ACTA MATHEMATICA VIETNAMICA
Volume 17, Number 2, 1992

ON MULTISTEP SEIDEL-NEWTON METHODS FOR QUASILINEAR OPERATOR EQUATIONS

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1. Introduction

In this paper we consider the following operator equation :

$$Ax = Fx, \tag{1.0}$$

where A is a bounded linear Fredholm operator (index zero) and F is a nonlinear operator from a Banach space X to another Banach space Y. It is well known that by the assumptions of A, we have $X = X_1 \oplus X_2,\ Y = Y_1 \oplus Y_2,\ X_2 = \text{Ker } A,\ Y_1 = \text{Im } A$, dim $X_2 < +\infty$ and Y_1, is closed in Y. Further, we can conclude that dim $X_2 =$ codim $Y_1 = m < +\infty$ and the restriction $\widetilde{A}$ of A on X_1 has a bounded inverse $\widetilde{A}^{-1}$.

Let us denote by P a bounded linear projection from Y on $Y_1,\ PY = Y_1,\ Q = (I - P)$, where I is the identity operator in Y. Then equation (1.0) is equivalent to the system :

$$\begin{cases} \widetilde{A}u = PF(u+v), \\ QF(u+v) = 0, \end{cases} \tag{1.1}$$

where $u \in X_1,\ v \in X_2$.

Note that the operator equation (1.0) has been investigated by many authors (see [1-5] for instance). In [2] it has been solved by an usual Seidel-Newton method and convergence theorems have been obtained. In this paper we will study the above mentioned equation (1.0) by using two multistep Seidel-Newton

Received January 10, 1992

methods, and we shall show that under some assumptions on F the rate of convergence of the approximate solutions to the exact one is quadratic.

2. First multistep Seidel-Newton method

Given initial value x_0, let us construct a sequence $(x_n)_n$ by using the following relations :

$$\begin{aligned} u_{n,1} &= \widetilde{A}^{-1}PFx_n, \\ u_{n,i+1} &= \widetilde{A}^{-1}PF(u_{n,i}+v_n),\ i=1,2,...,k-1, \\ u_{n,k} &= u_{n,1} = u_{n+1,0}, \\ v_{n+1} &= v_n - [QF'(u_{n+1}+v_n)]^{-1}_{X_2}QF(u_{n+1}+v_n), \\ x_{n+1} &= u_{n+1}+v_{n+1}, \end{aligned} \tag{2.1}$$

where $u_{n,1} \in X_1,\ \forall n \geq 0,\ i=1,...,k$ and $v \in X_2$.

THEOREM 1.1. *Let $F(x)$ be continuously differentiable (in the Gâteaux sense) on each segment $[a,b]$ lying in the ball $B=\{x|\ ||x-x_0||<R\}$ and for all $x \in B$,*

$$||PF'x||| \leq \alpha,\ ||QF'x|| \leq \beta$$

Assume that the restriction of $QF'x$ on X_2 has a uniform bounded inverse $[QF'x]^{-1}_{X_2}$, $|||[QF'x]^{-1}_{X_2}|| \leq \gamma$ and $||QF'x - QF'y|| \leq \rho(||x-y||)$, where $\rho : [0,\infty) \to [0,\infty)$ is a continuous nondecreasing function with $\rho(0)=0$. Let α be small enough and x_0 be chosen such that the following relations hold:

$$\begin{aligned} q_k =&[(||\widetilde{A}^{-1}||\alpha)^k + \sum_{i=1}^{2k-1}(||\widetilde{A}^{-1}||\alpha)^i]\gamma\beta + \gamma\int_0^1 \rho(\delta_k t)dt < 1, \\ &2\delta_k(1-q_k)^{-1} < R, \\ \delta_k =&[\sum_{i=0}^{k-1}(||\widetilde{A}^{-1}||\alpha)^i]||\widetilde{A}^{-1}||.||(A-PF)x_0||\gamma\beta + \gamma||QFx_0||. \end{aligned}$$

Then the sequence $(x_n)_n$ constructed by (2.1) converges to the solutions x^* of equation (1.0) and we obtain

$$||x_n - x^*|| \leq Rq_k^n. \tag{2.2}$$

PROOF: We will show by induction the following relations: $x_n \in B$, $\forall n \geq 0$, $x_{n,i} = u_{n,i} + v_n \in B\ \forall n \geq 0$, $i = 1, ..., k$, $||u_{n,1} - u_n|| \leq \delta_k q_k^n$, $||u_{n+1} - u_n|| \leq \delta_k q_k^n$, $||v_{n+1} - v_n|| \leq \delta_k q_k^n$.

It is clear that $x_0 \in B$. We have $||u_{0,1} - u_0|| = ||\widetilde{A}^{-1} PFx_0 - \widetilde{A}^{-1} Ax_0|| \leq ||A^{-1}\ ||||(PF - A)x_0|| < \delta_k$, thus $x_{0,1} \in B$. It is easy to see that

$$\begin{aligned}||u_{0,i+1} - u_{0,i}|| &\leq ||\widetilde{A}^{-1}||\ ||PFx_{0,i} - PFx_{0,i-1}|| \\ &\leq (||\widetilde{A}^{-1}||\alpha)^i ||u_{0,1} - u_0||,\ i = 1, ..., k-1.\end{aligned}$$

Hence

$$\begin{aligned}||x_{0,i+1} - x_0|| &\leq ||x_{0,i+1} - x_{0,i}|| + ... + ||x_{0,1} - x_0|| \\ &\leq (||\widetilde{A}^{-1}||\alpha)^i ||u_{0,1} - u_0|| + ... + ||u_{0,1} - u_0||.\end{aligned}$$

Therefore

$$||x_{0,i+1} - x_0|| \leq ||\widetilde{A}^{-1}||\ ||(PF - A)x_0||\ [1 + ||A^{-1}||\alpha + ... + (||A^{-1}||\alpha)^i].$$

It follows that $||x_{0,i+1} - x_0|| \leq \delta_k$, hence $x_{0,i+1} \in B$, $i = 1, ..., k-1$. When $i = k-1$, we have $||x_{0,k} - x_0|| \leq \delta_k < R$.

Let us estimate $||v_1 - v_0||$. Obviously

$$||v_1 - v_0|| \leq \gamma ||QF(x_{0,k})|| \leq \gamma\beta ||u_1 - u_0|| + \gamma ||QFx_0|| = \delta_k.$$

Whence $||x_1 - x_0|| \leq 2\delta_k < R$, i.e. $x_1 \in B$.

Assuming that the assertion is valid for $m \leq n-1$, we shall prove it for n. Indeed, since $x_{n-1,i} \in B$, $i = 1, ..., k$, we have $x_{n-1,k} = u_n + v_{n-1} \in B$ and

$$||u_n - u_{n-1}|| \leq \delta_k q_k^{n-1},\ ||v_n - v_{n-1}|| \leq \delta_k^{n-1}.$$

Therefore

$$||x_n - x_0|| \leq \sum_{j=0}^{n-1} ||x_{j+1} - x_j|| \leq 2\delta_k \sum_{j=0}^{n-1} q_k^j < R,$$

i.e. $x_n \in B$. Now let us estimate $||u_{n,1} - u_n||$. We have

$$||u_{n,1} - u_n|| = ||\widetilde{A}^{-1}PFx_n - \widetilde{A}^{-1}PFx_{n-1,k-1}|| \leq$$
$$\leq ||\widetilde{A}^{-1}||\alpha[||u_{n-1,k} - u_{n-1,k-1}|| + ||v_n - v_{n-1}||]$$
$$(||\widetilde{A}^{-1}||\alpha)^k||u_{n-1,1} - u_{n-1}|| + ||\widetilde{A}^{-1}||\alpha||v_n - v_{n-1}|| \leq \delta_k q_k^n.$$

It is not difficult to see that

$$||u_{n,i+1} - u_{n,i}|| \leq (||\widetilde{A}^{-1}||\alpha)^i||u_{n,1} - u_n|| \text{ for } i = 1, ..., k-1.$$

Hence

$$||u_{n,i+1} - u_{n,i}|| \leq (||\widetilde{A}^{-1}||\alpha)^{k+i}\delta_k q_k^{n-1} + (||\widetilde{A}^{-1}||\alpha)^{i+1}\delta_k q_k^{n-1},$$

for $i = 1, ..., k-1$. It follows that

$$||u_{n,i+1} - n_n|| \leq \sum_{j=0}^{i} ||u_{n,j-1} - n_{n,j}|| \leq \delta_k q_k^{n-1}[||\widetilde{A}^{-1}||\alpha+$$
$$+ (||\widetilde{A}^{-1}||\alpha)^2 + ... + (||\widetilde{A}^{-1}||\alpha)^{i+1} + (||\widetilde{A}^{-1}||\alpha)^k + (||\widetilde{A}^{-1}||\alpha)^{k+1} + ...+$$
$$+ (||\widetilde{A}^{-1}||\alpha)^{k+1}] \leq \delta_k q_k^n \text{ for } i = 1, ..., k.$$

Thus $x_{n,i} \in B, i = 1, .., (k-1)$. For $i = k$ we have $||u_{n+1} - u_n|| \leq \delta_k q_k^n$. Furthermore, we obtain

$$||v_{n+1} - v_n|| \leq \gamma||QF(x_{n,k})|| \leq \gamma||QF(x_{n,k}) - QFx_n|| + \gamma||QFx_n||$$
$$\leq \gamma\beta||v_{n+1} - v_n|| + \gamma||QFx_n -$$
$$- QF(x_{n-1,k}) - QF'(x_{n-1,k})(v_n - v_{n-1})||$$
$$\leq \gamma\beta||v_{n+1} - v_n|| + \gamma \int_0^1 ||QF'[x_{n-1,k} + t(v_n - v_{n-1})] -$$
$$- QF'(x_{n-1,k})||\ ||v_n - v_{n-1}||dt$$
$$\leq \gamma\beta[(||\widetilde{A}^{-1}||\alpha)^k + \sum_{i=1}^{2k-1} (||\widetilde{A}^{-1}||\alpha)^i]\delta_k q_k^{n-1} + (\gamma \int_0^1 \rho(t\delta_k)dt)\delta_k q_k^{n-1}$$
$$\leq \delta_k q_k^n,$$

i.e. $x_{i+1} \in B$ (Note that $||x_n - x_0|| \leq \sum_{i=0}^{n-1} ||x_{i+1} - x_i|| \leq 2\delta_k \sum_{i=1}^{n} q_k^i < R$). Therefore

$$||x_{n+m} - x_n|| \leq \sum_{i=0}^{m-1} ||x_{n+i+1} - x_{n+i}|| \leq 2\delta_k q_k^n (1 - q_k^m)(1 - qq_k)^{-1} < Rq_k^n.$$

Passing to the limit as $m \to \infty$ we get

$$||x_n - x^*|| \leq Rq_k^n.$$

Taking into account the continuity of A and F we can show that x^* is the solution of Equation (1.0). The proof is complete.

THEOREM 1.2. *Let $F(x)$ be continuously differentiable (in the Gâteaux sense) on each segment $[a, b]$ in a neighbourhood of the solution x^* of equation (1.0). Moreover, assume that the restriction of the $QF'(x^*)$ to X_2 has a bounded inverse and*

$$||QF'x^*|| \, ||[QF'x^*]_{X_2}^{-1}||[\sum_{i=1}^{2k-1} (||\widetilde{A}^{-1}|| \, ||PF'x^*||)^i + (||\widetilde{A}^{-1}|| \, ||PF'x^*||)^k] < 1.$$

If x_0 is sufficiently close to x^*, then the sequence $(x_n)_n$ constructed by (2.1) converges to x^* and we have the estimate (2.2) with $0 < q_k < 1$.

THEOREM 1.3. *(Rate of convergence). Let $F(x)$ be continuously differentiable (in the Gâteaux sense) on each segment $[a, b]$ in a neighbourhood Ω of the solution x^* of Equation (1.0). Assume that PF' is Lipschitz continuous on each segment $[a, b]$ in Ω with constant K and $PF'x^* = 0$. Furthermore, assume that $||QF'x|| \leq \beta$, $\forall x \in \Omega$ and $QF'x$ is Lipschitz continuous with constant L and $||[QF'x]_{X_2}^{-1}|| \leq \gamma$, $\forall x \in \Omega$. If x_0 is sufficiently close to x^*, then the sequence $(x_n)_n$ constructed by (2.1) converges to x^* with quadratic rate.*

PROOF.: By the assumptions, there exists a ball $B(x^*, R) \subset \Omega$ such that $||PF'x|| \leq \alpha$, $||QF'x|| \leq \beta$, $||[QF'x]_{X_2}^{-1} \leq \gamma$, $||QF'x - QF'y|| \leq \epsilon$ for all $x, y \in B(x^*, R)$ and

$$q_k = \beta\gamma[\sum_{i=1}^{2k-1} (||\widetilde{A}^{-1}||\alpha)^i + (||\widetilde{A}^{-1}||\alpha)^k] + \epsilon\gamma < 1.$$

Choose x_0 such that $||x_0-x^*|| \leq ||u_0-u^*||+||v_0-v^*|| < \delta_k$ with $2\delta_k(1-q_k)^{-1} < R$. It can be verified that $x_n, x_{n,i}$ are defined and belong to $B(x^*, R)$ for all $n \geq 0,\ i = 1, 2, ..., k$. Then we have

$$\begin{aligned}||u_{0,1} - u^*|| &= ||\widetilde{A}^{-1}PFx_0 - \widetilde{A}^{-1}PFx^*||\\ &\leq ||\widetilde{A}^{-1}||. \int_0^1 ||PF'[x^* + t(x_0 - x^*)](x_0 - x^*)||dt\\ &\leq ||\widetilde{A}^{-1}|| \int_0^1 ||PF'[x^* + t(x_0 - x^*)] - PF'(x^*)||\ ||x_0 - x^*||dt\\ &\leq ||\widetilde{A}^{-1}||K.||x_0 - x^*||/2 \leq (||\widetilde{A}^{-1}||K)\delta_k/2.\end{aligned}$$

Consequently,

$$||x_{0,1} - x^*|| \leq \delta_k(||\widetilde{A}^{-1}||K\delta_k/2 + 1).$$

From this we have

$$||u_{0,2} - u^*|| \leq (K||\widetilde{A}^{-1}||/2)||x_{0,1} - x^*||^2 \leq \delta^2(||\widetilde{A}^{-1}||K/2)\delta_k + ||G||]^2.$$

If we set $||\widetilde{A}^{-1}||K/2 = f_1,\ f_1(Rf_1 + 1)^2 = f_2$, then

$$||u_{0,1} - u^*|| \leq f_1.\delta_k^2,\ ||u_{0,2} - u^*|| \leq f_2\delta_k^2.$$

By recursion it is not difficult to show that $||u_{0,k} - u^*|| = ||u_1 - u^*|| \leq \delta_k^2 f_k$ with $f_i = f_1[Rf_{i-1} + ||G||],\ i = 2, ..., k$. Clearly,

$$||v_1 - v^*|| \leq \gamma\beta||u_1 - u^*|| + \gamma\frac{L}{2}(||u_1 - u^*|| + ||v_0 - v^*||)\ ||v_0 - v^*||.$$

Hence

$$||v_1 - v^*|| \leq \delta_k^2[\gamma\beta f_k + \gamma\frac{L}{2}f_kR + \gamma\frac{L}{2}||G||^2],$$

where G is a bounded linear projection from X on $X_2, G(x) = v$ for each $x = u + v,\ u \in X_1,\ v \in X_2$. If we choose δ_k such that at the same time we have $\delta_k(1 - q_k)^{-1} < R$ and $\delta_k C < 1$ with $C = [f_k + \gamma\beta f_k + \gamma L f_k R + \gamma L||G||^2]$, then we get

$$||x_1 - x^*|| \leq C\delta_k^2 = (1/C)\omega^2, \quad \text{with} \quad \omega = C\delta_k < 1.$$

Now, by induction we will prove that $||x_n - x^*|| \le (1/C)\omega^{2^n}$. Assume that the assertion holds $n-1$. Then

$$\begin{aligned}||u_{n-1,1} - u^*|| &\le (||\widetilde{A}^{-1}||K/2)||x_{n-1} - x^*||^2 = f_1||x_{n-1} - x^*||^2,\\ ||u_{n-1,2} - u^*|| &\le (||\widetilde{A}^{-1}||K/2)||x_{n-1} - x^*||^2[((||\widetilde{A}^{-1}||K)/2)||x_{n-1} - x^*||+\\ &\quad + ||G||]^2\\ &\le ||x_{n-1} - x^*||^2 f_1.[f_1.R + ||G||] = ||x_{n-1} - x^*||^2 f_2,\end{aligned}$$

and so on, and $||u_{n-1,k} - u^*|| \le ||x_{n-1} - x^*||^2 f_k$ with f_k defined as above. Moreover,

$$||v_n - v^*|| \le \gamma\beta||u_n - u^*|| + \gamma\frac{L}{2}||u_n - u^*||\ ||v_{n-1} - v^*|| + \gamma\frac{L}{2}||v_{n-1} - v^*||^2.$$

It follows that

$$||v_n - v^*|| \le [\gamma\beta f_k + \gamma\frac{LR}{2}f_k + \gamma\frac{L}{2}||G||^2]||x_{n-1} - x^*||^2.$$

From this we get

$$||x_n - x^*|| \le ||x_{n-1} - x^*||^2[f_k + \gamma\beta f_k + \gamma LR f_k + \gamma\frac{L}{2}||G||^2]$$

i.e. $||x_n - x^*|| \le (1/C)\omega^{2^n}$. The proof is complete.

REMARK I: 1) The Seidel-Newton method used in [2] is a special case of method (2.1) when $k = 1$. Theorems 2.1 and 2.2 of [2] are special cases of Theorems 1.1 and 1.2 in this paper when $k = 1$. However, Theorem 1.3 concerning the rate of convergence is new, it says that the rate is quadratic.

2) Under the above assumptions, the $(k+1)$-step method is, in general, better than the k-step method in the following sense: If the k-step method is applicable to a class of operators then $(k+1)$-step method can also be applied to this class. Moreover, for the following problem the k-step method (for some $k \ge 2$) is applicable, whereas the Seidel-Newton method used in [2] (the 1-step method) is not applicable.

Consider a nonlinear equation

$$Ax = Fx \tag{2.1}$$

in the real Hilbert space ℓ^2, where $Ax = (0, \xi_2, \xi_3, ..., \xi_k, ...)$ and $F(x) = (\frac{\xi_1}{100} + \frac{\xi_2}{30}, \frac{1}{12}(\xi_2 + \sin\xi_2), ..., \frac{1}{12}(\xi_k + \sin\xi_k), ...)$ for $x = (xi_1, \xi_2, .., \xi_k, ...) \in \ell^2$.

It can be verified that

$$\mathrm{Ker}A = X_2 = \{x \in \ell^2 | x = (\xi_1, 0, ..., 0, ...)\} = Y_2,$$

$$X_1 = Y_1 = \{x \in \ell^2 | x = (0, \xi_2, ..., \xi_k, ...)\}, \ ||A|| = 1, \ ||\hat{A}^{-1}|| = 1$$

$$QF(x) = (\frac{\xi_1}{100} + \frac{\xi_2}{30}, 0, ..., 0, ...), \ PFx = (0, \frac{1}{12}(\xi_2 + \sin\xi_2), ..., \frac{1}{12}(\xi_k + \sin\xi_k), ...).$$

This problem has the solution $x^* = (0, 0, ..., 0, ...)$ and $||[QF'(x^*)]^{-1}_{X_2}|| = 100$, $\frac{1}{30} \leq ||QF'(x^*)|| \leq \frac{\sqrt{109}}{300}$, $||PF'(x^*)|| = \frac{1}{6}$. It is obvious that

$$2||\hat{A}^{-1}||.||PF'(x^*)||.||QF'(x^*)|| \ ||[QF'x^*]^{-1}_{X_2}|| \geq \frac{10}{9} > 1$$

and for each $k \geq 2$,

$$||QF'(x^*)||.||[QF'x^*]^{-1}_{X_2}||.[\sum_{i=1}^{2k-1}(||\hat{A}^{-1}||.||PF'x^*||)^i + (||\hat{A}^{-1}||.||PF'x^*||)^k] < 1.$$

Hence using Theorem 1.2, it is easy to see that if the initial approximation x_0 is sufficiently close to x^*, then the sequence (x_n), constructed by the formula (2.1) converges to a solution of (1.1). Observe further that the Seidel-Newton method in [2] is not applicable.

3. Second multiple Seidel-Newton method

Given initial value x_0, let us construct the sequence $(x_n)_n$ by using the folowing relations:

$$\begin{aligned}
&v_{n,1} = v_n - [QF'x_n]^{-1}_{X_2} QFx_n,\\
&v_{n,i+1} = v_{n,i} - [QF'(u_n + v_{n,i})]^{-1}_{X_2} QF(u_n + v_{n,i}), \ i = 0, ..., k-1,\\
&v_{n,k} = v_{n+1} = v_{n+1,0},\\
&u_{n+1} = -\widetilde{A}^{-1} PF(u_n + v_{n+1}),\\
&x_{n+1} = u_{n+1} + v_{n+1},
\end{aligned} \tag{3.1}$$

where $v_{n,i} \in X_2, \ \forall n \geq 0, \ i = 0, ..., k, \ u_n \in X_1 \ \forall n \geq 0$

By an argument analogous to that used in the previous section, we get :

THEOREM 3.1. *(Convergence theorem). Let $F(x)$ be continuously differentiable (in the Gâteaux sense) on each segment $[a,b]$ in $B = \{x| \; ||x - x_0|| < R\}$ and $||PF'x|| \leq \alpha$, $||[QF'x]_{X_1}|| \leq \beta$, $\forall x \in B$. Assume that $[QF'x]^{-1}_{X_2}|| \leq \gamma$ and $[QF'x]_{X_2}$ is Lipschitz continuous with constant L in B (note that $[QF'x]_{X_2}$ is the restriction of $QF'x$ on the subset X_i). If*

$$\begin{aligned} q_k &= \max\{q_{k,1}, \; q_{k,2}\} < 1, \\ 2\delta_k(1 - q_k)^{-1} &< R, \\ \delta_k &= \max\{\delta_{k,1}, \; \delta_{k,2}\} \end{aligned}$$

with

$$\begin{aligned} q_{k,1} &= [\gamma\beta + (\delta_k\gamma L/2)^{2^{k-1}} + (\delta_k\gamma L/2)^{2^k-1}] \sum_{i=1}^{k-1} (\delta_k\gamma L/2)^{2^i-1}, \\ q_{k,2} &= ||\widetilde{A}^{-1}||\alpha + ||\widetilde{A}^{-1}||\alpha . q_{k,1}, \\ \delta_{k,1} &= \gamma||QFx_0|| \sum_{i=0}^{k-1} (\gamma L/2)^2\gamma||QFx_0||^{2^i-1}, \\ \delta_{k,2} &= ||\widetilde{A}^{-1}||\alpha\delta_{k,1} + ||\widetilde{A}^{-1}|| \; ||(A - PF)x_0||, \end{aligned}$$

then the sequence $(x_n)_n$ constructed by (3.0) converges to the solution x^* of equation (1.0) and we have

$$||x_n - x^*|| \leq Rq_k^n.$$

THEOREM 3.2. *Let $F(x)$ be continuously differentiable (in the Gâteaux sense) on each segment $[a,b]$ in a neighbourhood Ω of the solution x^* of equation (1.0). Moreover assume that the following inequalities hold*

$$\begin{aligned} &||[QF'x^*]^{-1}_{X_2}||.||[QF'x^*]_{X_1}|| < 1, \\ &||\widetilde{A}^{-1}||.||PF'x^*||(1 + ||[QF'x^*]^{-1}_{X_2}||.||[QF'x^*]_{X_1}||) < 1. \end{aligned}$$

If x_0 is sufficiently closed to x^*, then the sequence $(x_n)_n$ defined by (3.1) converges to x^* and we have the estimate

$$||x_n - x^*|| \le Cq^n,$$

with $0 < q < 1$, and C is a constant independent of n.

THEOREM 3.3. *(Rate of convergence). Let $F(x)$ be continuously differentiable (in the Gâteaux sense) on each segment $[a, b]$ in a neighbourhood Ω of the solution x^* of equation (1.0). Assume that $PF'x$ is Lipschitz continuous on each segment $[a, b]$ in Ω with constant K and $PF'x^* = 0$. Furthermore assume that $||[QF'x]_{X_1}|| \le \beta$, $||[QF'x]_{X_1}^{-1}|| \le \gamma$, $\forall x \in \Omega$, and $QF'x$ is Lipschitz continuous with constant L. If x_0 is sufficiently close to x^*, then the sequence $(x_n)_n$ defined by (3.1) converges to x^* with quadratic rate.*

REMARK II: 1) When $k = 1$, method (3.1) turns out to be another variant of the Seidel-Newton Method used in [2]. Nevertheless, Theorem 3.3 concerning the rate of convergence is new.

2) With $\gamma L\ \delta/2$ small enough (x_0 sufficiently closed to solution x^*), the $(k+1)$-Step Method is better than the k-Step Method (since $q_{k+1} < q_k$) in the sense stated in Remark I2).

3) It can be said that the two above multistep Seidel-Newton Methods are dual. To apply method (2.1) it is necessary that $||\widetilde{A}^{-1}||\alpha$ is sufficiently small, and for method (3.1) we need the smallness of $\gamma\beta$ and the Lipschitz-continuity of $[QF'x]_{X_2}$.

4. Periodic boundary-value problems

Consider the following periodic boundary-value problem

$$\begin{cases} \ddot{x} = f(t, x, \dot{x}, \ddot{x}), \ 0 < t < 1, \\ x(0) = x(1), \ \dot{x}(0) = \dot{x}(1). \end{cases} \tag{4.1}$$

Problem (4.1) may be reduced to the form (1.0) by introducing the following spaces and operators :

$$X = \{x \in C^2[0,1] |\ x(0) = x(1),\ \dot{x}(0) = \dot{x}(1)\},$$
$$Y = C[0,1],$$
$$||x||_X = \max_{0\le t\le 1} |x(t)| + \max_{0\le t\le 1} |\dot{x}(t)| + \max_{0\le t\le 1} |\ddot{x}(t)|,$$
$$||y||_Y = \max_{0\le t\le 1} |y(t)|,$$
$$X_1 = \{u \in X|\int_0^1 u(s)ds = 0\},\ Y_1 = \{y \in Y|\int_0^1 y(s)ds = 0\},$$
$$X_2 = Y_2 = \{\text{const}\},\ A : X \to Y,\ Ax = \ddot{x},\ F : X \to Y,$$
$$F(x) = f(t, x, \dot{x}, \ddot{x}).$$

It can be verified that A is a bounded linear Fredholm operator with Ker $A = X_2$, Im $A = Y_1$, $X = X_1 \oplus X_2$, $Y = Y_1 \oplus Y_2$. Moreover the restriction $\hat{A}$ of A to X_1 has a bounded inverse $\hat{A}^1$ and $||\hat{A}^{-1}|| \le \frac{25}{12}$ (see [1], [2]). Suppose that the funtion $f(t, x, \xi_1, \xi_2)$ is continuous in t and continuously differentiable in the remaining variables and for all pairs (t, x, ξ_1, ξ_2), $(t, \bar{x}, \bar{\xi}_1, \bar{\xi}_2) \in I$, where

$$I = \{(t, x, \xi_1, \xi_2) \mid\ 0 \le t \le 1,\ |x| \le R,\ |\xi_1| \le R,\ |xi_2| < R\},$$
$$|\ \frac{\partial f(t, x, \xi_1, \xi_2)}{\partial x}\ | < \alpha,\ |\ \frac{\partial f(t, x, \xi_1, \xi_2)}{\partial \xi_1}\ | < \alpha,\ |\ \frac{\partial f(t, x, \xi_1, \xi_2)}{\partial \xi_2}\ | \le \alpha,$$
$$|\ \frac{\partial f(t, x, \xi_1, \xi_2)}{\partial x} - \frac{\partial f(t, \bar{x}, \bar{\xi}_1, \bar{\xi}_2)}{\partial x}\ | \le L(|x - \bar{x}| + |\xi_1 - \bar{\xi}_1| + |\xi_2 - \bar{\xi}_2|),$$
$$|\ \frac{\partial f(t, x, \xi_1, \xi_2)}{\partial \xi_i} - \frac{\partial f(t, \bar{x}, \bar{\xi}_1, \bar{\xi}_2)}{\partial \xi_i}\ | \le L(|x - \bar{x}| + |\xi_1 - \bar{\xi}_1| + |\xi_2 - \bar{\xi}_2|), i = 1, 2.$$

Further, assume that $\frac{\partial f}{\partial x}(t, x, \xi_1, \xi_2) \ge a(t)$ for each $(t, x, \xi_1, \xi_2) \in I$, where $\int_0^1 a(s)ds \equiv \gamma^{-1} > 0$.

Then $F(x)$ is continuously differentiable in the closed ball $S = \{x \in X|\ ||x||_X \le R\}$ and for all $x, y \in S$, $||PF'x|| \le 2\alpha$, $||QF'x|| \le \alpha$, $||QF'x - QF'y|| \le L||x - y||$. Moreover, the restriction of $QF'(x)$ to X_2 has a uniformly bounded inverse $||[QF'x]^{-1}_{X_2}|| \le \gamma$ for each $x \in S$.

From Theorem 1.1 we get the following

THEOREM 4.1. *Suppose that the above conditions are satisfied. If moreover,*

$$q_k = [(\frac{25}{6}\alpha)^k + \sum_{i=1}^{2k-1}(\frac{25}{6}\alpha)^i]\alpha\gamma + \frac{\gamma L\delta}{2} < 1$$

$$2\delta_k(1-q_k)^{-1} < R$$

$$\delta_k = [\sum_{i=0}^{k-1}(\frac{25}{6}\alpha)^i]\frac{25}{12}\|f(t,0,0,0) - \int_0^t f(t,0,0,0)dt\|\|\gamma\alpha + \gamma|\int_o^1 f(t,0,0,0)dt|,$$

then the sequence $\{x_n\}$, constructed by the following formulas

$$\begin{cases} x_0 \equiv 0 \quad \text{for all} \quad t \in [0,1] \\ \ddot{u}_{n,1} = f(t, x_n, \dot{x}, \ddot{x}_n) - \int_0^1 f(s, x_n, \dot{x}_n, \ddot{x}_n)ds, \\ \ddot{u}_{n,i+1} = f(t, u_{n,i} + v_n, \dot{u}_{n,i}, \ddot{u}_{n,i}) - \int_0^1 f(s, u_{n,i}, \dot{u}_{n,i}, \ddot{u}_{n,i})ds, \\ \qquad i = 1, ..., (k-1), \ u_{n,k} = u_{n+1} \\ u_{n,i}(0) = u_{n,i}(1), \ \dot{u}_{n,i}(0) = \dot{u}_{n,i}(1), \ i = 1, ...k. \\ v_{n+1} = v_n - \frac{\int_0^1 f(s, u_{n+1}+v_n, \dot{u}_{n+1}, \ddot{u}_{n+1})ds}{\int_0^1 \frac{\partial f}{\partial x}(s, u_{n+1}+v_n, \dot{u}_{n+1}, \ddot{u}_{n+1})ds}, \\ x_{n+1}(t) = u_{n+1}(t) + v_{n+1}, \\ u_{n,i} \in X_1, \ \forall n \geq 0, \ i = 1...k, \ v_n \in X_2, \end{cases}$$

converges to a solution of (4.1) and the estimate (2.2) holds.

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