that $[D,T] = [D,X] = 0$. Then, since

$$[D,[T,X]] = [[D,T],X] + [T,[D,X]] = 0$$

we have $[T,X] = 0$. Now let us make the following transformation

$$\tilde{T} = T - \frac{\gamma}{\beta}D, \tilde{D} = D,$$

$$\tilde{Y} = Y,$$

$$\tilde{X} = X - \frac{\tau}{\beta}D, \tilde{Z} = Z.$$

It is easy to verify the equalities

$$ad^1_{\tilde{D}} = \begin{Vmatrix} \alpha & \beta \\ -\beta & \alpha \end{Vmatrix}, ad^1_{\tilde{T}} = \begin{Vmatrix} \tilde{\delta} & 0 \\ 0 & \tilde{\delta} \end{Vmatrix}, ad^1_{\tilde{X}} = \begin{Vmatrix} \tilde{\vartheta} & 0 \\ 0 & \tilde{\vartheta} \end{Vmatrix},$$

where $\tilde{\delta} = (\delta - \frac{\gamma\alpha}{\beta}), \tilde{\vartheta} = (\vartheta - \frac{\tau\alpha}{\beta})$, and

$$[\tilde{D}, \tilde{T}] = [\tilde{T}, \tilde{X}] = [\tilde{D}, \tilde{X}] = 0.$$

If $\tilde{\delta} = 0$, then $ad^1_{\tilde{T}} = 0$ and we have

$$\mathcal{G} = gen(\tilde{T}) \oplus gen(\tilde{D}, \tilde{X}, \tilde{Y}, \tilde{Z}).$$

If $\tilde{\delta} \neq 0$, then with $X' = \tilde{X} - \frac{\vartheta}{\delta}\tilde{T}$ we have $ad^1_{X'} = 0$ and $[X', \tilde{D}] = [X', \tilde{T}] = 0$. Therefore,

$$\mathcal{G} = gen(X') \oplus gen(\tilde{D}, \tilde{T}, \tilde{Y}, \tilde{Z}).$$

Thus, in both cases $\mathcal{G}$ is not irreducible.

2.2) Suppose that for the chosen basis $\{Y, Z\}$ of $\mathcal{G}^1$ we have

$$ad^1_D = \begin{Vmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{Vmatrix}, ad^1_T = \begin{Vmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{Vmatrix}, ad^1_X = \begin{Vmatrix} \vartheta_1 & 0 \\ 0 & \vartheta_2 \end{Vmatrix},$$

where $(\alpha_1 - \alpha_2)^2 + (\delta_1 - \delta_2)^2 + (\vartheta_1 - \vartheta_2)^2 \neq 0$. Since ad^1_D, ad^1_T, and ad^1_X are equivalent, we may assume that $\alpha_1 \neq \alpha_2$. Let us study the following possibilities:

2.2.i) $\alpha_1\alpha_2 \neq 0$. In this case, ad^1_D is regular. By arguments similar to those in 2.1) we may assume that

$$[D, T] = [T, X] = [X, D] = 0.$$

Suppose that $f = yY^* + zZ^*$ is an arbitrary linear form of $\mathcal{G}^{1*}$ and $\varphi(y, z)$ is the matrix of the form B_f for the basis $\{D, T, X, Y, Z\}$. Then we have

$$\varphi(y, z) = \begin{Vmatrix} 0 & 0 & 0 & \alpha_1 y & \alpha_2 z \\ 0 & 0 & 0 & \delta_1 y & \delta_2 z \\ 0 & 0 & 0 & \vartheta_1 y & \vartheta_2 z \\ -\alpha_1 y & -\delta_1 y & -\vartheta_1 y & 0 & 0 \\ -\alpha_2 y & -\delta_2 y & -\vartheta_2 y & 0 & 0 \end{Vmatrix}.$$

Since $rank\ \varphi(y, 0) \leq 2$ and $\mathcal{G}$ belongs to the class MD, $rank\ \varphi(y, z) \leq 2, \forall(x, y)$. From this it immediately follows

$$\frac{\alpha_1}{\alpha_2} = \frac{\beta_1}{\beta_2} = \frac{\vartheta_1}{\vartheta_2}.$$

Therefore, there exist two coëfficients t and x such that $ad^1_T = tad^1_D$ and $ad^1_X = xad^1_D$. Now by putting

$$\tilde{T} = T - tD, \tilde{D} = D,$$

$$\tilde{X} = X - xD, \tilde{Y} = Y, \tilde{Z} = Z,$$

we have $[\tilde{D}, \tilde{T}] = [\tilde{T}, \tilde{X}] = [\tilde{X}, \tilde{D}] = 0, ad^1_{\tilde{T}} = ad^1_{\tilde{X}} = 0$, and consequently, $\mathcal{G} = gen(\tilde{T}, \tilde{X}) \oplus gen(\tilde{D}, \tilde{Y}, \tilde{Z})$. Thus $\mathcal{G}$ is not irreducible and the case i) is excluded.

2.2.ii) $\alpha_1 = 0, \alpha_2 \neq 0$ (the case $\alpha_1 \neq 0, \alpha_2 = 0$ is the same). Changing T by $(T - \frac{\delta_2}{\alpha_2} D)$ and X by $(X - \frac{\vartheta_2}{\alpha_2} D)$ we may assume that

$$ad^1_D = \begin{Vmatrix} 0 & 0 \\ 0 & \alpha_2 \end{Vmatrix}, ad^1_T = \begin{Vmatrix} \delta_1 & 0 \\ 0 & 0 \end{Vmatrix}, ad^1_X = \begin{Vmatrix} \vartheta_1 & 0 \\ 0 & 0 \end{Vmatrix}.$$

If $\delta_1 \neq 0$ (or $\vartheta_1 \neq 0$), we consider the following basis

$$\tilde{D} = D + T \text{ (or } D + X), \tilde{T} = T, \tilde{X} = X, \tilde{Y} = Y, \tilde{Z} = Z.$$

We have

$$ad^1_{\tilde{D}} = \begin{Vmatrix} \delta_1 & 0 \\ 0 & \alpha_2 \end{Vmatrix}, \text{ (or } ad^1_{\tilde{D}} = \begin{Vmatrix} \vartheta_1 & 0 \\ 0 & \alpha_2 \end{Vmatrix}), ad^1_{\tilde{D}T} = \begin{Vmatrix} \delta_1 & 0 \\ 0 & 0 \end{Vmatrix}, ad^1_{\tilde{X}} = \begin{Vmatrix} \vartheta_1 & 0 \\ 0 & 0 \end{Vmatrix}.$$

Thus, we return to case i) which is excluded because $\mathcal{G}$ is not irreducible.

Suppose now that $\delta_1 = \vartheta_1 = 0$. Changing D by $\frac{D}{\alpha_2}$ (if necessary) we may assume that

$$ad^1_D = \begin{Vmatrix} 0 & 0 \\ 0 & 1 \end{Vmatrix}, ad^1_T = ad^1_X = 0.$$

Suppose that

$$[D, T] = a_1 Y + b_1 Z,$$

$$[T, X] = a_2 Y + b_2 Z,$$

$$[D, X] = a_3 Y + b_3 Z,$$

$$f = yY^* + zZ^* \in \mathcal{G}^{1*}.$$

Let $\varphi(y,z)$ be the matrix of the form B_f for the basis $\{D,T,X,Y,Z\}$. Then we have

$$\varphi(y,Z)=\begin{Vmatrix} 0 & (a_1y+b_1z) & (a_3y+b_3z) & 0 & z \\ -(a_1y+b_1z) & 0 & (a_2y+b_2z) & 0 & 0 \\ -(a_3y+b_3z) & -(a_2y+b_2z) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -z & 0 & 0 & 0 & 0 \end{Vmatrix}.$$

Since $rank\ \varphi(y,0)\le 2$ and $\mathcal{G}$ belongs to the class MD, it follows that $rank\ \varphi(y,z)\le 2, \forall(y,z)$. Using this condition, we can show that $a_2=b_2=0$, so that $[T,X]=0$. Put

$$\tilde{T}=T-b_1Z, \tilde{D}=D, \tilde{Z}=z,$$

$$\tilde{X}=X-b_3Z, \tilde{Y}=Y.$$

Obviously, the system $\{\tilde{D},\tilde{T},\tilde{X},\tilde{Y},\tilde{Z}\}$ is a new basis for which we have

$$ad^1_{\tilde{D}}=\begin{Vmatrix} 0 & 0 \\ 0 & 1 \end{Vmatrix}, ad^1_{\tilde{T}}=ad^1_{\tilde{X}}=0, [\tilde{T},\tilde{X}]=0, [\tilde{D},\tilde{T}]=a_1\tilde{Y}, [\tilde{D},\tilde{X}]=a_3\tilde{Y}.$$

If $a_1=0$, then

$$\mathcal{G}=gen(\tilde{T})\oplus gen(\tilde{D},\tilde{X},\tilde{Y},\tilde{Z}).$$

If $a_1\neq 0$, then with $X'=\tilde{X}-\frac{a_3}{a_1}\tilde{T}$, we get

$$\mathcal{G}=gen(X')\oplus gen(\tilde{D},\tilde{T},\tilde{Y},\tilde{Z}).$$

Thus $\mathcal{G}$ is not irreducible and this case is excluded.

2.3) Suppose that for the basis $\{Y,Z\}$ of $\mathcal{G}^1$ we have

$$ad^1_D=\begin{Vmatrix} \alpha & \beta \\ 0 & \alpha \end{Vmatrix}, ad^1_T=\begin{Vmatrix} \delta & \gamma \\ 0 & \delta \end{Vmatrix}, ad^1_D=\begin{Vmatrix} \vartheta & \tau \\ 0 & \vartheta \end{Vmatrix}.$$

Let us consider the following possibilities:

2.3.i) $\alpha\neq 0$ (the case $\delta\neq 0$ or $\vartheta\neq 0$ is similar). Then ad^1_D is regular and by arguments analogous to those in 2.1) we may assume that

$$[D,T]=[T,X]=[X,D]=0.$$

Putting

$$\tilde{T} = T - \frac{\delta}{\alpha}D, \tilde{D} = D, \tilde{X} = X - \frac{\vartheta}{\alpha}D, \tilde{Y} = Y, \tilde{Z} = Z,$$

we obtain

$$ad^1_{\tilde{D}} = \begin{Vmatrix} \alpha & \beta \\ 0 & \alpha \end{Vmatrix}, ad^1_{\tilde{T}} = \begin{Vmatrix} 0 & \tilde{\gamma} \\ 0 & 0 \end{Vmatrix}, ad^1_{\tilde{X}} = \begin{Vmatrix} 0 & \tau \\ 0 & 0 \end{Vmatrix}, (\tilde{\gamma} = \gamma - \frac{\delta}{\alpha}\beta, \tilde{\vartheta} = \tau - \frac{\vartheta}{\alpha}\beta).$$

If $\tilde{\gamma} = 0$, then $\mathcal{G} = gen(\tilde{T}) \oplus gen(\tilde{D}, \tilde{X}, \tilde{Y}, \tilde{Z})$.

If $\tilde{\gamma} \neq 0$, then with $X' = \tilde{X} - \frac{\tilde{\tau}}{\tilde{\gamma}}$, we have

$$\mathcal{G} = gen(X') \oplus gen(\tilde{D}, \tilde{T}, \tilde{Y}, \tilde{X}).$$

This means that $\mathcal{G}$ is not irreducible and the case is excluded.

2.3.ii) $\alpha = \delta = \vartheta = 0$. This case may be divided into the following subcases.

2.3.ii.1) $\beta \neq 0$ (the case $\gamma \neq 0$ or $\tau \neq 0$ is similar). Changing T by $(T - \frac{\gamma}{\beta}D)$, X by $(X - \frac{\tau}{\beta}D)$ and D by $\frac{1}{\beta}D$ we may assume

$$ad^1_D = \begin{Vmatrix} 0 & 1 \\ 0 & 0 \end{Vmatrix}, ad^1_T = ad^1_X = 0,$$

$$[D, T] = a_1Y + b_1Z, [T, X] = a_2Y + b_2Z, [D, T] = a_3Y + b_3Z.$$

Let $f = yY^* + zZ^*$ be an arbitrary linear form of $\mathcal{G}^{1*}$ and $\varphi(y, z)$ the matrix of the form B_f for the basis $\{D, T, X, Y, Z\}$. Then we have

$$\varphi(y, z) = \begin{Vmatrix} 0 & (a_1y + b_1z) & (a_3y + b_3z) & 0 & y \\ -(a_1y + b_1z) & 0 & (a_2y + b_2z) & 0 & 0 \\ -(a_3y + b_3z) & -(a_2y + b_2z) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -y & 0 & 0 & 0 & 0 \end{Vmatrix}.$$

Since $rank\ \varphi(0, z) \leq 2$ and $\mathcal{G}$ is MD's Lie algebra, $rank\ \varphi(y, z) \leq 2, \forall (y, z)$. From this condition it follows that $a_2 = b_2 = 0$ and $[T, X] = 0$. Now put

$$\tilde{T} = T - a_1Z, \tilde{D} = D, \tilde{Z} = Z, \tilde{X} = X - a_3Z, \tilde{Y} = Y.$$

Then $\{\tilde{D},\tilde{T},\tilde{X},\tilde{Y},\tilde{Z}\}$ is a basis of $\mathcal{G}$ for which we have

$$ad^1_{\tilde{T}} = ad^1_{\tilde{X}} = 0, ad^1_{\tilde{D}} = \begin{Vmatrix} 0 & 1 \\ 0 & 0 \end{Vmatrix}, [\tilde{D},\tilde{T}] = b_1\tilde{Z}, [\tilde{D},\tilde{X}] = b_3\tilde{Z}, [\tilde{T},\tilde{X}] = 0.$$

If $b_1 \neq 0$, then $\mathcal{G} = gen(\tilde{T}) \oplus gen(\tilde{D},\tilde{X},\tilde{Y},\tilde{Z})$. If $b_1 \neq 0$, then with $X' = \tilde{X} - \frac{b_3}{b_1}\tilde{T}$ we have $[X',\tilde{D}] = 0$. Hence

$$\mathcal{G} = gen(X') \oplus gen(\tilde{D},\tilde{T},\tilde{Y},\tilde{Z}).$$

This means that $\mathcal{G}$ is not irreducible. The case is excluded.

2.3.ii.2) $\beta = \gamma = \tau = 0$. Then $ad^1_D = ad^1_T = ad^1_X = 0$. Because the vectors $[D,T], [T,X]$ and $[X,D]$ generate the space $\mathcal{G}^1$, there exist among them two linearly independent vectors, say $[D,T]$ and $[T,X]$. Then $[X,D] = a[D,T] + b[T,X]$ for some a and b. Consider the transformation

$$\tilde{D} = D + bT, \tilde{X} = X + aT, \tilde{T} = T, \tilde{Y} = [D,T], \tilde{Z} = [X,T].$$

One checks that this transformation is regular so that $\{\tilde{D},\tilde{T},\tilde{X},\tilde{Y},\tilde{Z}\}$ is a basis of $\mathcal{G}$. Moreover, $\mathcal{G}^1 = gen(\tilde{Y},\tilde{Z}), ad^1_{\tilde{D}} = ad^1_{\tilde{T}} = ad^1_{\tilde{X}} = 0, [\tilde{D},\tilde{X}] = 0, [\tilde{D},\tilde{T}] = \tilde{Y}, [\tilde{X},\tilde{T}] = \tilde{Z}$. Thus $\mathcal{G}$ is isomorphic to $\mathcal{G}_{5.2}$.

3/ Suppose that $dim\mathcal{G}^1 = 3$ and $\mathcal{G}^1 = gen(X,Y,Z) \cong R^3$.

Note that the transformation ad^1_D and ad^1_T are commutative.

LEMMA 3. *There exists in $\mathcal{G}^1$ a basis for which the pair of matrices (ad^1_D, ad^1_T) have one of the following forms up to a permutation:*

i/ $\begin{Vmatrix} \alpha & \beta & 0 \\ -\beta & \alpha & 0 \\ 0 & 0 & \lambda \end{Vmatrix} (\beta > 0), \begin{Vmatrix} \delta & \gamma & 0 \\ -\gamma & \delta & 0 \\ 0 & 0 & \mu \end{Vmatrix}.$

ii/ $\begin{Vmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{Vmatrix} (\lambda_i \neq \lambda_j, i \neq j), \begin{Vmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{Vmatrix}.$

iii/ $\begin{Vmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{Vmatrix} (\lambda_1 \neq \lambda_2), \begin{Vmatrix} \mu_1 & \alpha & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{Vmatrix} (\mu_2 = \mu_1 \text{ or } \mu_2 = \mu_3).$

$$iv/ \left\|\begin{matrix}\lambda_1 & 0 & 0\\ 0 & \lambda_1 & 0\\ 0 & 0 & \lambda_2\end{matrix}\right\| (\lambda_1 \neq \lambda_2), \left\|\begin{matrix}\mu_1 & \alpha & 0\\ 0 & \mu_1 & 0\\ 0 & 0 & \mu_2\end{matrix}\right\|.$$

$$v/ \left\|\begin{matrix}\lambda & 0 & 1\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{matrix}\right\|, \left\|\begin{matrix}\mu & \alpha & \beta\\ 0 & \mu & \gamma\\ 0 & 0 & \mu\end{matrix}\right\|.$$

$$vi/ \left\|\begin{matrix}\lambda & 1 & 0\\ 0 & \lambda & 1\\ 0 & 0 & \lambda\end{matrix}\right\|, \left\|\begin{matrix}\mu & \alpha & \beta\\ 0 & \mu & \alpha\\ 0 & 0 & \mu\end{matrix}\right\|.$$

PROOF. If one of the transformations ad^1_D and ad^1_T has a complex eigenvalue $\alpha + \beta\sqrt{-1}, \beta > 0$, then it must have a real proper value λ and we can choose a basis of $\mathcal{G}^1$ for which the pair of matrices (ad^1_D, ad^1_T) have the form in i/ up to a permutation.

Suppose that ad^1_D and ad^1_T have only real eigenvalues. If one of these matrices has three real pairwise different eigenvalues, then we can find a basis of $\mathcal{G}^1$ such that the pair (ad^1_D, ad^1_T) have the form in ii/ up to a permutation. If none of them has three real pairwise different eigenvalues, but one of them has a simple one, then we can get the forms in iii/ or iv/ according to the real normal Jordan form of this transformation.

Finally, if every transformation ad^1_D and ad^1_T has only multiple eigenvalue, we come to the forms in v/ or vi/, according to the real normal Jordan matrix.

Let us now consider in details every subcase of Lemma 3. Since D and T are equivalent, we may always assume that ad^1_D is the first matrix and ad^1_T is the second in every pair.

3.1) Suppose that in the chosen basis $\{X, Y, Z\}$ of $\mathcal{G}^1$ we have

$$ad^1_D = \left\|\begin{matrix}\alpha & \beta & 0\\ -\beta & \alpha & 0\\ 0 & 0 & \lambda\end{matrix}\right\|, \beta > 0, ad^1_T = \left\|\begin{matrix}\delta & \gamma & 0\\ -\gamma & \delta & 0\\ 0 & 0 & \mu\end{matrix}\right\|.$$

Changing T by $(T - \frac{\gamma}{\beta}D)$ we may assume $\gamma = 0$ and

$$ad^1_D = \left\|\begin{matrix}\alpha & \beta & 0\\ -\beta & \alpha & 0\\ 0 & 0 & \lambda\end{matrix}\right\|, \beta > 0, ad^1_T = \left\|\begin{matrix}\delta & 0 & 0\\ 0 & \delta & 0\\ 0 & 0 & \mu\end{matrix}\right\|.$$

According to the value of λ we have two following possibilities.

3.1.1) $\lambda \neq 0$. Then ad_D^1 is regular. By arguments analogous to those in 2.1) we can assume that $[D,T]=0$. Let $f = xD^* + yY^* + zZ^*$ be an arbitrary linear form of $\mathcal{G}^{1*}$ and $\varphi(x,y,z)$ the matrix of the form B_f in the basis $\{D,T,X,Y,Z\}$. Then we have

$$\varphi(x,y,z) = \begin{Vmatrix} 0 & 0 & (\alpha x - \beta y) & (\beta x + \alpha y) & \lambda z \\ 0 & 0 & \delta x & \delta x & \mu z \\ (\beta y - \alpha x) & -\delta x & 0 & 0 & 0 \\ -(\beta x + \alpha y) & -\delta y & 0 & 0 & 0 \\ -\lambda z & -\mu z & 0 & 0 & 0 \end{Vmatrix}.$$

Since $rank\, \varphi(0,0,z) \leq 2$ and $\mathcal{G}$ belongs to the class MD, $rank\, \varphi(x,y,z) \leq 2, \forall (x,y,z)$. It is easy to obtain $\delta\mu = 0$ and so that $ad_T^1 = 0$. Combining this with $[D,T]=0$ we get the decomposition

$$\mathcal{G} = gen(T) \oplus gen(D,X,Y,Z).$$

Thus this case is excluded.

3.1.2) $\lambda = 0$. This means

$$ad_D^1 = \begin{Vmatrix} \alpha & \beta & 0 \\ -\beta & \alpha & 0 \\ 0 & 0 & 0 \end{Vmatrix}, \beta > 0, ad_T^1 = \begin{Vmatrix} \delta & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & \mu \end{Vmatrix}.$$

Assume that $[D,T] = aX + bY + cZ, f = xX^* + yY^* + zZ^* \in \mathcal{G}^{1*}$ and $\varphi(x,y,z)$ is the matrix of the form B_f in the basis $\{D,T,X,Y,Z\}$. Then we have

$$\varphi(x,y,z) = \begin{Vmatrix} 0 & (ax+by+cz) & (\alpha x - \beta y) & (\beta x + \alpha y) & 0 \\ -(ax+by+cz) & 0 & \delta x & \delta y & \mu z \\ (\beta y - \alpha x) & -\delta x & 0 & 0 & 0 \\ -(\beta x + \alpha y) & -\delta y & 0 & 0 & 0 \\ 0 & -\mu z & 0 & 0 & 0 \end{Vmatrix}.$$

It is easy to see that $rank\, \varphi(x,y,z) \leq 2, \forall (x,y,z)$. From this it follows that $\delta = \mu = 0$ and $ad_T^1 = 0$.

On the other hand, since $[D,X]$ and $[D,Y]$ are linearly independent in the subspace $L = L(x,y)$, there exist coefficients a_1 and b_1 such that $[D,T] = a_1[D,X]+b_1[D,Y]+cZ$. Therefore $[D,T-a_1X-b_1Y] = cZ$. Because $[D,T]$ and $ad_D(\mathcal{G}^1)$ generate $\mathcal{G}^1$, c must be nonzero. Let us now make the transformation

$$\tilde{D} = \frac{D}{\sqrt{\alpha^2+\beta^2}}, \tilde{X} = X, \tilde{Z} = \frac{cZ}{\sqrt{\alpha^2+\beta^2}}, \tilde{T} = T - a_1X - b_1Y,$$

$$\tilde{Y} = Y, \cos\varphi = \frac{\alpha}{\sqrt{\alpha^2+\beta^2}}, \sin\varphi = \frac{\beta}{\sqrt{\alpha^2+\beta^2}}, (\varphi \in (0,\pi)).$$

It is obvious that the system $\{\tilde{D},\tilde{T},\tilde{X},\tilde{Y},\tilde{Z}\}$ is independent and $\mathcal{G}^1 = gen(\tilde{X},\tilde{Y},\tilde{Z})$. Moreover, we see that $[\tilde{D},\tilde{T}] = \tilde{Z}$.

$$ad^1_{\tilde{D}} = \begin{Vmatrix} \cos\varphi & -\sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & \lambda_3 \end{Vmatrix}, ad^1_{\tilde{T}} = 0.$$

This means $\mathcal{G} \cong \mathcal{G}_{5.3.1}$.

3.2) Suppose that

$$ad^1_D = \begin{Vmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{Vmatrix}, \lambda_i \neq \lambda_j, i \neq j,$$

$$ad^1_T = \begin{Vmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{Vmatrix}.$$

If $[D,T] = aX + +bY + cZ, f = xX^* + yY^* + zZ^* \in \mathcal{G}^{1*}$, and $\varphi(x,y,z)$ is the matrix of the form B_f in the basis $\{D,T,X,Y,Z\}$, then we have

$$\varphi(x,y,z) = \begin{Vmatrix} 0 & (ax+by+cz) & \lambda_1 y & \lambda_2 y & \lambda_3 z \\ -(ax+by+cz) & 0 & \mu_1 x & \mu_2 y & \mu_3 z \\ -\lambda_1 x & -\mu_1 x & 0 & 0 & 0 \\ -\lambda_2 x & -\mu_2 x & 0 & 0 & 0 \\ -\lambda_3 x & -\mu_3 x & 0 & 0 & 0 \end{Vmatrix}.$$

Since $rank\ \varphi(0,0,z) \leq 2$ and $\mathcal{G}$ is a MD's Lie algebra, $rank\ \varphi(x,y,z) \leq 2$, $\forall (x,y,z)$. From this it is easy to see that

$$\frac{\mu_1}{\lambda_1} = \frac{\mu_2}{\lambda_2} = \frac{\mu_3}{\lambda_3}.$$

That means $ad_T = t ad_D$, for some $t \in R$ and $ad^1(T - tD) = 0$.

According to the values $\lambda_1, \lambda_2, \lambda_3$ we shall consider the following possibilities:

3.2.1) $\lambda_1\lambda_2\lambda_3 \neq 0$. Then ad^1_D is a regular transformation. By similar arguments as used in 3.1.1) we can show that $\mathcal{G}$ is not irreducible. So this case is excluded.

3.2.2) $\lambda_1\lambda_2 \neq 0, \lambda_3 = 0$. In this case, since $[D,T]$ and $[D,\mathcal{G}']$ generate $\mathcal{G}^1$, c must be different from zero. Let us make the transformation

$$\tilde{T} = T - tD - \frac{a}{\lambda_1}X - \frac{b}{\lambda_2}Y,$$

$$\tilde{D} = \frac{1}{\lambda_2}D, \tilde{X} = X, \tilde{Y} = Y, \tilde{Z} = \frac{cZ}{\lambda_2}, \lambda = \frac{\lambda_1}{\lambda_2} \neq 0, 1.$$

It is easy to verify that $\{\tilde{D}, \tilde{T}, \tilde{X}, \tilde{Y}, \tilde{Z}\}$ is a basis of $\mathcal{G}$ with $\mathcal{G}^1 = gen(\tilde{X}, \tilde{Y}, \tilde{Z})$. Besides, in this basis we have

$$[\tilde{D}, \tilde{T}] = \tilde{Z}, ad^1_{\tilde{D}} = \begin{Vmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{Vmatrix}, (\lambda_1 \neq 0; 1), ad^1_{\tilde{T}} = 0.$$

This means $\mathcal{G} \cong \mathcal{G}_{5.3.2}(\lambda \neq 1)$.

3.3) Suppose that we are in case iii) of Lemma 3. That means

$$ad_D = \begin{Vmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{Vmatrix}, \lambda_1 \neq \lambda_2, ad^1_T = \begin{Vmatrix} \mu_1 & \alpha & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{Vmatrix}.$$

Let $[D,T] = aX + bY + cZ$ be a decomposition of $[D,T]$, $f = xX^* + yY^* + zZ^*$ an arbitrary linear form of $\mathcal{G}^{1*}$, and $\varphi(x,y,z)$ the matrix of the form B_f in the basis $\{D, T, X, Y, Z\}$. Then we have

$$\varphi(x,y,z) = \begin{Vmatrix} 0 & (ax+by+cz) & \lambda_1 x & \lambda_1 y & \lambda_2 z \\ -(ax+by+cz) & 0 & \mu_1 x & (\alpha x + \mu_2 y) & \mu_3 z \\ -\lambda_1 x & -\mu_1 y & 0 & 0 & 0 \\ -\lambda_1 y & -(\alpha x + \mu_2 y) & 0 & 0 & 0 \\ -\lambda_2 z & -\mu_3 z & 0 & 0 & 0 \end{Vmatrix}.$$

It is obvious that $rank\ \varphi(0,0,z) \leq 2$, from which it follows that $rank\ \varphi(x, y, z) \leq 2$, for all (x,y,z). By a simple argument one can show that $\alpha = 0, \mu_1 = \mu_2$ and $\dfrac{\lambda_1}{\lambda_2} = \dfrac{\mu_1}{\mu_2}$. Therefore $ad^1_T = t ad^1_D$ for some $t \in R$.

Changing T by $(T - tD)$ if necessary we may assume $ad^1_T = 0$. Thus, we have

$$ad^1_D = \begin{Vmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{Vmatrix}, \lambda_1 \neq \lambda_2, ad^1_T = 0.$$

According to the values of λ_1, λ_2 we shall consider the following subcases.

3.3.1) $\lambda_1\lambda_2 \neq 0$. Then ad^1_D is regular. By the method used in 3.1.1) we prove that $\mathcal{G}$ is not irreducible and this case is excluded.

3.3.2) $\lambda_1 = 0, \lambda_2 \neq 0$. Then $[D,T]$ and $ad_D(\mathcal{G}^1)$ can not generate the space $\mathcal{G}^1$. Thus, this case is excluded, too.

3.3.3) $\lambda_1 \neq 0, \lambda_2 = 0$. Since $[D,T]$ and $ad^1_D(\mathcal{G}^1)$ generate $\mathcal{G}^1, c \neq 0$. Let us make the regular transformation

$$\tilde{D} = \frac{1}{\lambda_1} D,$$

$$\tilde{X} = X,$$

$$\tilde{T} = T - \frac{a}{\lambda_1} X - \frac{b}{\lambda_1} Y, \tilde{Y} = Y, \tilde{Z} = \frac{c}{\lambda_1} Z.$$

It is easy to see that $[\tilde{D}, \tilde{T}] = \tilde{Z}, ad^1_{\tilde{T}} = 0$ and $ad^1_{\tilde{D}} = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{Vmatrix}$. Therefore $\mathcal{G} \cong \mathcal{G}_{5.3.2}$ (according to $\lambda = 1$) .

3.4) Suppose that with the chosen basis $\{X, Y, Z\}$ of $\mathcal{G}^1$ we have

$$ad_D^1 = \begin{Vmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{Vmatrix}, \lambda_1 \neq \lambda_2, ad_T^1 = \begin{Vmatrix} \mu_1 & \alpha & 0 \\ 0 & \mu_1 & 0 \\ 0 & 0 & \mu_2 \end{Vmatrix}.$$

By similar arguments as in 3.3) we can show that $ad_T^1 = t ad_D^1$ for some $t \in R$. Then changing T by $(T - tD)$ we may assume $ad_T^1 = 0$.

Let us study the following possibilities.

3.4.1) $\lambda_1 \lambda_2 \neq 0$. This means ad_D^1 is regular. By the method used in 3.1.1) one can show that $\mathcal{G}$ is not irreducible. So this case is excluded.

3.4.2) $\lambda_1 = 0, \lambda_2 \neq 0$ and $[D, T] = aX + bY + cZ$. Since $[D, T]$ and $ad_D(\mathcal{G}^1)$ generate $\mathcal{G}^1$, b must be nonzero. By the transformation

$$\tilde{D} = D, \tilde{X} = bX,$$

$$\tilde{T} = T - aY - \frac{c}{\lambda_2} Z, \tilde{Y} = bY, \tilde{Z} = Z,$$

we obtain $[\tilde{D}, \tilde{T}] = \tilde{Y}, \mathcal{G}^1 = gen(\tilde{X}, \tilde{Y}, \tilde{Z})$ and

$$ad_{\tilde{D}}^1 = \begin{Vmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda_2 \end{Vmatrix}, \lambda_2 \neq 0, ad_{\tilde{T}}^1 = 0.$$

This means $\mathcal{G} \cong \mathcal{G}_{5.3.3}$.

3.4.3) Suppose that $\lambda_1 \neq 0, \lambda_2 = 0$. Without loss of generality we may assume that $\lambda_1 = 1$. If $[D, T] = aX + bY + cZ$,then $c \neq 0$ because $[D, T]$ and $ad_D(\mathcal{G}^1)$ generate $\mathcal{G}^1$. By the transformation

$$\tilde{D} = D, \tilde{X} = X,$$

$$\tilde{T} = T - (a - b)X - bY, \tilde{Y} = Y, \tilde{Z} = cZ,$$

we get

$$[\tilde{D}, \tilde{T}] = \tilde{Z}, \mathcal{G}^1 = gen(\tilde{X}, \tilde{Y}, \tilde{Z}), ad_{\tilde{D}}^1 = \begin{Vmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{Vmatrix}, ad_{\tilde{T}}^1 = 0.$$

From this it follows $\mathcal{G} \cong \mathcal{G}_{5.3.4}$.

3.5) Suppose now that with the chosen basis $\{X, Y, Z\}$ of $\mathcal{G}^1$ we have

$$ad^1_D = \begin{Vmatrix} \lambda & 0 & 1 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{Vmatrix}, ad^1_T = \begin{Vmatrix} \mu & \alpha & \beta \\ 0 & \mu & \gamma \\ 0 & 0 & \mu \end{Vmatrix}.$$

Assume that $[D, T] = aX + bY + cZ, f \neq xX^* + yY^* + zZ^* \in \mathcal{G}^{1*}$ and $\varphi(x, y, z)$ is the matrix of the form B_f in the basis $\{D, T, X, Y, Z\}$.

Then

$$\varphi(x,y,z) = \begin{Vmatrix} 0 & (ax+by+cz) & \lambda x & \lambda y & x+\lambda z \\ -(ax+by+cz) & 0 & \mu x & (\alpha x+\mu y) & \beta x+\gamma y+\mu z \\ -\lambda x & -\mu y & 0 & 0 & 0 \\ -\lambda_y & -(\alpha x+\mu y) & 0 & 0 & 0 \\ -(x+\lambda z) & -(\beta x+\gamma y+\mu z) & 0 & 0 & 0 \end{Vmatrix}.$$

Since $rank\ \varphi(0, 0, z) \leq 2$ and $\mathcal{G}$ is a MD's Lie algebra, $rank\ \varphi(x, y, z) \leq 2, \forall (x, y, z)$. From this it is easy to see that $\alpha = 0, \lambda\gamma = 0$ and $\lambda\beta = \mu$.

Now we shall consider the following possibilities.

3.5.1) $\lambda \neq 0$. Then $\gamma = 0$ and $ad^1_T = \beta ad^1_D$. Changing T by $(T - \beta D)$ if necessary, we can assume $ad^1_T = 0$. Moreover, since $\lambda \neq 0$, ad^1_D is regular. By the method used in 3.1.1) we can show that $\mathcal{G}$ is not irreducible. Thus, this case is excluded.

3.5.2) $\lambda = 0$. Then $\mu = 0$. Changing T by $(T - \beta D)$ we can assume

$$ad^1_T = \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & \gamma \\ 0 & 0 & 0 \end{Vmatrix}.$$

Since $[D, T], ad_D(\mathcal{G}^1)$ and $ad_T(\mathcal{G}^1)$ generate the space $\mathcal{G}^1$, one has $\gamma \neq 0$ and $c \neq 0$. Put

$$\tilde{D} = D + \frac{b}{\gamma}X, \tilde{Z} = \frac{c}{\gamma}Z, \tilde{T} = \frac{1}{\gamma}(T - aZ), \tilde{Y} = \frac{c}{\gamma}Y.$$

It is easy to verify that

$$\mathcal{G}^1 = gen(\tilde{X},\tilde{Y},\tilde{Z}), \quad [\tilde{D},\tilde{T}] = \tilde{Z}, ad^1_{\tilde{D}} = \begin{Vmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix}, ad^1_{\tilde{T}} = \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{Vmatrix}.$$

Therefore $\mathcal{G} \cong \mathcal{G}_{5.3.5}$.

3.6) Suppose that with the basis $\{X, Y, Z\}$ of $\mathcal{G}^1$ we have

$$ad^1_D = \begin{Vmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{Vmatrix}, ad^1_T = \begin{Vmatrix} \mu & \alpha & \beta \\ 0 & \mu & \alpha \\ 0 & 0 & \mu \end{Vmatrix}.$$

Let $[D,T] = aX + bY + cZ$ be the decomposition of $[D,T]$, $f = xX^* + yY^* + zZ^*$ an arbitrary linear form of $\mathcal{G}^{1*}$ and $\varphi(x,y,z)$ the matrix of the form B_f in the basis $\{D, T, X, Y, Z\}$. Then

$$\varphi(x,y,z) = \begin{Vmatrix} 0 & (ax+by+cz) & \lambda x & (x+\lambda y) & (x+\lambda z) \\ -(ax+by+cz) & 0 & \mu x & (\alpha x+\mu y) & (\beta x+\alpha y+\mu z) \\ -\lambda x & -\mu x & 0 & 0 & 0 \\ -(x+\lambda y) & -(\alpha x+\mu y) & 0 & 0 & 0 \\ -(y+\lambda z) & -(\beta x+\alpha y+\mu z) & 0 & 0 & 0 \end{Vmatrix}.$$

It is obvious that $rank\ \varphi(0,0,z) \leq 2$. Therefore $rank\ \varphi(x,y,z) \leq 2$, $\forall(x,y,z)$. From this it is easy to see that $\beta = 0, \lambda\alpha = \mu$.

By changing T by $(T - \alpha D)$ we may assume $ad^1_T = 0$. One has the following cases:

3.6.1) $\lambda \neq 0$. Then ad^1_D is regular. By similar arguments as in 3.1.1) we can show that $\mathcal{G}$ is not irreducible, and this case is excluded.

3.6.2) $\lambda = 0$. It is clear that if $[D,T] = aX + bY + cZ$, then $c \neq 0$. Now put

$$\tilde{D} = D, \tilde{X} = cX, \tilde{Z} = cZ,$$

$$\tilde{T} = T - aY - bZ, \tilde{Y} = cY.$$

One easily shows that

$$\mathcal{G}^1 = gen(\tilde{X}, \tilde{Y}, \tilde{Z}), [\tilde{D}, \tilde{T}] = \tilde{Z}, ad^1_{\tilde{T}} = 0, \text{ and } ad^1_{\tilde{D}} = \begin{Vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{Vmatrix}.$$

Therefore $\mathcal{G} \cong \mathcal{G}_{5.3.6}$.

4/ Now suppose that $dim\mathcal{G}^1 = 3$ and $\mathcal{G}^1$ is not commutative. Since $\mathcal{G}^1$ is a nilpotent Lie algebra, $\mathcal{G}^1$ must be the 3-dimensional Heisenberg Lie algebra. So we may assume that $\mathcal{G}^1 = gen(X, Y, Z)$ with $[X, Y] = Z$ and $[X, Z] = [Y, Z]] = 0$. Let $\tilde{\mathcal{G}}$ be the factor Lie algebra $\mathcal{G}/\mathcal{G}^2$ and $\pi : \mathcal{G} \to \tilde{\mathcal{G}}$ the natural epimorphism from $\mathcal{G}$ to $\tilde{\mathcal{G}}$. By Theorem 2, the Lie algebra $\tilde{\mathcal{G}}$ is a 4-dimensional MD's Lie algebra.

LEMMA 4. *Under the present conditions $\tilde{\mathcal{G}}$ is irreducible.*

PROOF. Suppose $\tilde{\mathcal{G}}$ is not irreducible. Then $\tilde{\mathcal{G}} = \mathcal{G}_1 \oplus \mathcal{G}_2$, where $\mathcal{G}_1$ is a 1-dimensional ideal and $\mathcal{G}_2$ is a 3-dimensional ideal in $\mathcal{G}$. It is obvious that $\pi(\mathcal{G}^1) = \tilde{\mathcal{G}^1} = \mathcal{G}_2^1$. Thus, we may assume that the basis $\{D, T, X, Y, Z\}$ of $\mathcal{G}$ satisfies the conditions

$$\begin{aligned} \mathcal{G}^1 &= gen(X, Y, Z); [X, Y] = Z; [X, Z] = [Y, Z] = 0, \\ \mathcal{G}_1 &= gen(\pi(D), \\ \mathcal{G}_2 &= gen(\pi(T), \pi(X), \pi(Y)). \end{aligned}$$

Now suppose that

$$[D, T] = d_1 Z; [D, X] = d_2 Z; [D, Y] = d_3 Z; [D, Z] = d_4 Z; [T, Z] = tZ.$$

Let us consider the vector $D' = D - d_3 X + d_2 Y$. It is clear that $[D', Z] = d_4 Z$, and since $[X, Y] = Z, [D', X] = [D', Y] = 0$.

Assume that $[D', T] = aX + bY + cZ, f = xX^* + yY^* + zZ^* \in \mathcal{G}^{1*}$ and $\varphi(x, y, z)$ is the matrix of the form B_f in the basis $\{D', T, X, Y, Z\}$. Then we have

$$\varphi(x,y,z) = \begin{Vmatrix} 0 & (ax+by+cz) & 0 & 0 & d_4 z \\ -(ax+by+cz) & 0 & \langle f,[T,X]\rangle & \langle f,[T,Y]\rangle & tz \\ 0 & \langle f,[X,T]\rangle & 0 & z & 0 \\ 0 & \langle f,[Y,T]\rangle & -z & 0 & 0 \\ -d_4 z & -tz & 0 & 0 & 0 \end{Vmatrix}$$

It is evident that there are no 4-dimensional K-orbits in $\tilde{\mathcal{G}}^*$. Hence, there are no 4-dimensional K-orbits in $\mathcal{G}^*$. This means $rank\ \varphi(x,y,z) \leq 2, \forall (x,y,z)$, from which we immediately obtain $a = b = c = d_4 = t = 0$. Therefore, $\mathcal{G} = gen(D') \oplus gen(T,X,Y,Z)$. This contradicts the irreducibility of $\mathcal{G}$. We have just proved Lemma 4.

From Lemma 4 we see that $\tilde{\mathcal{G}}$ can be isomorphic only to one of the following Lie algebras:

$$i/\quad \mathcal{G}_{4.2.2}.$$

$$ii/\quad \mathcal{G}_{4.2.3}.$$

$$iii/\quad \mathcal{G}_{4.2.1} = Aff(C).$$

LEMMA 5. *The cases i/ and ii/ are impossible.*

PROOF. Suppose that $\tilde{\mathcal{G}} \cong \mathcal{G}_{4.2.2}$. We can choose a basis $\{D,T,X,Y,Z\}$ of $\mathcal{G}$ such that the following conditions are satisfied:

$$\mathcal{G}^1 = gen(X,Y,Z); [X,Y] = Z; [X,Z] = [Y,Z] = 0,$$

$$\tilde{\mathcal{G}} = gen(\pi(D),\pi(T),\pi(X),\pi(Y)), [\pi(D),\pi(T)] = \pi(X)$$

$$ad^1_{\pi(D)} = \begin{Vmatrix} 0 & 0 \\ 0 & 1 \end{Vmatrix}; ad^1_{\pi(T)} = 0.$$

One can easily verify the equalities $[T,Z] = 0, [D,Z] = Z$. But then

$$[D,[T,Y]] + [T,[Y,D]] + [Y,[D,T]] = -Z \neq 0.$$

Thus, case i/ is impossible. By the same arguments one shows that case ii/ is impossible, too.

Let us now consider the last case $\tilde{\mathcal{G}} \cong Aff(C)$. It is allways possible to find a basis $\{D, T, X, Y, Z\}$ of $\mathcal{G}$ such that

$$\mathcal{G}^1 = gen(X, Y, Z); [X, Y] = Z, [X, Z] = [Y, Z] = 0,$$

$$\tilde{\mathcal{G}} = gen(\pi(D), \pi(T), \pi(X), \pi(Y)), [\pi(D), \pi(T)] = 0,$$

$$ad^1_{\pi(D)} = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}, ad^1_{\pi(T)} = \begin{Vmatrix} 0 & 0 \\ 0 & 1 \end{Vmatrix}.$$

Suppose that

$$\begin{aligned} &[D, T] = aZ, && [D, Z] = dZ, \\ &[D, X] = -Y + bZ, && [T, X] = X + eZ, \\ &[D, Y] = X + cZ, && [T, Y] = Y + fZ, \\ &[T, Z] = gZ, && (a, b, .., f, g \in R). \end{aligned}$$

From the Jacobi identities we obtain $d = 0, g = 2, e = -c, f = b$. Therefore

$$\begin{aligned} &[D, T] = aZ, && [D, Z] = 0, && [X, Y] = Z, \\ &[D, X] = -Y + bZ, && [T, X] = X - cZ, && [T, Z] = 2Z, \\ &[D, Y] = X + cZ, && [T, Y] = Y + bZ, && [X, Z] = [Y, Z] = 0. \end{aligned}$$

Let us make the transformation

$$\tilde{D} = D + \frac{a}{2}Z, \tilde{X} = X + cZ, \tilde{Y} = Y - bZ, \tilde{T} = T, , \tilde{Z} = Z.$$

It is clear that the vectors $\tilde{D}, \tilde{T}, \tilde{X}, \tilde{Y}, \tilde{Z}$ are also linearly independent and

$$\begin{aligned} &[\tilde{D}, \tilde{T}] = 0, && [\tilde{D}, \tilde{Z}] = 0, && [\tilde{X}, \tilde{Y}] = \tilde{Z}, \\ &[\tilde{D}, \tilde{X}] = -Y, && [\tilde{T}, \tilde{X}] = \tilde{X}, && [\tilde{T}, \tilde{Z}] = 2\tilde{Z}, \\ &[\tilde{D}, \tilde{Y}] = X, && [\tilde{T}, \tilde{Y}] = \tilde{Y}, && [\tilde{X}, \tilde{Z}] = [\tilde{Y}, \tilde{Z}] = 0. \end{aligned}$$

That means $\mathcal{G} \cong \mathcal{G}_{5.3.7}$.

REMARK. In this case the non trivial K-orbits in $\mathcal{G}^*$ are 4-dimensional. Moreover, $\mathcal{G}_{5.3.7}$ has no centre.

Let us now pass to the last case:

5/ $dim\mathcal{G}^1 = 4$. First we shall prove the following lemma

LEMMA 6. *$\mathcal{G}^1$ is a 4-dimensional MD's Lie algebra.*

PROOF. Let f be a linear form of $\mathcal{G}^*$ and B_f the bilinear form corresponding to the form f in $\mathcal{G}$. Let $f_1 = f|\mathcal{G}^1$ be the restriction of f on $\mathcal{G}^1$ and $B^1_{f_1}$ the bilinear form corresponding to f_1 on $\mathcal{G}^1$. It is obvious that $rankB_f \geq rankB^1_{f_1}$. Since $dim\mathcal{G} = 5$ and $dim\mathcal{G}^1 = 4$, we see that if f satisfies the conditions $f|\mathcal{G}^1 \neq 0$ and $f|\mathcal{G}^2 = 0$, then $rankB_f = 2$. Therefore, every K-orbit in $\mathcal{G}^{1*}$ must be 0-dimensional or 2-dimensional. Thus Lemma 6 is proved.

By Theorems 3 and 4, there are only three nilpotent 4-dimensional MD's Lie algebras: $\mathcal{G}_{4.2.3}, R^1 \oplus \mathcal{H}_3, R^4$.

LEMMA 7. *There are no 5-dimensional MD's Lie algebras whose commutant is isomorphic to $\mathcal{G}_{4.2.3}$ or $R^1 \oplus \mathcal{H}_3$.*

PROOF. Suppose $\mathcal{G} = gen(D, T, X, Y, Z)$ is a 5-dimensional MD's Lie algebra and its commutant $\mathcal{G}^1 = gen(T, X, Y, Z)$ is isomorphic to $\mathcal{G}_{4.2.3}$. We may assume that the basis vectors D, T, X, Y, Z satisfy the equalities $[T, X] = Z; [T, Z] = Y; [T, Y] = [X, Y] = [X, Z] = [Y, Z] = 0$. Suppose that the matrix ad^1_D in the basis $\{T, X, Y, Z\}$ has the form

$$ad_D = \begin{Vmatrix} d_{11} & d_{12} & d_{13} & d_{14} \\ d_{21} & d_{22} & d_{23} & d_{24} \\ d_{31} & d_{32} & d_{33} & d_{34} \\ d_{41} & d_{42} & d_{43} & d_{44} \end{Vmatrix}.$$

It is obvious that $ad_D(\mathcal{G}^1) + \mathcal{G}^2 = \mathcal{G}^1$. Let $f = tT^* + xX^* + yY^* + zZ^*$ be an arbitrary linear form of $\mathcal{G}^{1*}$ and $\varphi(t, x, y, z)$ the matrix of the form B_f in the basis $\{D, T, X, Y, Z\}$. Then we have

$$\varphi(x, y, z, t) = \begin{Vmatrix} 0 & \langle f, [D,T]\rangle & \langle f, [D,X]\rangle & \langle f, [D,Y]\rangle & \langle f, [D,Z]\rangle \\ \langle f, [T,D]\rangle & 0 & z & 0 & y \\ \langle f, [X,D]\rangle & -z & 0 & 0 & 0 \\ \langle f, [Y,D]\rangle & 0 & 0 & 0 & 0 \\ \langle f, [Z,D]\rangle & -y & 0 & 0 & 0 \end{Vmatrix}.$$

Since $rank\ \varphi(t,x,0,0) \leq 2$, $rank\ \varphi(t,x,y,z) \leq 2, \forall(t,x,y,z)$. From this we get $y\langle f,[D,Y]\rangle = 0, \quad z\langle f,[D,Z]\rangle - y\langle f,[D,X]\rangle = 0, \forall(t,x,y,z)$, and therefore

$$d_{13} = d_{23} = d_{33} = d_{43} = 0,$$

$$d_{14} = d_{24} = d_{44} = 0,$$

$$d_{12} = d_{22} = d_{32} = 0, d_{42} = d_{34}.$$

On the other hand, from the Jacobi identity written for the triplet (D,T,X) we have $d_{11} = 0$. So, ad_D^1 has the form

$$ad_D^1 = \begin{Vmatrix} 0 & 0 & 0 & 0 \\ d_{21} & 0 & 0 & 0 \\ d_{31} & 0 & 0 & d_{34} \\ d_{41} & d_{42} & 0 & 0 \end{Vmatrix}.$$

This means $ad_D^1(\mathcal{G}^1) \subset gen(X,Y,Z)$. Besides, since $\mathcal{G}^2 = gen(X,Y)$, $ad_D^1(\mathcal{G}^1) + \mathcal{G}^2 \not\subseteq \mathcal{G}^1$. This is a contradiction. We have just proved the first part of our Lemma.

Now suppose that $\mathcal{G} = gen(D,T,X,Y,Z)$ is a 5-dimensional MD's Lie algebra such that $\mathcal{G} = gen(T,X,Y,Z) = R^1 \oplus \mathcal{H}_3$ and the basis vectors D,T,X,Y,Z satisfy the equalities:

$$[X,Y] = Z; [T,X] = [T,Y] = [T,Z] = [X,Z] = [Y,Z] = 0.$$

Let $f = tT^* + xX^* + yY^* + zZ^*$ be an arbitrary linear form of $\mathcal{G}^{1*}$ and $\varphi(t,x,y,z)$ the matrix of the form B_f in the chosen basis $\{D,T,X,Y,Z\}$. Then we have

$$\varphi(x,y,z,t) = \begin{Vmatrix} 0 & \langle f,[D,T]\rangle & \langle f,[D,X]\rangle & \langle f,[D,Y]\rangle & \langle f,[D,Z]\rangle \\ \langle f,[T,D]\rangle & 0 & 0 & 0 & 0 \\ \langle f,[X,D]\rangle & 0 & 0 & z & 0 \\ \langle f,[Y,D]\rangle & 0 & -z & 0 & 0 \\ \langle f,[Z,D]\rangle & 0 & 0 & 0 & 0 \end{Vmatrix}.$$

It is obvious that $rank\ \varphi(t,x,y,z) \leq 2, \forall(t,x,y,z)$. Therefore we have

$$z\langle f,[D,Z]\rangle = 0, z\langle f,[D,T]\rangle = 0, \forall(t,x,y,z).$$

From these equalities it follows $[D, Z] = [D, T] = 0$, hence $rank\ ad_D^1 \leq 2$.

On the other hand, we see that $\mathcal{G}^2 = gen(Z)$. This means $ad_D^1(\mathcal{G}^1) + \mathcal{G}^2 \not\subseteq \mathcal{G}^1$. Thus, we obtain a contradiction. Lemma 7 is completely proved.

From Lemma 7 we obtain that if $\mathcal{G}$ is a 5-dimensional MD's Lie algebra with $dim\mathcal{G}^1 = 4$, then $\mathcal{G}^1$ must be isomorphic to R^4.

Suppose that $\{D, T, X, Y, Z\}$ is a basis of $\mathcal{G}$ and $\{T, X, Y, Z\}$ is a basis in $\mathcal{G}^1$ such that the matrix of ad_D^1 is one of the following real normal Jordan matrices:

$$i/ \left\| \begin{matrix} \alpha_1 & \beta_1 & 0 & 0 \\ -\beta_1 & \alpha_1 & 0 & 0 \\ 0 & 0 & \alpha_2 & \beta_2 \\ 0 & 0 & -\beta_2 & \alpha_2 \end{matrix} \right\|, (\beta_1, \beta_2 > 0), \quad ii/ \left\| \begin{matrix} \alpha & \beta & 1 & 0 \\ -\beta & \alpha & 0 & 1 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\beta & \alpha \end{matrix} \right\|, (\beta > 0),$$

$$iii/ \left\| \begin{matrix} \alpha & \beta & 0 & 0 \\ -\beta & \alpha & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{matrix} \right\|, \quad \begin{matrix} (\beta > 0, \\ \lambda_1\lambda_2 \neq 0), \end{matrix} \quad iv/ \left\| \begin{matrix} \alpha & \beta & 0 & 0 \\ -\beta & \alpha & 0 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{matrix} \right\|, (\beta > 0, \lambda \neq 0),$$

$$v/ \left\| \begin{matrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{matrix} \right\|, (\lambda_1\lambda_2\lambda_3\lambda_4 \neq 0), \quad vi/ \left\| \begin{matrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{matrix} \right\|, (\lambda_1\lambda_2\lambda_3 \neq 0),$$

$$vii/ \left\| \begin{matrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{matrix} \right\|, (\lambda_1\lambda_2 \neq 0), \quad viii/ \left\| \begin{matrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & \lambda_2 \end{matrix} \right\|, (\lambda_1\lambda_2 \neq 0),$$

$$ix/ \left\| \begin{matrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{matrix} \right\|, (\lambda \neq 0).$$

Now, changing D by

$$\frac{D}{\sqrt{\alpha_1^2+\beta_1^2}} \text{ in case i/,}$$
$$\frac{D}{\sqrt{\alpha^2+\beta^2}} \text{ in cases ii/,iii/,iv/,}$$
$$\frac{D}{\lambda_1} \text{ in cases v/,vi/,vii/,viii/,}$$
$$\frac{D}{\lambda} \text{ in the last case}$$

and making some simple transformations of the basis $\{T, X, Y, Z\}$, we can show that $\mathcal{G}$ is isomorphic to one of the following Lie algebras:

$$\mathcal{G}_{5.4.1}; \mathcal{G}_{5.4.2}; \mathcal{G}_{5.4.3}; \mathcal{G}_{5.4.4}; \mathcal{G}_{5.4.5}; \mathcal{G}_{5.4.6}; \mathcal{G}_{5.4.7}; \mathcal{G}_{5.4.8}; \mathcal{G}_{5.4.9}.$$

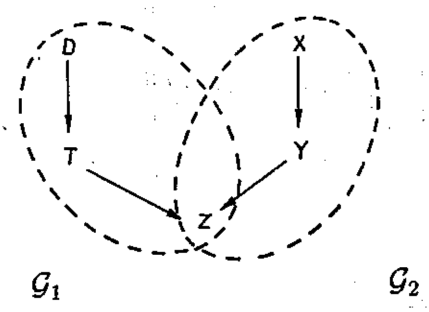
The proof of Theorem 5 is now complete.

REMARK.

1. Among the Lie algebras listed in Theorem 5 there are only two for which the non trivial K-orbits are 4-dimensional. They are $\mathcal{G}_{5.1} = \mathcal{H}_5$ and $\mathcal{G}_{5.3.7}$.

2. The structure of the Lie algebras $\mathcal{G}_{5.1}, \mathcal{G}_{5.2}$ and $\mathcal{G}_{5.3.5}$ may be described by the following schemes

$\mathcal{G}_{5.1} = \mathcal{H}_5$:

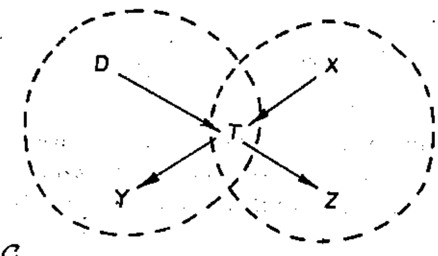


$$\mathcal{G}_1 \cong \mathcal{G}_2 \cong \mathcal{H}_3$$

$$\mathcal{H}_5 = \mathcal{G}_1 + \mathcal{G}_2$$

$$z(\mathcal{H}_5) = gen(Z) \text{ (centre of } \mathcal{H}_5)$$

$\mathcal{G}_{5.2}$:



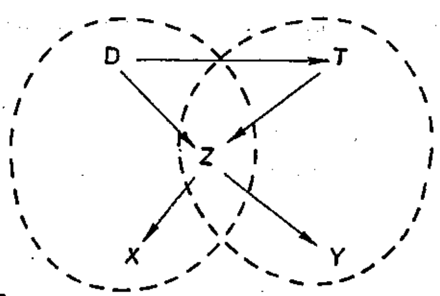
$\mathcal{G}_1$ $\mathcal{G}_2$

$$\mathcal{G}_1 \cong \mathcal{G}_2 \cong \mathcal{H}_3$$

$$\mathcal{G}_{5.2} = \mathcal{G}_1 + \mathcal{G}_2$$

$$\mathcal{G}^1_{5.2} = gen(Y, Z) \text{ is the centre of } \mathcal{G}_{5.2}$$

$\mathcal{G}_{5.3.5}$:



$\mathcal{G}_1$ $\mathcal{G}_2$

$$\mathcal{G}_1 \cong \mathcal{G}_2 \cong \mathcal{H}_3$$

$$[D, T] = Z, \qquad \mathcal{G}_{5.3.5} = \mathcal{G}_1 + \mathcal{G}_2$$

$$gen(X, Y) \text{ is the centre of } \mathcal{G}_{5.3.5}$$

3. The Lie algebras $\mathcal{G}_{5.3.7}, \mathcal{G}_{5.4.1}, \mathcal{G}_{5.4.2}, \mathcal{G}_{5.4.9}$ have no centre.

The Lie algebras $\mathcal{G}_{5.1}, \mathcal{G}_{5.3.1}, \mathcal{G}_{5.3.2}, \mathcal{G}_{5.3.4}, \mathcal{G}_{5.3.6}$ have one-dimensional centres.

The Lie algebras $\mathcal{G}_{5.2}$ and $\mathcal{G}_{5.3.5}$ have 2-dimensional centres.

4. By the method of this paper one can principally classify the MD's Lie Algebras of higher dimensions. Recently the 6-dimensional Lie Algebras of the class MD have been completely classified in [9].

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