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HAMILTON CYCLES IN CUBIC $(4, n)$-METACIRCULANT GRAPHS

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Abstract. It is shown that every connected cubic $(4, n)$-metacirculant graph has a Hamilton cycle.

1. Introduction

There are only four known nontrivial vertex-transitive graphs which do not have any Hamilton cycle. These graphs are the Petersen graph, the Coxeter graph [8, p. 241] and the graphs obtained from them by replacing every vertex by a triangle. L. Babai [6] has asked whether there are infinitely many such graphs. C. Thomassen [7, p. 163] has conjectured that there are not. None of these four graphs is a Cayley graph so that it may be conjectured that every connected Cayley graph on a finite group has a Hamilton cycle. This has been shown to be true at least for abelian groups [9] and for some other special groups [10, 12].

The class of (m, n)-metacirculant graphs was introduced in [1] as an interesting class of vertex-transitive graphs which includes many non-Cayley graphs and probably further examples of non-hamiltonian graphs. In particular, the Petersen graph is $(2, 5)$-metacirculant. It is reasonable to ask [1, 2] whether every connected (m, n)-metacirculant graph which is different than the Petersen graph has a Hamilton cycle.

There are several papers that study the above question. In [2, 3] an affirmative answer was obtained for prime n. Connected cubic (m, n)-metacirculant graphs different than the Petersen graph are also proved to be hamiltonian for m odd [11] and $m = 2$ [4, 11]. In this paper we will consider the above question

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for connected cubic $(4, n)$-metacirculants. We will prove that every connected cubic $(4, n)$- metacirculant graph has a Hamilton cycle.

2. Preliminaries

a) *(m, n)-metacirculant graphs*

The reader is referred to [1] for basic properties of (m, n)-metacirculant graphs. Here we will only describe their construction.

We will denote the ring of integers modulo n by Z_n and the multiplicative group of units in Z_n by Z_n^*. Let m and n be two positive integers, $\alpha \in Z_n^*$, $\mu = \lfloor m/2 \rfloor$, and $S_0, S_1, \ldots, S_\mu$ subsets of Z_n satisfying the following conditions:

(1) $0 \notin S_0 = -S_0$.

(2) $\alpha^m S_r = S_r$ for $0 \leq r \leq \mu$.

(3) If m is even, then $\alpha^\mu S_\mu = -S_\mu$.

Then we define the (m, n)-metacirculant graph $G = MC(m, n, \alpha, S_0, S_1, \ldots, S_\mu)$ to be the graph with vertex-set $V(G) = \{v_j^i \mid i \in Z_m; j \in Z_n\}$ and edge-set $E(G) = \{v_j^i v_h^{i+r} \mid 0 \leq r \leq \mu; \quad i \in Z_m; \quad h, j \in Z_n \text{ and } (h - j) \in \alpha^i S_r\}$, where superscripts and subscripts are always reduced modulo m and modulo n, respectively.

The set $V^i = \{v_j^i \mid j \in Z_n\}$ is called the i-th block of vertices of G.

The above construction is designed to allow the permutations ρ and τ on $V(G)$ defined by $\rho(v_j^i) = v_{j+1}^i$ and $\tau(v_j^i) = v_{\alpha j}^{i+1}$ to be automorphisms of G. It is clear that the group $< \rho, \tau >$ generated by ρ and τ is transitive on $V(G)$. So G is vertex-transitive.

b) *Quotient graphs*

The concept of quotient graphs with respect to a semiregular automorphism was introduced in [5].

A permutation β is said to be semiregular if all cycles in the disjoint cycle decomposition of β have the same length. If a graph G has a semiregular automorphism β, then the quotient graph G/β is defined as follows. The vertices of G/β are the orbits of the subgroup $< \beta >$ generated by β and two such vertices are adjacent if and only if there is an edge in G joining a vertex of a corresponding orbit to a vertex in the other orbit.

Because of the action of β, a vertex of G/β corresponds to a circulant subgraph of G and an edge of G/β corresponds to a perfect matching between the corresponding orbits of $< \beta >$.

Let β be of order t and $G^0, G^1, \ldots, G^h$ be the subgraphs induced by G on the orbits of $< \beta >$. Let $v_0^i, v_1^i, \ldots, v_{t-1}^i$ be a cyclic labelling of the vertices of G^i under the action of β and $C = G^0 G^i G^j \ldots G^r G^0$ be a cycle of G/β. We consider paths of G arising from a lifting of C. Such a path starts from v_0^0 and goes to a vertex v_a^i of G^i, then to a vertex v_b^j of G^j following G^i in C, and so on until it returns to a vertex v_d^0 of G_0. The set of all paths that can be constructed in this way is called [5] the coil of C and is denoted by coil (C).

LEMMA 1. *Let t be the order of a semiregular automorphism β of a graph G and G^0 be the subgraph induced by G on an orbit of $< \beta >$. If there exists a Hamilton cycle C in G/β such that coil (C) contains a path P whose terminal vertices are distance d apart in G^0 where P starts and terminates and gcd $(d,t) = 1$, then G has a Hamilton cycle.*

PROOF: Let $C = G^0 G^i G^j \ldots G^r G^0$ be a Hamilton cycle of G/β and $P = v_0^0 v_a^i v_b^j \ldots v_c^r v_d^0$ a path in coil (C) with gcd $(d,t) = 1$. Denote by $P(v_e^0)$ the path $v_e^0 v_{a+e}^i v_{b+e}^j \cdots v_{c+e}^r$. Since $\gcd(d,t) = 1$, the vertices $v_d^0, v_{2d}^0, v_{3d}^0, \ldots, v_{(t-1)d}^0$, $v_{td}^0 = v_0^0$ are all the vertices of G^0. Therefore $P(v_0^0)P(v_d^0)P(v_{2d}^0)\ldots P(v_{(t-1)d}^0)v_0^0$ is a Hamilton cycle of G.

c) *Condition for connectedness*

LEMMA 2. *Let $G = MC(4, n, \alpha, S_0, S_1, S_2)$ be a $(4, n)$ -metacirculant graph such that $S_0 = \emptyset, S_1 = \{s\}$ with $0 \le s < n$ and $S_2 = \{k\}$ with $0 \le k < n$. Then G is connected if and only if gcd $(k - s(1+\alpha), n) = 1$.*

PROOF: To begin with, we note that if $v_{a(1)}^1$ and $v_{a(2)}^1$ of the block V^1 are joined by a path $v_{a(1)}^1 v_{b(1)}^2 v_{c(1)}^3 v_{a(2)}^1$, then $a(2) \equiv a(1) + \alpha s + \alpha^2 s + \alpha^3 k \equiv a(1) - \alpha(k - s(1+\alpha))(\bmod n)$, i.e., they are distance f apart in V^1 with f to be a multiple of $(k - s(1+\alpha))$. Similarly, if $v_{a(1)}^1$ and $v_{a(2)}^1$ of the block V^1 are joined by a path $v_{a(1)}^1 v_{c(1)}^3 v_{b(1)}^2 v_{a(2)}^1$, then $a(2) \equiv a(1) + \alpha k - \alpha^2 s - \alpha s \equiv a(1) + \alpha(k - s(1+\alpha))$ (mod n), i.e., they are also distance f apart in V^1 with f to be a multiple of $(k - s(1+\alpha))$. Denote by $\overline{Q}(v_{a(i)}^1)$ the path $v_{a(i)}^1 v_{b(i)}^2 v_{c(i)}^3$ and by $\overline{\overline{Q}}(v_{a(i)}^1)$ the path $v^1_{a(i)}v^3_{c(i)}v^2_{b(i)}$. Let $v^1_{a(1)}$ and $v^1_{a(d)}$ be two vertices of V^1 joined by a path Q not containing any vertices of V^0. Then it is not difficult to see that Q has either the form

$$Q = \overline{Q}(v^1_{a(1)})\overline{Q}(v^1_{a(2)})\dots\overline{Q}(v^1_{a(d-1)})v^1_{a(d)}$$

or the form

$$Q = \overline{\overline{Q}}(v^1_{a(1)})\overline{\overline{Q}}(v^1_{a(2)})\dots\overline{\overline{Q}}(v^1_{a(d-1)})v^1_{a(d)}.$$

Therefore, they are distance f apart in V^1 with f to be a multiple of $(k-s(1+\alpha))$. Similar conclusions are also true for vertices from the block V^2 and the block V^3.

Assume that gcd $(k-s(1+\alpha),n) = e > 1$. Consider a path

$$P = v^0_0 v^{i(1)}_{a(1)} v^{i(2)}_{a(2)} \dots v^{i(h-1)}_{a(h-1)} v^0_{a(h)}$$

such that every $v^{i(x)}_{a(x)}$ is not a vertex of V^0. Then P must be of one of the types listed in Table 1. By the remarks in the preceding paragraph we can calculate $a(h)$, the subscript of $v^0_{a(h)}$ of P. These detailed calculations for the respective cases are given in Table 2.

From Table 2 we see that in such a path P, the vertex v^0_0 is joined to v^0_a with a to be a multiple of $(k-s(1+\alpha))$. Therefore, among vertices of V^0, v^0_0 is joined by a path in G only to a vertex v^0_a with a to be a multiple of $(k-s(1+\alpha))$. Since $\gcd(k-s(1+\alpha),n) = e > 1$, among vertices of V^0, there exists a vertex v^0_a which is not joined to v^0_0 by any path in G. This means that G is not connected.

Conversely, assume that gcd $(k-s(1+\alpha),n) = 1$. Let

$$R(v^0_a) = v^0_a v^2_{(a+k)} v^1_{(a+k-\alpha s)} v^0_{(a+k-s(1+\alpha))}.$$

Then we can join v^0_0 to $v^0_{(k-s(1+\alpha))}$ by $R(v^0_0), v^0_{(k-s(1+\alpha))}$ to $v^0_{2(k-s(1+\alpha))}$ by $R(v^0_{(k-s(1+\alpha))})$, $v^0_{2(k-s(1+\alpha))}$ to $v^0_{3(k-s(1+\alpha))}$ by $R(v^0_{2(k-s(1+\alpha))}), \dots$. Therefore, every vertex of V^0 can be joined to v^0_0 by a path in G because $\gcd(k-s(1+\alpha),n) = 1$. Now we can easily see that G is connnected. Lemma 2 is proved.

1. $P = v_0^0 v_{a(1)}^1 \cdots v_{a(h-1)}^1 v_{a(h)}^0$
2. $P = v_0^0 v_{a(1)}^1 \cdots v_{a(h-2)}^1 v_{a(h-1)}^2 v_{a(h)}^0$
3. $P = v_0^0 v_{a(1)}^1 \cdots v_{a(h-3)}^1 v_{a(h-2)}^2 v_{a(h-1)}^3 v_{a(h)}^0$
4. $P = v_0^0 v_{a(1)}^1 \cdots v_{a(h-2)}^1 v_{a(h-1)}^3 v_{a(h)}^0$
5. $P = v_0^0 v_{a(1)}^1 \cdots v_{a(h-3)}^1 v_{a(h-2)}^3 v_{a(h-1)}^2 v_{a(h)}^0$
6. $P = v_0^0 v_{a(1)}^2 \cdots v_{a(h-1)}^2 v_{a(h)}^0$
7. $P = v_0^0 v_{a(1)}^2 \cdots v_{a(h-2)}^2 v_{a(h-1)}^1 v_{a(h)}^0$
8. $P = v_0^0 v_{a(1)}^2 \cdots v_{a(h-3)}^2 v_{a(h-2)}^1 v_{a(h-1)}^3 v_{a(h)}^0$
9. $P = v_0^0 v_{a(1)}^2 \cdots v_{a(h-2)}^2 v_{a(h-1)}^3 v_{a(h)}^0$
10. $P = v_0^0 v_{a(1)}^2 \cdots v_{a(h-3)}^2 v_{a(h-2)}^3 v_{a(h-1)}^1 v_{a(h)}^0$
11. $P = v_0^0 v_{a(1)}^3 \cdots v_{a(h-1)}^3 v_{a(h)}^0$
12. $P = v_0^0 v_{a(1)}^3 \cdots v_{a(h-2)}^3 v_{a(h-1)}^2 v_{a(h)}^0$
13. $P = v_0^0 v_{a(1)}^3 \cdots v_{a(h-3)}^3 v_{a(h-2)}^2 v_{a(h-1)}^1 v_{a(h)}^0$
14. $P = v_0^0 v_{a(1)}^3 \cdots v_{a(h-2)}^3 v_{a(h-1)}^1 v_{a(h)}^0$
15. $P = v_0^0 v_{a(1)}^3 \cdots v_{a(h-3)}^3 v_{a(h-2)}^1 v_{a(h-1)}^2 v_{a(h)}^0$

Table 1. Types of P

3. Main result

The purpose of this section is to prove the following result.

THEOREM 1. *Let G be a connected cubic $(4, n)$-metacirculant graph. Then G possesses a Hamilton cycle.*

PROOF: Let $G = MC(4, n, \alpha, S_0, S_1, S_2)$ be a connected cubic $(4, n)$-metacirculant graph. Then 4 divides the order $4n$ of G. But 4 does not divide 10 which is the order of the Petersen graph. So if $S_0 \neq \emptyset$, then the graph G has a Hamilton cycle [11].

We assume now that $S_0 = \emptyset$. Since G is a cubic $(4, n)$-metacirculant graph, only the following cases may happen:

1. $a(h) \equiv s + f_1(k - s(1+\alpha)) - s \equiv f_1(k - s(1+\alpha)) (\bmod n)$

2. $a(h) \equiv s + f_2(k - s(1+\alpha)) + \alpha s - k \equiv (f_2 - 1)(k - s(1+\alpha)) (\bmod n)$

3. $a(h) \equiv s + f_3(k - s(1+\alpha)) + \alpha s + \alpha^2 s + \alpha^3 s \equiv$
$\equiv (f_3 - \alpha^2 - 1)(k - s(1+\alpha)) (\bmod n)$

4. $a(h) \equiv s + f_4(k - s(1+\alpha)) + \alpha k + \alpha^3 s \equiv$
$\equiv (f_4 - \alpha^2 + \alpha - 1)(k - s(1+\alpha)) (\bmod n)$

5. $a(h) \equiv s + f_5(k - s(1+\alpha)) + \alpha k - \alpha^2 s - k \equiv$
$\equiv (f_5 + \alpha - 1)(k - s(1+\alpha)) (\bmod n)$

6. $a(h) \equiv k + f_6(k - s(1+\alpha)) - k \equiv f_6(k - s(1+\alpha)) (\bmod n)$

7. $a(h) \equiv k + f_7(k - s(1+\alpha)) - \alpha s - s \equiv$
$\equiv (f_7 + 1)(k - s(1+\alpha)) (\bmod n)$

8. $a(h) \equiv k + f_8(k - s(1+\alpha)) - \alpha s + \alpha k + \alpha^3 s \equiv$
$\equiv (f_8 - \alpha^2 + \alpha)(k - s(1+\alpha)) (\bmod n)$

9. $a(h) \equiv k + f_9(k - s(1+\alpha)) + \alpha^2 s + \alpha^3 s \equiv$
$\equiv (f_9 - \alpha^2)(k - s(1+\alpha)) (\bmod n)$

10. $a(h) \equiv k + f_{10}(k - s(1+\alpha)) + \alpha^2 s - \alpha k - s \equiv$
$\equiv (f_{10} - \alpha + 1)(k - s(1+\alpha)) (\bmod n)$

11. $a(h) \equiv -\alpha^3 s + f_{11}(k - s(1+\alpha)) + \alpha^3 s \equiv f_{11}(k - s(1+\alpha)) (\bmod n)$

12. $a(h) \equiv -\alpha^3 s + f_{12}(k - s(1+\alpha)) - \alpha^2 s - k \equiv$
$\equiv (f_{12} + \alpha^2)(k - s(1+\alpha)) (\bmod n)$

13. $a(h) \equiv -\alpha^3 s + f_{13}(k - s(1+\alpha)) - \alpha^2 s - \alpha s - s \equiv$
$\equiv (f_{13} + \alpha^2 + 1)(k - s(1+\alpha)) (\bmod n)$

14. $a(h) \equiv -\alpha^3 s + f_{14}(k - s(1+\alpha)) - \alpha k - s \equiv$
$\equiv (f_{14} + \alpha^2 - \alpha + 1)(k - s(1+\alpha)) (\bmod n)$

15. $a(h) \equiv -\alpha^3 s + f_{15}(k - s(1+\alpha)) - \alpha k + \alpha s - k \equiv$
$\equiv (f_{15} + \alpha^2 - \alpha)(k - s(1+\alpha)) (\bmod n)$

Table 2. Values of $a(h)$ for respective cases of P

1) $S_0 = \emptyset, S_1 = \emptyset$ and $| S_2 |= 3$.

2) $S_0 = \emptyset, | S_1 |= 1$ and $| S_2 |= 1$.

Case 1 does not occur because G is connected.

Consider now Case 2. Let $S_1 = \{s\}$ with $0 \leq s < n$ and $S_2 = \{k\}$ with $0 \leq k < n$. Let ρ be the automorphism of G defined as in Section 2, i.e., $\rho(v_j^i) = v_{j+1}^i$ for every vertex $v_j^i \in V(G)$. Then ρ is semiregular because $\rho = (v_0^0 v_1^0 \dots v_{n-1}^0)(v_0^1 v_1^1 \dots v_{n-1}^1)(v_0^2 v_1^2 \dots v_{n-1}^2)(v_0^3 v_1^3 \dots v_{n-1}^3)$ is the disjoint cycle decomposition of ρ and all these four cycles have the same length n. Since any power of a semiregular automorphism is semiregular, this implies that $\rho^{\alpha-1}$ is also semiregular. Thus, we can construct the quotient graph $G/\rho^{\alpha-1}$ of G. It is not difficult to verify that the quotient graph $G/\rho^{\alpha-1}$ is isomorphic to the $(4, \overline{n})$-metacirculant graph $\overline{G} = MC(4, \overline{n}, \overline{\alpha}, \overline{S}_0, \overline{S}_1, \overline{S}_2)$, where $\overline{n} =$ gcd $(\alpha - 1, n), 1 = \overline{\alpha} \equiv \alpha(\bmod \overline{n}), \overline{S}_0 = \emptyset, \overline{S}_1 = \{\overline{s}\}$ with $\overline{s} \equiv s(\bmod \overline{n})$ and $0 \leq \overline{s} < \overline{n}$ and $\overline{S}_2 = \{\overline{k}\}$ with $\overline{k} \equiv k(\bmod \overline{n})$ and $0 \leq \overline{k} < \overline{n}$. Therefore, we can from now on identify $G/\rho^{\alpha-1}$ with $\overline{G}$ and in order to avoid the confusion between vertices of G and $\overline{G}$ we assume that $V(\overline{G}) = \{w_j^i \mid i = 0, 1, 2, 3,$ and $j \in Z_{\overline{n}}\}$. From the connectedness of G it follows that $\overline{G}$ is connected. by Lemma 2,

$$\gcd(\overline{k} - \overline{s}(1 + \overline{\alpha}), \overline{n}) = \gcd(\overline{k} - 2\overline{s}, \overline{n}) = 1. \tag{1}$$

By the definition of $(4, \bar{n})$-metacirculant graphs, we have $\overline{\alpha}^2 \overline{k} \equiv -\overline{k}(\bmod \overline{n}) \Leftrightarrow 2\overline{k} \equiv 0(\bmod \overline{n})$. This means that

$$2\overline{k} = u\overline{n} \tag{2}$$

for some integer u.

Assume first that n is odd. Then $\overline{n} = \gcd(\alpha - 1, n)$ is odd. If $\overline{k}$ were not equal to 0, then in (2) u might be even and greater than 0. Therefore, from (2) it would follow that $\overline{k} = (u/2)\overline{n}$ with $u/2 \geq 1$. This would contradict $\overline{k} < \overline{n}$. Thus, $\overline{k} = 0$. From this and (1) we have $\gcd(2\overline{s}, \overline{n}) = 1$. Therefore, $\gcd(4\overline{s}, \overline{n}) = 1$ and in this subcase $\overline{G}$ possesses the Hamilton cycle $C_1 = Q(w_0^0)Q(w_{4\overline{s}}^0)Q(w_{8\overline{s}}^0)\dots Q(w_{(\overline{n}-1)4\overline{s}}^0)$, where $Q(w_j^0) = w_j^0 w_{j+\overline{s}}^1 w_{j+2\overline{s}}^2 w_{j+3\overline{s}}^3$.

Let $\overline{\rho}$ and $\overline{\tau}$ be the automorphisms of $\overline{G}$ defined by $\overline{\rho}(w_j^i) = w_{j+1}^i$ and $\overline{\tau}(w_j^i) = w_j^{i+1}$ for every $w_j^i \in V(\overline{G})$. Set $\overline{\gamma} = \overline{\rho}^{\overline{s}}\overline{\tau}$. Then $\overline{\gamma}$ is an automorphism of $\overline{G}$ and $\overline{\gamma}(w_j^i) = \overline{\rho}^{\overline{s}}\overline{\tau}(w_j^i) = \overline{\rho}^{\overline{s}}(w_j^{i+1}) = w_{j+\overline{s}}^{i+1}$. That is, $\overline{\gamma}$ maps every vertex of $\overline{G}$ to its following vertex in C_1. This means that $\overline{G}$ is a circulant graph. Therefore, since $\overline{G}$ in cubic, it is not difficult to see that that w_0^0 is adjacent to $w_{2\overline{n}\overline{s}}^2 (= w_0^2)$ and $w_{2\overline{n}\overline{s}-\overline{s}}^1 (= w_{\overline{n}-\overline{s}}^1)$ is adjacent to $w_{4\overline{n}\overline{s}-\overline{s}}^3 (= w_{\overline{n}-\overline{s}}^3)$. So we can construct the following Hamilton cycle C of $\overline{G}$ from C_1 (see Figure 1). Start C at w_0^0. Extend it by going around C_1 from w_0^0 in the direction of $w_{\overline{s}}^1$ until reaching $w_{2\overline{n}\overline{s}-\overline{s}}^1$. Take now the edge $w_{2\overline{n}\overline{s}}^1 w_{4\overline{n}\overline{s}-\overline{s}}^3$ and then proceed C by going around C_1 from $w_{4\overline{n}\overline{s}-\overline{s}}^3$ in the direction of $w_{4\overline{n}\overline{s}-2\overline{s}}^2$ until reaching $w_{2\overline{n}\overline{s}}^2$. To return to w_0^0 we take the edge $w_{2\overline{n}\overline{s}}^2 w_0^0$.

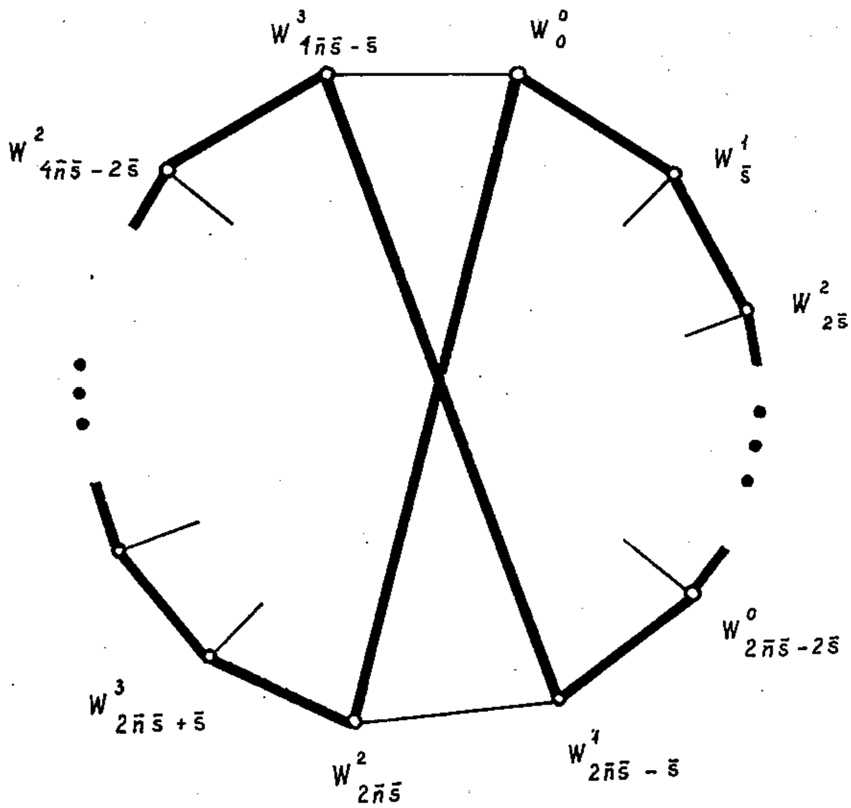

Fig. 1

Let P be the path of coil (C) which starts at v_0^0. This path terminates at

v_a^0 with

$$a \equiv s + \alpha s + \alpha^2 s + \alpha^3 s + s + \cdots + \alpha^3 s + s + \alpha k - \alpha^2 s - \alpha s - s - \alpha^3 s - \ldots$$

$$-s - \alpha^3 s - \alpha^2 s - k \pmod{n},$$

where the numbers of s and $(-\alpha^2 s)$ are $\lfloor \overline{n}/2 \rfloor + 1$, while the numbers of $\alpha s, \alpha^2 s$, $\alpha^3 s, -s, -\alpha s, -\alpha^3 s$ are $\lfloor \overline{n}/2 \rfloor$ and the numbers of αk and $-k$ are 1. Therefore,

$$a \equiv s - \alpha^2 s + \alpha k - k \equiv s(1-\alpha)(1+\alpha) + k(\alpha - 1)$$

$$\equiv (\alpha - 1)(k - s(1+\alpha)) \equiv (\alpha - 1)d \pmod{n},$$

where $d = (k - s(1+\alpha))$. It is not difficult to see that $\rho^{\alpha-1}$ has the order $t = n/\overline{n}$. Since G is connected, by Lemma 2, we have $\gcd(k - s(1+\alpha), n) = 1$. Therefore, $\gcd(d,t) = 1$. By Lemma 1, G has a Hamilton cycle.

Assume now that n is even. Then α must be odd and therefore $\overline{n} = \gcd(\alpha - 1, n)$ is even. If $\overline{k}$ were equal to 0, then from $\overline{\alpha} = 1$ it would follow that 2 would divide $\gcd(\overline{k} - \overline{s}(1+\overline{\alpha}), \overline{n}) = \gcd(2\overline{s}, \overline{n})$. This would contradict (1). Thus, $\overline{k} \neq 0$. From (2) and $\overline{k} < \overline{n}$ it follows that $\overline{k} = \overline{n}/2$. Moreover, because of (1), $\overline{k} = \overline{n}/2$ must be odd. From (1) it also follows that $\gcd(2\overline{s}, \overline{n}/2) = 1$. Hence $\gcd(4\overline{s}, \overline{n}) = 2$. Consequently, $C_1 = Q(w_0^0)Q(w_{4\overline{s}}^0)Q(v_{8\overline{s}}^0)\ldots Q(w_{2\bar{n}\bar{s}-4\overline{s}}^0)$ and $C_2 = Q(w_1^0)Q(w_{1+4\overline{s}}^0)Q(w_{1+8\overline{s}}^0)\ldots Q(w_{1+2\bar{n}\bar{s}-4\overline{s}}^0)$ are cycles of G. Moreover, $V(C_1) \cap V(C_2) = \emptyset$ and $V(\overline{G}) = V(C_1) \cup V(C_2)$. Since $\overline{\alpha} = 1$ and $\overline{k} = \overline{n}/2$, the vertex w_0^0 of C_1 is adjacent to $w_{(\overline{n}/2)}^2$ of C_2 and the vertex $w_{2\bar{n}\bar{s}-\overline{s}}^3 (= w_{\overline{n}-\overline{s}}^3)$ of C_1 is adjacent to $w_{(\overline{n}/2)-\overline{s}}^1$ of C_2. So we can construct the following Hamilton cycle C of $\overline{G}$ from C_1 and C_2 (see Figure 2). Start C at w_0^0. Extend it by going around the cycle C_1 from w_0^0 in the direction of $w_{\overline{s}}^1$ until reaching $w_{2\bar{n}\bar{s}-\overline{s}}^3$. Proceed it by taking the edge $w_{2\bar{n}\bar{s}-\overline{s}}^3 w_{(\overline{n}/2)-\overline{s}}^1$ and then by going around the cycle C_2 from $w_{(\overline{n}/2)-\overline{s}}^1$ in the direction of $w_{(\overline{n}/2)-2\overline{s}}^0$ until reaching $w_{\overline{n}/2}^2$. Finally, to return to w_0^0 we take the edge $w_{\overline{n}/2}^2 w_0^0$.

Let P be the path of coil (C) which starts at v_0^0. This path terminates at v_a^0 with

$$a \equiv s + \alpha s + \alpha^2 s + \alpha^3 s + s + \cdots + s + \alpha s + \alpha^2 s + \alpha^3 s - s - \alpha^3 s - \alpha^2 s - \alpha s - s$$

$$- \cdots - \alpha^3 s - \alpha^2 s + \alpha^2 k \pmod{n},$$

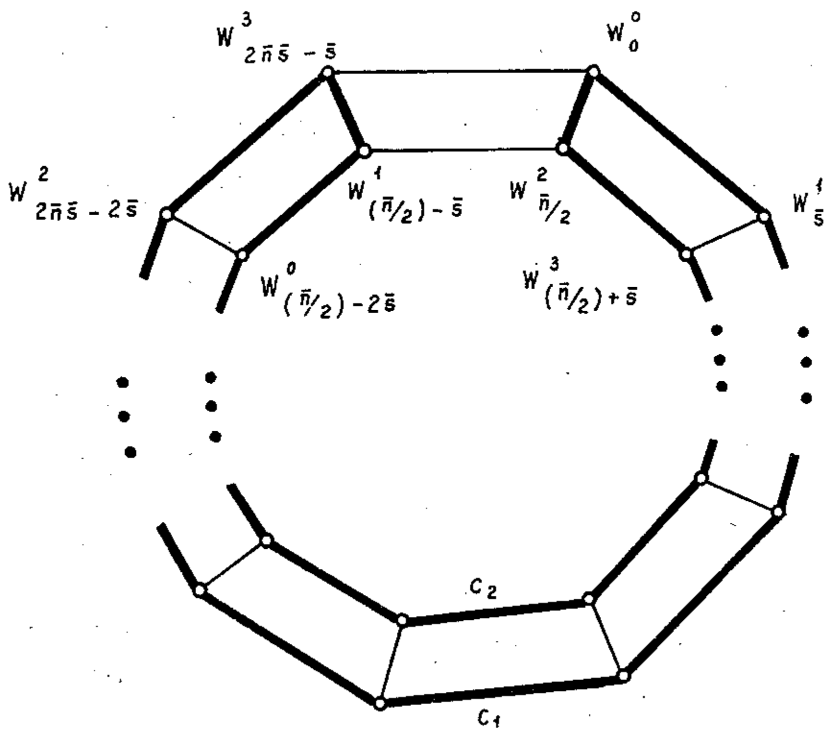

Fig. 2

where the numbers of $s, \alpha s, \alpha^2 s, -s, -\alpha^2 s, -\alpha^3 s$ are $(\overline{n}/2)$, while the numbers of $\alpha^3 s, -\alpha s$ are $(\overline{n}/2) - 1$ and the numbers of $\alpha^3 k, \alpha^2 k$ are 1. Therefore,

$$a \equiv \alpha s - \alpha^3 s + \alpha^3 k + \alpha^2 k \equiv \alpha s(1-\alpha^2) - \alpha k + \alpha^2 k$$

$$\equiv \alpha(\alpha - 1)(k - s(1+\alpha)) \equiv (\alpha - 1)d(\bmod n),$$

where $d = \alpha(k - s(1+\alpha))$. The automorphism $\rho^{\alpha-1}$ has the order $t = n/\overline{n}$. By definition, gcd $(\alpha, n) = 1$ and since G is connected, by Lemma 2, gcd $(k - s(1+\alpha), n) = 1$. Therefore, gcd $(d, t) = 1$. By Lemma 1, G has a Hamilton cycle . The proof of Theorem 1 is completed.

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