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AN INEXACT SEIDEL-NEWTON METHOD FOR NONLINEAR BOUNDARY-VALUE PROBLEMS

PHAM KY ANH AND BUI DUC TIEN

0. Introduction

This paper is motivated by previous works of O. Axelsson [1] and D. Sweet [2], and it is closely related to our recent results on the Seidel-Newton method for nonlinear boundary-value problems (BVP) [3-8]. The principal result of this paper is to present an iterative method for finding solutions of nonlinear differential equations which satisfy nonlinear boundary conditions. Besides, no special assumption on the structure of linear parts is required.

This paper is organized as follows: The first part deals with a modification of the Seidel-Newton method for solving nonlinear operator equations involving a linear Fredholm part. In the second part, an application to nonlinear BVPs is considered. In particular, an inaccuracy in [2] is shown. Finally, in the third part, some illustrative examples are given.

1. Inexact Seidel-Newton method

Let X and Y be two real Banach spaces. Consider the nonlinear operator equation

$$\mathcal{A}x = F(x), \tag{1.1}$$

where $\mathcal{A} : X \to Y$ is a bounded linear Fredholm operator (of index zero), and $F : X \to Y$ a possibly nonlinear operator. For any linear operator $T, \mathcal{N}(T)$ and $\mathcal{R}(T)$ will denote the null space and range of T respectively.

Since $\mathcal{A}$ is a Fredholm operator, X and Y can be written as direct sums: $X = X_1 \oplus X_2; Y = Y_1 \oplus Y_2$, where $Y_1 = \mathcal{R}(\mathcal{A}) \subset Y$ is a closed subspace, $X_2 = \mathcal{N}(\mathcal{A}) \subset X$ is a finite dimensional subspace and codim $Y_1 = \dim X_2 < +\infty$.

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Moreover, the restriction $\widehat{\mathcal{A}}$ of $\mathcal{A}$ to X_1 has a bounded inverse. Denote by P and Q the bounded linear projections satisfying conditions: $\mathcal{R}(P) = \mathcal{N}(Q) = Y_1; \mathcal{R}(Q) = \mathcal{N}(P) = Y_2$. Clearly, $P + Q = I$ is the identity operator.

For the Seidel-Newton method [3-8], at each step, we have to solve a linear system of equation. This can be expensive and moreover, may not be justified when approximate solutions are still far from an exact solution. Hence, it may be of interest to consider the following inexact Seidel-Newton method.

Suppose that the n-th approximate solution is found: $x_n = u_n + v_n$ ($u_n \in X_1; v_n \in X_2$). We construct the first component of the $(n+1)$-th approximate solution by the formula

$$\widehat{\mathcal{A}} u_{n+1} = PF(x_n). \tag{1.2a}$$

Then putting

$$\tilde{x}_n = u_{n+1} + v_n. \tag{1.2b}$$

We find $\mathcal{M}_n \in X_2$ such that

$$||[QG'(\tilde{x}_n)]_{X_2} \mathcal{M}_n + QF(\tilde{x}_n)|| \leq \tau ||QF(\tilde{x}_n)||, \tag{1.2c}$$

where G is a continuously Fréchet differentiable operator and $\tau \in (0,1)$ is a fixed number.

The second component v_{n+1} is defined as follows:

$$v_{n+1} = v_n + \mathcal{M}_n. \tag{1.2d}$$

Finally, let

$$x_{n+1} = u_{n+1} + v_{n+1}. \tag{1.2e}$$

We have the following result.

THEOREM 1.1. *Assume that the mapping F is of the form $F = G + H$, where $G : X \to Y$ is continuously Fréchet differentiable in an an open set D which includes the closed ball S with center at x_0 and radius $r > 0$ and H is continuous on D. Further, suppose that F, G, H satisfy the conditions*

(i) $\quad ||PF(x) - PF(y)|| \leq \alpha ||x - y||;$

$\quad ||QH(x) - QH(y)|| \leq \epsilon ||x - y||;$

$$||QG'(x) - QG'(y)|| \leq \rho(||x-y||),\ (x, y \in S),$$

where ρ is a Dini function (i.e. ρ is nonnegative, nondecreasing, continuous and $\rho(0) = 0$).

(ii) $||QG'(x)|| \leq \beta;\ ||[QG'(x)]_{X_2}]^{-1}|| \leq \gamma \qquad (x \in S)$.

If the coefficients α, ϵ are sufficiently small, and the initial approximation x_0 is good enough such that

$$q_0 := 2\alpha\gamma(\beta+\epsilon)||\widehat{\mathcal{A}}^{-1}|| + \epsilon\gamma^2\beta + \beta\gamma^2 \int_0^1 \rho(t\delta_0)dt < 1, \tag{1.3}$$

and

$$2\delta_0(1-q_0)^{-1} < r, \tag{1.4}$$

where $\delta_0 = \gamma(\beta+\epsilon)||\widehat{\mathcal{A}}^{-1}||.||\mathcal{A}x_0 - PF(x_0)|| + ||QF(x_0)||]$, then the sequence $\{x_n\}$ constructed by (1.2a)-(1.2e) with a suitable choice of $\tau \in (0,1)$ converges to a solution $x^* \in S$ of (1.1) and

$$||x_n - x^*|| < rq^n, \tag{1.5}$$

where $q \in (0,1)$ is a constant.

PROOF: Let $\delta = (1+\tau)\delta_0$ and

$$\begin{aligned} q = {} & 2\alpha\gamma(\beta+\epsilon)||\widehat{\mathcal{A}}^{-1}|| + \epsilon\beta\gamma^2 + \tau[2\alpha\gamma(\beta+\epsilon)||\widehat{\mathcal{A}}^{-1}|| + \\ & + \beta\gamma(1+\tau)(1-\tau)^{-1} + \epsilon\beta\gamma^2(3+\tau)(1-\tau)^{-1}] + \\ & + \beta\gamma^2(1+\tau^2)(1-\tau)^{-1}\int_0^1 \rho(t\delta)dt, \end{aligned}$$

where $\tau \in (0,1)$ is an arbitrary fixed number. Since $\delta \to \delta_0, q \to q_0$ when $\tau \to 0$, and by virtue of (1.3), (1.4), there exists $\tau \in (0,1)$ such that $q < 1$ and

$$0 < 2\delta(1-q)^{-1} < r. \tag{1.6}$$

Let $\lambda_n = u_{n+1} - u_n = \tilde{x}_n - x_n, \mathcal{M}_n = v_{n+1} - v_n = x_{n+1} - \tilde{x}_n$. Suppose that for all $n > 0, x_n, \tilde{x}_n \in S$. Then

$$||\lambda_n|| = ||\widehat{\mathcal{A}}^{-1}[PF(x_n) - PF(x_{n-1}]|| \leq \alpha||\widehat{\mathcal{A}}^{-1}||(||\lambda_{n-1}|| + ||\mathcal{M}_{n-1}||). \tag{1.7}$$

Using the condition (ii) and the relation (1.2c) we get:

$$||\mathcal{M}_n|| \leq \gamma||[QG'(\tilde{x}_n)]_{X_2}\mathcal{M}_n|| \leq (1+\tau)\gamma||QF(\tilde{x}_n)||. \tag{1.8}$$

Further,

$$||QF(\tilde{x}_n)|| \leq ||QF(\tilde{x}_n) + [QG'(\tilde{x}_n)]_{X_2}\mathcal{M}_n|| + ||[QG'(\tilde{x}_n)]_{X_2}\mathcal{M}_n|| \leq$$
$$\leq \tau||QF(\tilde{x}_n)|| + \beta||\mathcal{M}_n||.$$

Therefore

$$||QF(\tilde{x}_n)|| \leq \beta(1-\tau)^{-1}||\mathcal{M}_n||. \tag{1.9}$$

On the other hand,

$$||QF(x_{n+1})|| \leq ||QG(x_{n+1}) - QG(\tilde{x}_n) - [QG'(\tilde{x}_n)]_{X_2}\mathcal{M}_n||+$$
$$+ ||QH(x_{n+1}) - QH(\tilde{x}_n)|| + ||QF(\tilde{x}_n) - [QG'(\tilde{x}_n)]_{X_2}\mathcal{M}_n|| \leq$$
$$\leq \int_0^1 \rho(t||\mathcal{M}_n||)||\mathcal{M}_n||dt + \epsilon||\mathcal{M}_n|| + \tau||QF(\tilde{x}_n)||.$$

Hence

$$||QF(x_{n+1})|| \leq e_n||QF(\tilde{x}_n)||, \tag{1.10}$$

where

$$e_n = \tau + (\epsilon + \int_0^1 \rho(t||\mathcal{M}_n||)dt).||\mathcal{M}_n||/||QF(\tilde{x}_n)||. \tag{1.11}$$

Further,

$$||QF(\tilde{x}_{n+1})|| \leq ||QF(x_{n+1})|| + (\beta+\epsilon)||\lambda_{n+1}||. \tag{1.12}$$

Thus, by (1.8), (1.10), (1.12) we get the estimate

$$||\mathcal{M}_{n+1}|| < (1+\tau)\gamma||QF(\tilde{x}_{n+1})|| \leq (1+\tau)\gamma\{||QF(x_{n+1})|| + (\beta+\epsilon)||\lambda_{n+1}||\}$$
$$\leq \gamma(1+\tau)[(\beta+\epsilon)||\lambda_{n+1}|| + e_n||QF(\tilde{x}_n)||].$$

Combining the last inequality with (1.9), we obtain:

$$||\mathcal{M}_{n+1}|| \leq \gamma(1+\tau)(\beta+\epsilon)\{||\lambda_{n+1}|| + \beta e_n(\beta+\epsilon)^{-1}(1-\tau)^{-1}||\mathcal{M}_n||\}, \tag{1.13}$$

Next, we prove by induction the following relations

$$x_n, \tilde{x}_n \in S, \tag{14}$$

$$||\lambda_n|| \le \delta q^n; \qquad ||\mathcal{M}_n|| \le \delta q^n. \tag{15}$$

For $n = 0, x_0 \in S$ and $||\lambda_0|| = ||u_1 - u_0|| \le ||\widehat{\mathcal{A}}^{-1}|| \, ||\mathcal{A}x_0 - PF(x_0)|| \le \delta_0 \le \delta$. Further, by (1.2c) and (1.6),

$$||\mathcal{M}_0|| \le \gamma(1+\tau)||QF(\tilde{x}_0)|| \le \gamma(1+\tau)\{||QF(x_0)|| + (\beta+\epsilon)||\lambda_0||\} \le$$
$$\le \gamma(1+\tau)\{||QF(x_0)|| + (\beta+\epsilon)||\widehat{\mathcal{A}}^{-1}||||\mathcal{A}x_0 - PF(x_0)||\} = \delta < r.$$

Thus, (1.14) and (1.15) hold for $n = 0$. Assuming that (1.14), (1.15) hold for all $k \le n$, we shall prove that they also hold for $k = n+1$. Since

$$||x_{n+1} - x_0|| \le \sum_{k=0}^{n} ||x_{k+1} - x_k|| \le \sum_{k=0}^{n}(||\lambda_k|| + ||\mathcal{M}_k||) \le$$
$$\le 2\delta \sum_{k=0}^{n} q^k < 2\delta(1-q)^{-1} < r,$$

it follows that $x_{n+1} \in S$. Further, (1.7) implies that

$$||\lambda_{n+1}|| \le \alpha||\widehat{\mathcal{A}}^{-1}||(||\lambda_n|| + ||\mathcal{M}_n||) \le 2\alpha||\widehat{\mathcal{A}}^{-1}||\delta q^n \le \delta q^{n+1}. \tag{1.16}$$

From (1.16), (1.6) we get

$$||\tilde{x}_{n+1} - x_0|| \le ||\tilde{x}_{n+1} - x_{n+1}|| + ||x_{n+1} - x_0|| \le$$
$$\le ||\lambda_{n+1}|| + \sum_{k=0}^{n}(||\lambda_k|| + ||\mathcal{M}_k||) \le$$
$$\le \delta q^{n+1} + 2\delta \sum_{k=0}^{n} q^k < 2\delta(1-q)^{-1} < r.$$

Therefore $\tilde{x}_{n+1} \in S$. Using (1.13), (1.16) and the inductive assumption we find

$$||\mathcal{M}_{n+1}|| \le \gamma(1+\tau)(\beta+\epsilon)\{2\alpha||\widehat{\mathcal{A}}^{-1}||\delta q^n + \beta e_n(\beta+\epsilon)^{-1}(1-\tau)^{-1}\delta q^n\} = \delta q^n T_n,$$

where $T_n = 2\alpha\gamma(1+\tau)(\beta+\epsilon)||\widehat{\mathcal{A}}^{-1}|| + \beta\gamma(1+\tau)(1-\tau)^{-1}e_n$. By (1.8), (1.11) and the inductive assumption we have

$$e_n \le \tau + (\epsilon + \int_0^1 \rho(t\delta q^n)dt)(1+\tau)\gamma \le \tau + \gamma(1+\tau)(\epsilon + \int_0^1 \rho(t\delta)dt).$$

Hence

$$T_n \le 2\alpha\gamma(1+\tau)(\beta+\epsilon)||\widehat{\mathcal{A}}^{-1}|| +$$
$$+ \beta\gamma(1+\tau)(1-\tau)^{-1}\{\tau + \gamma(1+\tau)(\epsilon + \int_0^1 \rho(t\delta)dt)\} == q.$$

Thus, $||\mathcal{M}_{n+1}|| \le \delta q^{n+1}$ and relations (1.14), (1.15) are proved for $k = n+1$. Finally, since $||x_{n+1} - x_n|| \le ||\lambda_n|| + ||\mathcal{M}_n|| \le 2\delta q^n$, $\{x_n\}$ is a Cauchy sequence. Evidently, $x^* = \lim x_n$ is a solution of (1.1) in S, and the estimate (1.5) holds.

2. Nonlinear boundary-value problems

Consider the BVP

$$\dot{x} = A(t)x + f(t, x, \dot{x}) \qquad t \in (0,1), \tag{2.1}$$

$$\Gamma x = \Lambda g(t, x, \dot{x}), \tag{2.2}$$

where $A \in C([0,1], \mathbf{R}^{n\times n})$ is a continuous $(n\times n)$ matrix, $\Gamma, \Lambda : C([0,1], \mathbf{R}^{n\times n} \to \mathbf{R}^n$ are bounded linear operators, and f, g are vector-valued functions.

Let $X = C^1([0,1], \mathbf{R}^n), Y = C([0,1], \mathbf{R}^n), Z = Y \times \mathbf{R}^n$ be Banach spaces with the norms

$$||y|| = \max_{0\le t\le 1} |y(t)|, (y \in Y); |||x||| = ||x|| + ||\dot{x}||, (x \in X),$$

$||z|| = ||\binom{y}{r}|| = ||y|| + |r|, (z \in Z)$. Here, $|.|$ denotes the maxnorm of vectors and the corresponding norm for matrices. The transpose of a matrix or vector will have a superscript T. The scalar product in Y will denote by $< \varphi, \psi > = \int_0^1 \varphi^T \psi dt$.

We are particularly interested in the resonance case of (2.1)-(2.2), i.e. when the homogenous BVP

$$\dot{x} = A(t)x; \qquad \Gamma x = 0$$

has nontrivial solutions. The main difficulty of this case is that the solvability of (2.1), (2.2) mainly depends on the structure of the nonlinear parts f and g.

Problem (2.1), (2.2) can be reduced to the operator form (1.1), where

$$\mathcal{A}x = \begin{pmatrix} \dot{x} - A(.)x \\ \Gamma x \end{pmatrix}, \qquad F(x) = \begin{pmatrix} f(.,x,\dot{x}) \\ \Lambda g(.,x,\dot{x}) \end{pmatrix}.$$

Let $U(t)$ be the principal solution of $\dot{x} = A(t)x$, i.e.

$$\dot{U} = A(.)U, \qquad U(0) = E,$$

where E is the identity matrix. By Riesz's representation theorem, $\Gamma x = \int_0^1 d\eta(t)x(t)$, where $\eta \in BV([0,1], \mathbb{R}^{n\times n})$ is a matrix, whose entries are of bounded variation. Denote by D the so-called determining matrix $D = \int_0^1 d\eta(t)U(t)$ and let rank $D = n - \nu$ $(0 \leq \nu \leq n)$. Then Problem (2.1), (2.2) is at resonance if and only if $\nu > 0$. For convenience, we assume that $0 < \nu < n$. The remaining cases $\nu = 0$ and $\nu = n$ are more simple and may be considered by the same arguments.

Denote by $\{x_i\}_1^\nu, \{w_i\}_1^\nu, \{y_j\}_{\nu+1}^n$ fixed bases of the subspaces $\mathcal{N}(D), \mathcal{N}(D^T)$ and $\mathcal{R}(D)$, respectively. Without loss of generality we can assume that $w_i^T w_j = \delta_{ij}$ $(i,j = \overline{1,\nu})$; $y_k^T y_s = \delta_{ks}$ $(k,s = \overline{\nu+1,n})$.

Let $x_j (j = \overline{\nu+1,n})$ be any fixed vectors satisfying conditions $Dx_j = y_j$ $(j = \overline{\nu+1,n})$. Let $\varphi_i = U(t)x_i$ $(i = \overline{1,\nu})$ and $\Phi(t) = (\varphi_1,...,\varphi_\nu)$. Concerning operator $\mathcal{A}$, we have the following result [2].

LEMMA 2.1. *The following statements hold:*

(1) $\mathcal{N}(\mathcal{A}) = \{\Phi(.)a \mid a \in \mathbb{R}^\nu\}$; *dim* $\mathcal{N}(\mathcal{A}) = \nu\}$.

(2) $\binom{h}{u} \in \mathcal{R}(\mathcal{A})$ *if and only if* $< y, h > + w^T u = 0$ *for any* $w \in \mathcal{N}(D^T)$ *and* $y^T = w^T \int_0^1 d\eta(s)U(s)U^{-1}(t)$.

(3) Let $Q_0 x = \Phi(.)(\int_0^1 \Phi^T(s)\Phi(s)ds)^{-1} \int_0^1 \Phi^T(s)x(s)ds$. *Then* Q_0 *is a bounded linear projection,* $\mathcal{R}(Q_0) = \mathcal{N}(\mathcal{A})$ *and* $x = \mathcal{N}(Q_0) \oplus \mathcal{N}(\mathcal{A})$.

Notice that the matrix $M = \int_0^1 \Phi^T(s)\Phi(s)ds = (<\varphi_i, \varphi_j>)_1^\nu$ *is a Gramm matrix. Therefore the linear-independence of* $\varphi_i = U(t)x_i$ $(i = \overline{1,\nu})$ *implies the nonsingularity of* M.

LEMMA 2.2. $\mathcal{A}: X \to Z$ *is a bounded linear Fredholm operator.*

PROOF: By Lemma 2.1, $\mathcal{N}(\mathcal{A})$ is ν-dimensional subspace and $\mathcal{R}(\mathcal{A})$ is a closed subspace. Hence, to prove that $\mathcal{A}$ is a Fredholm operator, it will be sufficient to show that $Z = Z_1 \oplus Z_2$, where $Z_1 = \mathcal{R}(\mathcal{A})$ and dim $Z_2 = \nu$. Putting $Z_2 = \text{span}\{\binom{0}{w_i}\}_1^\nu$, we have dim $Z_2 = \nu$. Let $\binom{h}{u} \in Z_1 \cap Z_2$. Then $h = 0$ and $u = \sum_{i=1}^\nu c_i w_i$. Since $\binom{h}{u} \in \mathcal{R}(\mathcal{A})$, it follows that for any $w \in \mathcal{N}(D^T), 0 = <y, h> + w^T u = w^T u$. In particular, for $w = u \in \mathcal{N}(D^T)$, we have $u^T u = 0$, therefore $Z_1 \cap Z_2 = \{\binom{0}{0}\}$. Now for an arbitrary $\binom{h}{u} \in Z$, we define $c_i = <\psi_i, h> + w_i^T u$, where

$$\psi_i^T = w_i^T \int_0^t d\eta(s)U(s)U^{-1}(t), \qquad (i = \overline{1,\nu}).$$

Clearly, $\binom{h}{u - \sum_{i=1}^\nu c_i w_i} \in Z_1$, and $Z = Z_1 \oplus Z_2$.

LEMMA 2.3. *Let*

$$Q\begin{pmatrix} h \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ \sum_{i=1}^\nu c_i w_i \end{pmatrix}, \tag{2.3}$$

where $c_i = <\psi_i, h> + u^T w_i$ $(i = \overline{1,\nu})$. *Then* Q *is a bounded linear projection,* $\mathcal{N}(Q) = \mathcal{R}(\mathcal{A})$ *and* $Z = \mathcal{R}(\mathcal{A}) \oplus \mathcal{R}(Q)$.

PROOF: Since $Q^2\binom{h}{u} = \binom{0}{\sum_i c_i' w_i}$, where $c_i' = <0, \psi_i> + \sum_j c_j w_j^T w_i = c_i$, we get $Q^2 = Q$. Further, $\mathcal{N}(Q) = \{\binom{h}{u} : <h, \psi_i> + u^T w_i = 0, \ (i = \overline{1,\nu})\}$. By Lemma 2.1, $\mathcal{N}(Q) = \mathcal{R}(\mathcal{A})$ and it follows from the proof of Lemma 2.2 that $Z = \mathcal{R}(\mathcal{A}) + \mathcal{R}(Q)$.

REMARK: D. Sweet [2] considered the following projection $P_0 : Z \to Z$ defined by

$$P_0\begin{pmatrix} h \\ u \end{pmatrix} = \begin{pmatrix} \Psi^T(\int_0^1 \Psi(s)\Psi^T(s)ds)^{-1}(\int_0^1 \Psi(s)h(s)ds + Wu) \\ 0 \end{pmatrix},$$

where

$$\Psi = \begin{pmatrix} \Psi_1^T \\ \cdots \\ \Psi_\nu^T \end{pmatrix} \quad ; \quad W = \begin{pmatrix} w_1^T \\ \cdots \\ w_\nu^T \end{pmatrix}.$$

Clearly, P_0 is more complicated than the above mentioned projection Q. Moreover, P_0 is not always well-defined. The necessary and sufficient condition for the existence of P_0 is $\mathcal{N}(D^T) \cap \mathcal{N} = \{0\}$, where $\mathcal{N} = \{a \in \mathbb{R}^n : a^T \int_0^t d\eta(s)U(s)ds = 0 \text{ (a.e.)}\}$. In what follows, we shall use the following notations:

$$P = I - Q, X_1 = \mathcal{N}(Q_0), X_2 = \mathcal{N}(\mathcal{A}), Z_1 = \mathcal{R}(\mathcal{A}), Z_2 = \mathcal{R}(Q).$$

It follows from Lemma 2.1 and 2.3 that $Z_1 = \mathcal{N}(Q)$ and $X = X_1 \oplus X_2, Z = Z_1 \oplus Z_2$. Now we shall construct the inverse operator $\widehat{\mathcal{A}}^{-1}$ and estimate its norm. Let $\mathcal{A}x = \binom{h}{u}$. Then

$$x = U(t)\{x_0 + \int_0^t U^{-1}(s)h(s)ds\} \tag{2.4}$$

and

$$\Gamma x = \int_0^1 d\eta(t)x(t) = u. \tag{2.5}$$

From (2.4), (2.5) it follows that

$$Dx_0 = u - \int_0^1 d\eta(t)U(t)\int_0^t U^{-1}(s)h(s)ds = \sum_{j=\nu+1}^{n} \beta_j y_j, \tag{2.6}$$

where

$$\beta_j = y_j^T(u - \int_0^1 d\eta(t)U(t)\int_0^1 U^{-1}(s)h(s)ds) \qquad (j = \overline{\nu+1, n}). \tag{2.7}$$

Let

$$x_0 = \sum_{i=1}^{\nu} \alpha_i x_i + \sum_{j=\nu+1}^{n} \beta_j x_j, \tag{2.8}$$

where the coefficients $\{\alpha_i\}$ are undetermined. Then (2.6) is evidently satisfied. We choose α_i such that $x(t)$, defined by (2.4), belongs to $X_1 = \mathcal{N}(Q_0)$, i.e.

$Q_0x = 0$, or equivalently

$$\sum_{i=1}^{\nu} \alpha_i Q_0 U(.)x_i + p(t) = 0, \tag{2.9}$$

where $p(t) = Q_0U(t)\{\sum_{j=\nu+1}^{n} \beta_j x_j + \int_0^t U^{-1}(s)h(s)ds\}$. Since $Q_0U(.)x_i = Q_0\varphi_i = \varphi_i \quad (i = \overline{1,\nu})$, we can rewrite (2.9) as

$$\sum_{i=1}^{\nu} \alpha_i \varphi_i + p(t) = 0.$$

Multiplying scalarly both sides of the last equation by φ_j $(j = \overline{1,\nu})$, we obtain the following linear system for determining α_i

$$\sum_{i=1}^{\nu} < \varphi_i, \varphi_j > \alpha_i = - < p, \varphi_j > \qquad (j = \overline{1,\nu}). \tag{2.10}$$

Let $\alpha = (\alpha_1, ..., \alpha_\nu), \hat{p} = -(< p, \varphi_1 >, ..., < p, \varphi_\nu >)^T$ and $M = \int_0^1 \Phi^T(t)\Phi(t)dt$. We can reduce (2.10) to the vector form $M\alpha = \hat{p}$. Since M is nonsingular, $\alpha = M^{-1}\hat{p}$. Now in order to get an estimation $||\widehat{A}^{-1}|| \leq \omega$, we first estimate the norm of $x = \widehat{A}^{-1}\binom{h}{u}$ and its derivative $\dot{x} = A(t)x + h$. It follows from (2.4) that

$$||x|| \leq \max_t |U(t)|(|x_0| + \max_t |U^{-1}(t)| \; ||h||). \tag{2.11}$$

Evidently, $||\dot{x}|| \leq \max_t |A(t)| \; ||x|| + ||h||$. This together with (2.11) implies

$$|||x||| = ||x|| + ||\dot{x}|| \leq c_0|x_0| + c_1||h||, \tag{2.12}$$

where

$$c_0 = (1 + \max_t |A(t)|) \max_t |U(t)|;$$
$$c_1 = 1 + (1 + \max_t |A(t)|) \max_t |U(t)| \max_t |U^{-1}(t)|.$$

Further, by (2.6) we have

$$|x_0| \leq c_2|\alpha| + c_3|\beta|, \tag{2.13}$$

where $c_2 = \sum_{i=1}^{\nu} |x_i|, c_3 = \sum_{j=\nu+1}^{n} |x_j|$ and $|\alpha| = \max_i |\alpha_i|, |\beta| = \max_j |\beta|$. From (2.7) it follows that

$$|\beta| \leq |Y_0^T|(|u| + |D| \max_t |U^{-1}(t)| \; ||h||), \tag{2.14}$$

where $Y_0 = (y_{v+1}, ..., y_n)$. Since $\alpha = M^{-1}\hat{p}$, we have $|\alpha| \leq |M^{-1}||\hat{p}| \leq |M^{-1}| \max_t |\Phi^T(t)| \times ||p||$. On the other hand, $||p|| \leq ||Q_0|| \max_t |U(t)| \{c_1|\beta| + \max_t |U^{-1}(t)| \quad ||h||\}$. It follow from Lemma 2.1 that $||Q_0|| \leq \max_t |\Phi(t)| \max_t |\Phi^T(t)||M^{-1}|$. Then $|\alpha| \leq c_4|\beta| + c_5||h||$, where $c_4 = c_1|M^{-1}|^2(\max_t |\Phi^T(t)|)^2 \max_t |\Phi(t)| \max_t |U(t)|$, and $c_5 = c_4 c_1^{-1} \max_t |U^{-1}(t)|$. Taking into account the last inequality and (2.13) we have

$$|x_0| \leq c_6|\beta| + c_7||h||, \tag{2.15}$$

where $c_6 = c_2c_4 + c_3; c_7 = c_2c_5$. Finally, using (2.12), (2.14), (2.15) we obtain

$$\begin{aligned} |||x|| &< c_0(c_6|\beta| + c_7||h||) + c_1||h|| \leq \\ &\leq c_0c_6|Y_0^T||u| + \{c_0c_6|Y_0^T||D| \max_t |U^{-1}(t)| + c_1 + c_0c_7\}||h|| \leq \omega|| \begin{pmatrix} h \\ u \end{pmatrix} ||, \end{aligned}$$

where

$$\omega = \max\{c_0c_6|Y_0^T|, \; c_1 + c_0c_7 + c_0c_6|Y_0^T||D| \max_t |U^{-1}(t)|\}. \tag{2.16}$$

Summarizing the above results we come to the following

THEOREM 2.1. *$\mathcal{A} : X \to Z$ is a bounded linear Fredholm operator with $\mathcal{R}(\mathcal{A}) = \mathcal{N}(Q)$. Moreover, the restriction $\widehat{\mathcal{A}}$ of $\mathcal{A}$ to $X_1 = \mathcal{N}(Q_0)$ has a bounded inverse defined by the formula*

$$\widehat{\mathcal{A}}^{-1} \begin{pmatrix} h \\ u \end{pmatrix} = U(t)\{x_0 + \int_0^t U^{-1}(s)h(s)ds\},$$

$x_0 = \sum_{i=1}^{\nu} \alpha_i x_i + \sum_{j=\nu+1}^{n} \beta_j x_j$, *where* $\beta_j = y_j^T(u - \int_0^1 d\eta(t)U(t) \int_0^t U^{-1}(s)h(s)ds)(j = \overline{\nu+1, n})$ *and* $\alpha = (\alpha_1, ..., \alpha_\nu)^T$ *satisfy the* *relations*

$$\Big(\int_0^1 \Phi^T(t)\Phi(t)dt\Big)\alpha = -\Big(\int_0^1 p^T\varphi_1 ds, \ldots, \int_0^1 p^T\varphi_\nu ds\Big),$$

$$p(t) = \Phi(t)\Big(\int_0^1 \Phi^T(s)\Phi(s)ds\Big)^{-1}\int_0^1 \Phi^T(s)U(s)\Big\{\sum_{j=\nu+1}^{n}\beta_j x_j +$$

$$+\int_0^s U^{-1}(\tau)h(\tau)d\tau\Big\}ds.$$

Moreover, $||\widehat{\mathcal{A}}^{-1}|| \le \omega$, *where* ω *is determined by (2.16).*

Now let us consider the nonlinear operator $F(x)$. *For the sake of simplicity, we shall restrict our consideration to the case where* f, g *are continuously differentiable function.*

Let $\Delta = \{(t,x,\xi) | t \in [0,1], x,\xi \in \mathbf{R}^n, |x|, |\xi| \le R\}$ *and* $\Omega = \{x \in X \mid |||x||| < R\}$.

LEMMA 2.4. *Suppose that* $f, g : \Delta \to \mathbf{R}^n$ *are continuous in the first variable and continuously differentiable in the remaining variables. Moreover, suppose that*

(i) $|f'_x(t,x,\xi)| \le a_1, |f'_\xi(t,x,\xi)| \le a_1, |g'_x(t,x,\xi)| \le a_2, |g'_\xi(t,x,\xi)| \le a_2$ *for any* $(t,x,\xi) \in \Delta$.

(ii) $||f'_x(t,x,\xi) - f'_x(t,y,\zeta)|| \le b_1(|x-y| + |\xi-\zeta|)$,
$||f'_\xi(t,x,\xi) - f'_\xi(t,y,\zeta)|| \le b_1(|x-y| + |\xi-\zeta|)$,
$||g'_x(t,x,\xi) - g'_x(t,y,\zeta)|| < b_2(|x-y| + |\xi-\zeta|)$,
$||g'_\xi(t,x,\xi) - g'_\xi(t,y,\zeta)|| < b_2(|x-y| + |\xi-\zeta|)$.

for all triples $(t,x,\xi), (t,y,\zeta) \in \Delta$.

Then the operator

$$F(x) = \begin{pmatrix} f(.,x,\dot{x}) \\ \wedge g(.,x,\dot{x}) \end{pmatrix} : X \to Z$$

is continuously Fréchet differentiable on Ω *and its derivative satisfies the relation*

$$||F'(x)|| \le a = a_1 + a_2||\wedge||,$$

$$||F'(x) - F'(y)|| \le b|||x-y||| = (b_1 + b_2||\wedge||)|||x-y|||$$

for all $x, y \in \Omega$. Moreover, the restriction $[QF'(x)]_{X_2}$ is of the form

$$[QF'(x)]_{X_2} h = \begin{pmatrix} 0 \\ \sum_{i=1}^{\nu} c_i w_i \end{pmatrix},$$

where $h = \sum_{j=1}^{\nu} d_j \varphi_j \in X_2; c_i = \sum_{j=1}^{\nu} s_{ij} d_j$, and

$$s_{ij} =< \psi_i, (f'_x + f'_\xi A)\varphi_j > + w'_i \wedge (g'_x + g'_\xi A)\varphi_j. \tag{2.17}$$

PROOF: For any $x \in \Omega, h \in X$, we have

$$F'(x)h = \begin{pmatrix} f'_x(t,x,\dot{x})h + f'_\xi(t,x,\dot{x})\dot{h} \\ \wedge g'_x(t,x,\dot{x})h + g'_\xi(t,x,\dot{x})\dot{h} \end{pmatrix}.$$

Further,

$$\begin{aligned} ||F'(x)h|| &= ||f'_x h + f'_\xi \dot{h}|| + |\wedge (g'_x h + g'_\xi \dot{h})| \leq \\ &\leq a_1(||h|| + ||\dot{h}||) + ||\wedge||a_2(||h|| + ||\dot{h}||) = a|||h|||. \end{aligned}$$

Analogously, for any $x, y \in \Omega, h \in X$,

$$\begin{aligned} &||[F'(x) - F'(y)]h|| = ||[f'_x(.,x,\dot{x}) - f'_x(.,y,\dot{y})]h + [f'_\xi(.,x,\dot{x}) - f'_\xi(.,y,\dot{y})]\dot{h}|| + \\ &+ |\wedge [g'_x(.,x,\dot{x}) - g'_x(.,y,\dot{y})]h + \wedge[g'_\xi(.,x,\dot{x}) - g'_\xi(.,y,\dot{y})]\dot{h}| \leq \\ &\leq \{b_1||h||\ |||x-y||| + b_1||\dot{h}||\ |||x-y|||\} + ||\wedge||\{b_2|||x-y|||(||h|| + ||\dot{h}||)\} = \\ &= b|||h|||\ |||x-y|||. \end{aligned}$$

Thus, $||F'(x) - F'(y)|| \leq b|||x-y|||$. Finally, for $h = \sum_{i=1}^{\nu} d_j \varphi_j \in X_2$, we have $QF'(x)h = \begin{pmatrix} 0 \\ \sum_{i=1}^{\nu} c_i w_i \end{pmatrix}$, where $c_i =< \psi_i, f'_x h + f'_\xi \dot{h} > + w_i^T \wedge (g'_x h + g'_\xi \dot{h})$. Since $\dot{h} = Ax$, it follows that $c_i = \sum_{j=1}^{\nu} s_{ij} d_j$, where s_{ij} are defined by (2.17).

LEMMA 2.5. *Let the matrix $S = (s_{ij})_1^{\nu}$ be nonsingular. Moreover, suppose that $|S^{-1}(x)| \leq \gamma_0$ $(\forall x \in \Omega)$. Then the restriction $[QF'(x)]_{X_2}$ of $QF'(x)$ to X_2 has a bounded inverse $||[QF'(x)]_{X_2}^{-1}|| \leq \gamma$, where $\gamma = \gamma_0(1 + \max_t |A(t)|)|W| \sum_{i=1}^{\nu} ||\varphi_i||$ and $W = \begin{pmatrix} w_i^T \\ \cdots \\ w_\nu^T \end{pmatrix}$.*

PROOF: Putting $d = (d_1, ..., d_\nu)^T, c = (c_1, ..., c_\nu)^T$, and $w = \sum_1^{\nu} c_i w_i$, we can rewrite (2.17) as $Sd = c$. Further, $||h|| = ||\sum_{j=1}^{\nu} d_j \varphi_j|| \leq |d| \sum_{j=1}^{\nu} ||\varphi_j|| \leq$

$\gamma_0|c|\sum_{j=1}^{\nu}||\varphi_j||$, hence $|||h||| \leq \gamma_0(1+\max_t|A(t)|)|c|\sum_1^{\nu}||\varphi_j||$. Since $c_i = w_i^T w_i$, we get $|c| < |W|\ |w|$. Therefore,

$$|||h||| = |||[QF'(x)]_{X_2}^{-1}\begin{pmatrix}0\\w\end{pmatrix}|| \leq \gamma_0(1+\max_t|A(t)|)\sum_{i=1}^{\nu}||\varphi_i||\ |W|\ |w| = \gamma|w|.$$

The following fact immediately follows from linear algebra.

LEMMA 2.6. *Let* $f(t,x,y) = M_1(t)x+N_1(t)y+\epsilon f_1(t,x,y)$, $g(t,x,y) = M_2(t)x+N_2(t)y+\epsilon g_1(t,x,y)$. *Suppose that the matrix* $S_0 = (s_{ij}^{(0)})$ *with entries of the form*

$$s_{ij}^{(0)} =< \psi_i, (M_1+N_1A)\varphi_j > +w_i^T \wedge (M_2+N_2A)\varphi_j$$

is nonsingular. Further, assume that f_1, g_1 *have uniformly bounded partial derivative in* Δ. *Then the matrix* S *defined by (2.17) has a uniformly bounded inverse whenever* $\epsilon > 0$ *is sufficiently small.*

Notice that for the operator Q *defined by (2.3), we have the estimation*

$$||Q|| \leq \tilde{c} = \sum_{i=1}^{\nu}|w_i|\max\{\max|\Psi(t)|, |W|\} \text{ and } ||P|| = ||I-Q|| \leq 1+c.$$

Applying the inexact Seidel-Newton method (1.2a)-(1.2e) to BVP (2.1)-(2.2), we obtain the following algorithm: Let an initial approximation $x^{(0)} \in \Omega$ *be given. Set* $r = R - |||x^{(0)}|||$, $f^{(0)}(t) = f(t, x^{(0)}, \dot{x}^{(0)})$, $g^{(0)}(t) = g(t, x^{(0)}, \dot{x}^{(0)})$, $\alpha = (1+\tilde{c})a$, $\beta = \tilde{c}a$, $L = \tilde{c}b$ and

$$\delta = \beta\gamma\omega\{\max_t|\dot{x}^{(0)} - A(t)x^{(0)} - f^{(0)}(t)| +$$
$$+ |\Gamma x^{(0)} - \wedge g^{(0)} + \sum_{i=1}^{\nu} c_i^{(0)} w_i|\} + \gamma|\sum_{i=1}^{\nu} c_i^{(0)} w_i|,$$

where $c_i^{(0)} =< f^{(0)}, \psi_i > +w_i^T \wedge g^{(0)}$.

Suppose that the k-th approximate solution is known

$$x^{(k)} = z^{(k)} + v^{(k)} \qquad (k \geq 0),$$

where $v^{(k)} = \Phi(t)(\int_0^1 \Phi^T(s)\Phi(s)ds)^{-1}\int_0^1 \Phi^T(s)x^{(k)}(s)ds$, *and* $z^{(k)}(t) = x^{(k)}(t) - v^{(k)}(t)$. *Letting* $f^{(k)}(t) = f(t, x^{(k)}, \dot{x}^{(k)}), g^{(k)}(t) = g(t, x^{(k)}, \dot{x}^{(k)}), c_i^{(k)} =< f^{(k)}, \psi_i > +w_i^T \wedge g^{(k)}$ *and putting*

$$\beta_j^{(k)} = y_j^T\{\wedge g^{(k)} - \sum_{i=1}^{\nu} c_i^{(k)} w_i - \int_0^1 d\eta(t)U(t)\int_0^t U^{-1}(s)f^{(k)}(s)ds\},$$

$$p^{(k)}(t) = \Phi(t)(\int_0^1 \Phi^T(s)\Phi(s)ds)^{-1}\int_0^1 \Phi^T(s)U(s)\{\sum_{j=\nu+1}^{n}\beta_j^{(k)}x_j + \int_0^s U^{-1}(\tau)f^{(k)}(\tau)d\tau\}ds,$$

$$\hat{p}^{(k)} = -(< p^{(k)}, \varphi_1 >, ..., < p^{(k)}, \varphi_\nu >)^T,$$

we can find $\alpha^{(k)} = (\alpha_1^{(k)}, ..., \alpha_\nu^{(k)})^T$ *from the system of equations* $(\int_0^1 \Phi^T(t)\Phi(t)dt)\alpha^{(k)} = \hat{p}^{(k)}$. *Then the first component* $z^{(k+1)}$ *can be determined as*

$$z^{(k+1)}(t) = U(t)\{\sum_{i=1}^{\nu}\alpha_i^{(k)}x_i + \sum_{j=\nu+1}^{n}\beta_j^{(k)}x_j + \int_0^t U^{-1}(s)f^{(k)}(s)ds\}. \quad (2.18)$$

Further, denote by $\tilde{f}^k, \tilde{g}^k, \tilde{f}_x^k, \tilde{f}_\xi^k, \tilde{g}_x^k, \tilde{g}_\xi^k$ *the values of* $f(t, x, \xi)$ *and* $g(t, x, \xi)$ *and their derivatives* $f'_x, f'_\xi, g'_x, g'_\xi$ *calculated at* $(t, \tilde{x}^{(k)}, \tilde{x}^{(k)})$, *where*

$$\tilde{x}^{(k)} = z^{(k+1)} + v^{(k)}. \quad (2.19)$$

Then the corrector $\mathcal{M}^{(k)}$ *for the second component of the desired* $(k+1)$*-th approximation can be found from the condition*

$$||[QF'(\tilde{x}^{(k)})]_{X_2}\mathcal{M}^{(k)} + QF(\tilde{x}^{(k)})|| \le \tau||QF(\tilde{x}^{(k)})||,$$

where $\tau \in (0,1)$ *is a fixed number. Letting* $d^{(k)} = (d_1^{(k)}, ..., d_\nu^{(k)}), \tilde{c}_i^{(k)} = < \psi_i, \tilde{f}^{(k)} > +w_i^T \wedge \tilde{g}^{(k)}$ $(i = \overline{1,\nu}), \tilde{c}^{(k)} = (\tilde{c}_1^{(k)}, ..., \tilde{c}_\nu^{(k)})^T, \tilde{w}^{(k)} = \sum_{i=1}^{\nu}\tilde{c}_i^{(k)}w_i$; $\mathcal{M}^{(k)} = \sum_{i=1}^{\nu} d_i^{(k)}\varphi_i$ *and denoting by* $M^{(k)}$ *the matrix whose elements are* $m_{ij}^{(k)} =< \psi_i, (\tilde{f}_x^{(k)} + \tilde{f}^{(k)}k_\xi A)\varphi_j > +w_i^T \wedge (\tilde{g}_x^{(k)} + \tilde{g}_\xi^{(k)}A)\varphi_j$, *we can choose* $d^{(k)}$ *such that*

$$|M^{(k)}d^{(k)} + \tilde{c}^{(k)}| \le \tau|\tilde{W}^{(k)}|\sum_{i=1}^{\nu}|w_i|)^{-1}. \quad (2.20)$$

Indeed, by virtue of (2.20) and the fact that $|M^{(k)}d^{(k)} + \tilde{c}^{(k)}| = \max_i |\sum_j m_{ij}^{(k)} d_j^{(k)} \tilde{c}_i^{(k)}|$, *we get*

$$|||[QF'(\tilde{x}^{(k)})]_{X_2}\mathcal{M}^{(k)} + QF(\tilde{x}^{(k)})||| = |\sum_{ij} m_{ij}^{(k)} d_j^{(k)} w_i + \sum_i \tilde{c}_i^{(k)} w_i| =$$

$$= |\sum_i (\sum_j m_{ij}^{(k)} d_j^{(k)} + \tilde{c}_i^{(k)}) w_i| \leq \max_i |\sum_{j=1}^{\nu} m_{ij}^{(k)} d_j^{(k)} + \tilde{c}_i^{(k)}| \sum_{i=1}^{\nu} |w_i| \leq$$

$$\leq \tau |w^{(k)}| = \tau ||QF(\tilde{x}^{(k)})||.$$

Finally we put

$$v^{(k+1)} = v^{(k)} + \mathcal{M}^{(k)}, \tag{2.21}$$

$$x^{(k+1)} = z^{(k+1)} + v^{(k+1)}. \tag{2.22}$$

The following convergence result follows from Theorems 1.1, 2.1 and Lemma 2.4, 2.5.

THEOREM 2.2. *Assume that all conditions of Lemmas 2.4 and 2.5 are satisfied. Moreover, suppose that* $q = 2\alpha\beta\omega^{\gamma} + L\beta\gamma^2\delta/2 < 1$ *and* $2\delta(1-q)^{-1} < r$. *Then there exists* $\tau \in (0,1)$ *such that the sequence* $\{x^{(k)}\}$ *constructed by (2.18)-(2.20) converges to a solution* x^* *of (2.1), (2.2) at the rate of geometric progression* $|||x^{(k)} - x^*||| < rq_1^k$, *where* $q_1 \in (0,1)$ *is a constant.*

3. Nonlinear BVP for Duffing-Van Der Pol's equations

Consider the following BVP

$$\begin{cases} y'' + y + \epsilon f(t, y, y') = 0, \\ -y(0) + y(2\pi) = \epsilon \int_0^{2\pi} g(s, y(s)) ds, \\ y'(0) = y'(2\pi), \end{cases} \tag{3.1}$$

where $f(t, x_1, x_2), g(t, x_1)$ are continuous in t and continuously differentiable in the remaining variables, and $\epsilon > 0$ is a small parameter. Putting $x(t) = (x_1, x_2)^T = (y, y')^T, \hat{f} = (0, -f(t,x))^T, \hat{g} = (g(t, x_1), 0)^T, \Gamma x = -x(0) + x(2\pi)$, $\Lambda x = \int_0^{2\pi} x(s) ds, A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ we can reduce problem (3.1) to the operator form

$$\mathcal{A}x = \epsilon F(x), \tag{3.2}$$

where

$$\mathcal{A}x = \begin{pmatrix} \dot{x} - Ax \\ \Gamma x \end{pmatrix}, \qquad F(x) = \begin{pmatrix} \hat{f}(.,x) \\ \Lambda\hat{g}(.,x) \end{pmatrix}.$$

Clearly, $U(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$ and $\Gamma x = \int_0^{2\pi} d\eta(t)x(t)$, where

$$\eta(t) = \begin{cases} 0 & t \in (0, 2\pi) \\ E & t = 0, 2\pi. \end{cases}$$

Since $D = \int_0^{2\pi} d\eta(t)U(t) = 0$, we have $\mathcal{R}(D) = \{0\}, \mathcal{N}(D) = \mathcal{N}(D^T) = \mathbb{R}^2$, and $\nu = n = 2$. Hence we can put $x_1 = w_1 = (1,0)^T, x_2 = w_2 = (0,1)^T, \varphi_1 = U(t)x_1 = (\cos t, -\sin t)^T, \varphi_2 = U(t)x_2 = (\sin t, \cos t)^T$. Further, set $\Phi = (\varphi_1, \varphi_2) = U, U^T = U^{-1}$. From Theorem 2.1 it follows that $\mathcal{A}$ is a bounded linear Fredholm operator, $\mathcal{N}(\mathcal{A}) = \{U(.)a \mid a \in \mathbb{R}^2\}$, dim $\mathcal{N}(\mathcal{A}) = 2$, and $Q_0 x = \frac{1}{2\pi}U(t)\int_0^{2\pi} U^T(s)x(s)ds$ is a bounded linear projection, where $X = \mathcal{N}(Q_0) \oplus \mathcal{N}(\mathcal{A}) \equiv X_1 \oplus X_2$. Further, $Z = \mathrm{R}(\mathcal{A}) \oplus \mathcal{R}(Q) \equiv Z_1 \oplus Z_2$, where $Q\binom{h}{u} = \binom{0}{(c_1,c_2)^T}$,

$$c_1 = u_1 + \int_0^{2\pi} (h_1 \cos t + h_2 \sin t)dt, \; c_2 = u_2 + \int_0^{2\pi} (h_2 \cos t - h_1 \sin t)dt.$$

Using Theorem 2.1 we can show that $\widehat{\mathcal{A}}^{-1}\binom{h}{u} = U(t)\{x_0 + \int_0^t U^{-1}(s)h(s)ds\}$ where $x_0 = -\frac{1}{2\pi}\int_0^{2\pi}(2\pi - s)U^T(s)h(s)ds$. Moreover, $\|\widehat{\mathcal{A}}^{-1}\| \le \omega = 5 + 4\pi$. For the nonlinear operator $F(x)$ we have $\psi_1 = (-\cos t, \sin t)^T, \psi_2 = (-\sin t, -\cos t)^T$, and

$$\hat{f}'_x = -\begin{pmatrix} 0 & 0 \\ f'_{x_1} & f'_{x_2} \end{pmatrix}, \qquad \hat{g}'_x = \begin{pmatrix} g'_{x_1} & 0 \\ 0 & 0 \end{pmatrix}.$$

A simple calculation shows that

$$s_{11} = \int_0^{2\pi} (f'_{x_2} \sin t - f'_{x_1} \cos t) \sin t dt + \int_0^{2\pi} g'_{x_1} \cos t dt,$$

$$s_{12} = -\int_0^{2\pi} (f'_{x_1} \sin t + f'_{x_2} \cos t) \sin t dt + \int_0^{2\pi} g'_{x_1} \sin t dt,$$

$$s_{21} = \int_0^{2\pi} (f'_{x_1} \cos t - f'_{x_2} \sin t) \cos t dt,$$
$$s_{22} = \int_0^{2\pi} (f'_{x_1} \sin t + f'_{x_2} \cos t) \cos t dt.$$

Let us consider Duffing's equations

$$\begin{cases} y'' + y + \epsilon(y' + y^3 - u(t)) = 0, \\ y(2\pi) - y(0) = \epsilon \int_0^{2\pi} y^3(t) \sin t dt, \\ y'(0) = y'(2\pi). \end{cases} \tag{3.3}$$

In this case, $s_{11} = \pi, s_{12} = 0, s_{21} = 3\int_0^{2\pi} x_1^2 \cos^2 t dt, s_{22} = \pi + \frac{3}{2}\int_0^{2\pi} x_1^2 \sin 2t dt$. Suppose that $x \in \Omega = \{x \in\in X : |||x||| < R\}$ where $R^2 < \frac{\pi}{3}$. Then

$$S^{-1} = \begin{pmatrix} 1/\pi & 0 \\ -s_{21}/\pi s_{22} & 1/s_{22} \end{pmatrix}$$

and $|S^{-1}| = \max\{\pi^{-1}; (\pi + |s_{21}|)(\pi|s_{22}|)^{-1}\}$. From $|s_{22}| \le 3R^2 \int_0^{2\pi} \cos^2 t dt = 3\pi R^2$ and $|s_{22}| \ge \pi - 3R^2$ it follows that

$$|S^{-1}| \le (1 + 3R^2)(\pi - 3R^2)^{-1} \qquad (\forall x \in \Omega).$$

THEOREM 3.1. *Suppose that for the given function $u(t)$ there are the relations*

$$\int_0^{2\pi} u(t) \cos t dt = \int_0^{2\pi} u(t) \sin t dt = 0.$$

Then there exist $\epsilon_0 > 0$ such that for any fixed $\epsilon \in (0, \epsilon_0)$ the sequence $\{x^{(k)}\}$ constructed by (2.18)-(2.22) converges to a solution $x^* \in \Omega$ of problem (3.3) in the vector form (3.2) at the rate of geometric progression.

PROOF: Letting $x^{(0)} = 0$ and observing that $q(\epsilon) = 0(\epsilon), \delta(\epsilon) = 0(\epsilon)$ we can choose $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0), q(\epsilon) < 1$, and $2\delta(\epsilon)(1 - q(\epsilon))^{-1} = 0(\epsilon) < r \equiv R$. Now Theorem 3.1 is an immediate consequence of Theorem 2.2.

We end this section by presenting some numerical results. First, we consider problem (3.3) with $u(t) = 1 + \epsilon^3$, where $0 < \epsilon < 1$. In this case, $y^*(t) =$

$\epsilon, x^*(t) = (\epsilon, 0)^T$ are the exact solutions of (3.3) and (3.2), respectively. A simple calculation by algorithm (2.18)-(2.22) shows that

$$x^{(0)} = (0,0)^T;\ x^{(1)} = (\epsilon + \epsilon^4)(1,0)^T;\ x^{(2)} = (\epsilon + 0(\epsilon^7))(1,0)^T.$$

Thus, $|||x^{(2)} - x^*||| = 0(\epsilon^7)$. Finally, letting $u(t) = \sin 2t \quad (0 \le t \le 2\pi)$ and applying algorithm (2.18)-(2.22) with the initial approximation $x^{(0)} = (0,0)^T$ to problem (3.3) we get the following approximate solutions

$$x^{(1)}(t) = -(\epsilon/3)(\sin 2t, 2\cos 2t)^T + (\epsilon^4/36)(\sin t, \cos t)^T,$$
$$x^{(2)}(t) = -((\epsilon/3)\sin 2t + (2\epsilon^2/9)\cos 2t, (2\epsilon/3)\cos 2t - (4\epsilon^2/9)\sin 2t.)$$

Evidently, $||\mathcal{A}x^{(1)} - \epsilon F(x^{(1)})|| = 0(\epsilon^2)$ and $||\mathcal{A}x^{(2)} - \epsilon F(x^{(2)})|| = 0(\epsilon^3)$.

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REFERENCES

[1] O. Axelsson, *On global convergence of iterative methods*, in: Iterative solution of nonlinear systems of equations, In Lect. Notes in Math. 953, Springer Verlag 1982, 1–9.
[2] D. Sweet, *An alternative method for weekly nonlinear BVPs*, Nonlinear Anal. Th. Meth. Appl. **8** (1984), 421–428.
[3] Pham Ky Anh, *On the Seidel-Newton method for solving quasilinear operator equations*, Acta Math. Viet. **7** (1982), 111–126.
[4] Pham Ky Anh and Vu Duy Tich, *An iterative method for general periodic BVPs*, Ukrain. Math. J. **35** (1983), 348–352.
[5] Pham Ky Anh and Tran Dang Hong, *An approximate method for a nonlinear periodic BVP*, Acta Math. Viet. **11** (1986), 156–171.
[6] Pham Ky Anh, *An approximate method for multipoint BVPs at resonance*, Ukrain. Math. J. **39** (1987), 619–624.
[7] Pham Ky Anh, *On two approximate methods for solving nonlinear Neumann problems*, Ukrain. Math. J. **40** (1988), 619–625.
[8] Pham Ky Anh, *An iterative method for nonlinear BVPs at resonance*, Ukrain. Math. J. **43** (1991), 663–674.

DEPARTMENT OF MATHEMATICS
UNIVERSITY, HANOI, VIETNAM