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A REPRESENTATION THEOREM FOR SYMMETRIC STABLE RANDOM OPERATORS

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Abstract. In this paper, it is shown that every summetric p-stable random operator has a representation of the form of a random integral with respect to an p-stable random measure. A necessary and sufficient condition is given for the existence of a representation of the form of a random series.

1. Introduction

Random operators in Hilbert spaces were investigated by Skorokhod [6]. They often arise in a variety of theoretical and applied contexts such as linear equations with random coefficients, stochastic integrals and stochastic differential equation in Hilbert spaces. The theory of random operators acting between Banach spaces have been developed in [2], [7], [8], [9].

Let X, Y be separable Banach spaces. In [8] it was shown that every symmetric Gaussian random operator A into Y admits an expansion of the form

$$Ax = \sum_{n=1}^{\infty} \gamma_n b_n x \tag{1.1}$$

Here (b_n) is a sequence in the space $L(X, Y)$ of nonrandom linear operators from X into Y and (γ_n) is a sequence of real-valued independent standard Gaussian random variables. The series (1.1) is a. s. convergent in Y.

A natural problem is to establish an analogous representation for symmetric p-stable (SpS) random operators. The purpose of this paper is to prove Theorem 3.2 which states that every SpS random operator can be represented as a random integral with respect to an SpS random measure. Moreover, an SpS random operator. A admits a representation of the form (1.1) if and only if the subspace

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of $L_0(\Omega)$ spanned by the r.v. 's $\{(Ax, y), x \in X, y \in Y'\}$ can be isometrically imbedded in to $\ell_p(1 < p < 2)$.

2. Preliminaries and notation

Let X, Y be two separable Banach spaces and Y' the dual space of Y. By a random mapping A from X into Y we mean a correspondence that associates to each element x in X an Y-valued random variable Ax. A random mapping A from X into Y is called a random operator if

1) For all x_1, x_2 in X and t_1, t_2 in R we have

$$A(t_1x_1 + t_2x_2) = t_1Ax_1 + t_2Ax_2 \quad a.s.$$

2) $\forall \epsilon > 0 \qquad \lim_{x_n \to x} P\{\| Ax_n - Ax \| > \epsilon\} = 0$

We say that a random operator B is a representation of the random operator A if for every finite sequence $\{(x_k, y_k)\}$ in $X \times Y'$, the laws of two random variables $\{(Ax_k, y_k)\}$ and $\{(Bx_k, y_k)\}$ are the same.

The random operator A is called an SpS random operator if for every finite sequence $\{(x_k, y_k)\}$ in $X \times Y'$ the law of the random variable $\{(Ax_k, y_k)\}$ is symmetric p-stable.

Let $(S, \sum, \mu)$ be a measurable space. A mapping $M : \sum \longrightarrow L_0(\Omega)$ is said to be an SpS random measure on $(S, \sum, \mu)$ if

1) For each $A \in \sum, M(A)$ is a random variable with the ch. f.

$$E\exp\{\mathrm{it}M(A)\} = \exp\{-\mu(A) \mid t \mid^p\}.$$

2)If $\{A_n\}$ is a sequence of disjoint sets in $\sum$, then $\{M(A_n)\}$ are independent and $M(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} M(A_n)$ a.s.

In [5] Rosinski introduced a random integral of Banach space valued functions with respect to an SpS random measure M. This integral is constructed as follows: Let E be a separable Banach space . For a simple function $f : S \to$

$E, f = \sum_{i=1}^{n} c_i \ell_{A_i}$, where (A_i) are disjoint sets in $\sum$ and $(c_i) \subset E$, we define

$$\int f dM = \sum_{i=1}^{n} c_i M(A_i)$$

A measurable function $f : S \longrightarrow E$ is said to be M-integrable if there exitst simple functions (f_n) such that $f_n \to f$ in μ-measure and the sequence $\{\int f_n dM\}$ converges in probability. If f is M-integrable, then we put $\int f dM = p - lim \int f_n dM$. This value does not depend on the choice of the approximating sequence (f_n). The set of M-integrable functions $f : S \to E$ is denoted by $\mathcal{L}_E(M)$. Some crucial properties of this p-stable random integral are listed in Theorem 2.1 and Theorem 2.2 below.

THEOREM 2.1. [5] 1) If $f \in \mathcal{L}_E(M)$, then $\int f dM$ an E-valued SpS random variable with the ch. f.

$$F(a) = \exp\{-\int |(f(t), a)|^p \, d\mu(t)\}, \quad a \in E'.$$

2) For $0 < r < p < 2$ and $f \in \mathcal{L}_E(M)$, we put

$$| f |_r = [E \| \int f dM \|^r]^{1/r}.$$

Then $| f |_r$ is a quasi-norm (a norm if $r \geq 1$) on $\mathcal{L}_E(M)$. For $1 < p < 2, \mathcal{L}_E(M)$ becomes a Banach space under the norm $| f |_r$. Moreover there exists a constant C depending only on r such that

$$[\int \| f \|^p \, d\mu]^{1/p} \leq C \, | f |_r$$

for all $f \in \mathcal{L}_E(M)$.

3)The mapping $f \to \int f dM$ is a linear continuous operator from $\mathcal{L}_E(M)$ into $L_E^0 = L^0(S, \sum, \mu, E)$.

THEOREM 2.2. [5] (Characterization of M-integrable functions) *A function $f : S \to E$ is M-integrable if and only if the function $F(a) = exp\{-\int |(f(t), a)|^p \, d\mu(t)\}, a \in E'$ is the ch. f. of an SpS measure on E.*

Following Linde [4] we say that an operator $T : E' \to L_p(S, \sum, \mu)$ is an Λ_p-operator if the function $F(a) = \exp\{- \| Ta \|^p\}$ is the ch.f. of SpS measure on E. It was shown (see Th. 7.4.6 of [4]) that if T is an Λ_p-operator $(0 < p < 2)$, then there is a function $f : S \to E$ such that for each $a \in E'$, $Ta(t) = (f(t), a)$ for μ-almost t. Combining this fact and Theorem 2.2 we get

THEOREM 2.3. *An operator $T : E' \to L_p(S, \sum, \mu)$ is an Λ_p -operator if and only if it is of the form*

$$Ta(\cdot) = (f(\cdot), a) \quad in L_p(S, \sum, \mu)$$

where $f \in \mathcal{L}_E(M)$ and M is an SpS random measure on $(S, \sum, \mu)$.

3. Representation theorem

PROPOSITION 3.1. *Let M be an SpS random measure on a measurable space $(S, \sum, \mu)$ and $G : X \to \mathcal{L}_Y(M)$ be a linear continuous operator. Then the random mapping B from X into Y given by*

$$Bx = \int GxdM \tag{3.1}$$

is an SpS random operator.

PROOF: By Theorem 2.1, the mapping $H : \mathcal{L}_Y(M) \to L_Y^0$ given by $Hf = \int fdM$ is linear and continuous. Hence $B = H \circ G$ is a linear continuous operator from X into L_Y^0, i.e. B is a random operator. Now for each finite sequence $\{(x_k, y_k)\}$ in $X \times Y'$, the linear combination $\sum t_k(Bx_k, y_k) = \int [\sum t_k(Gx_k, y_k)]dM$ is an SpS random variable. Consequently, B is an SpS random operator.

Of cause, every random operator which has a representation of the form (3.1) is an SpS random operator. The Representation theorem states that the converse is also true.

THEOREM 3.2. *Suppose that A is an SpS random operator from X into $Y (0 < p < 2)$. Then there exist an SpS random measure M on some measurable space $(S, \sum, \mu)$ and a linear continuous operator $G : X \to \mathcal{L}_Y(M)$ such that the SpS random operator B given by (3.1) is a representation of A.*

PROOF: Let $[A]$ denote the closed subspace of $L_0(\Omega)$ spanned by the random variables the $\{(Ax, y), x \in X, y \in Y'\}$. For each $h \in [A]$ we put

$$\rho(h) = [-\ln E\exp\{ih\}]^{1/p}.$$

By a theorem due to Bretagnolle et al. [1] (see also the Addendum of [3], $\rho(h)$ is a F-norm (a norm if $p \geq 1$) on $[A]$ such that $h_n \to h$ in $L_0(\Omega)$ if and only if $\rho(h_n - h) \to 0$. Moreover, $([A], \rho)$ embeds isometrically into some $L_p(S, \sum, \mu)$ by an isometry I.

Let $X \otimes Y'$ be the tensor product of X and Y'. By the property of the tensor product the bilinear mapping $(x, y) \to (Ax, y)$ induces a linear mapping $\hat{A} : X \otimes Y' \to [A]$ such that $\hat{A}(x \otimes y) = (Ax, y)$. Put $T = I \circ \hat{A} : X \otimes Y' \to L_p(S, \sum, \mu)$. For each $u \in X \otimes Y'$ we have

$$E\exp\{i\hat{A}u\} = \exp\{- \| I(\hat{A}u) \|^p\} = \exp\{- \| Tu \|^p\}. \tag{3.2}$$

In particular, for $u = x \otimes y$ we have

$$\exp\{- \| T(x \otimes y) \|^p\} = E\exp\{i(Ax, y)\}.$$

This equality shows that for each fixed $x \in X$ the mapping $y \to T(x \otimes y)$ from Y' into $L_p(S, \sum, \mu)$ is an Λ_p- operator. By Theorem 2.3 there exist an SpS random measure M on $(S, \sum, \mu)$ and a function, denoted by Gx, belonging to $\mathcal{L}_Y(M)$ such that for each $y \in Y'$,

$$T(x \otimes y)(\cdot) = (Gx(\cdot), y) \quad \mu - a.s. \tag{3.3}$$

We claim that the mapping $x \to Gx$ is a linear continuous operator from X into $\mathcal{L}_Y(M)$. Indeed, the linearity of G follows from the linearity of T and the separability of Y. In order to prove the continuity of G we use the closed graph theorem. Let $x_n \to x$ in X and $Gx_n \to f$ in $\mathcal{L}_Y(M)$. We have

$$(A(x_n - x), y) \stackrel{d}{=} \int (G(x_n - x), y) dM$$

for each $y \in Y'$. Because $(Ax_n - Ax, y) \xrightarrow{P} 0$ and $\int (Gx_n, y) dM \xrightarrow{P} \int (f, y) dM$ as $n \to \infty$, it follows that

$$\int (f, y) dM = \int (Gx, y) dM = p - \lim \int (Gx_n, y) dM$$

for each $y \in Y'$. Hence $\int Gx dM = \int f dM$, i.e. $Gx = f$ as desired. Put $Bx = \int Gx dM$. It remains to prove that B is a representation of A. Indeed, for each finite sequence $\{(x_k, y_k)\}$ in $X \times Y'$, from (3.2) and (3.3) we have

$$E\exp\{i \sum t_k (Bx_k, y_k)\} = E\exp\{i \int [\sum t_k (Gx_k, y_k)] dM\} =$$

$$\exp\{- \int | \sum t_k (Gx_k, y_k) |^p d\mu\} = \exp\{- \int | \sum t_k T(x_k \otimes y_k) |^p d\mu\} =$$

$$\exp\{- \int | Tu |^p d\mu\} = \exp\{- \| Tu \|^p\} =$$

$$E\exp\{i\hat{A}u\} = E\exp \quad i\{\sum t_k (Ax_k, y_k)\}.$$

(Here we put $u = \sum t_k (x_k \otimes y_k)$)

This shows that the random variables $\{(Ax_k, y_k)\}$ and $\{(Bx_k, y_k)\}$ have the same law. Thus Theorem 3.2 is proved.

Now we are going to find a condition ensuring that the SpS random operator A has a representation of the form of a random series.

PROPOSITION 3.3. *Let (θ_n) be a sequence of i.i.d. random variables with the ch.f. $\exp\{- | t |^p\}$ and (b_n) a sequence in $L(X, Y)$ such that for each $x \in X$ the series $\sum_{n=1}^{\infty} \theta_n b_n x$ is a. s. convergent in Y. Then the random mapping B from X into Y given by*

$$Bx = \sum_{n=1}^{\infty} \theta_n b_n x \tag{3.4}$$

is an SpS random operator from X into Y. Moreover, if a random operator A has a representation B of the form (3.4), then [A] embeds isometrically into ℓ_p.

PROOF: Put $B_n x = \sum_{k=1}^{n} \theta_k b_k x$. Clearly, B_n is a random operator for each n and we have $Bx = p - \lim B_n x$ for all $x \in X$. By [7, Thm.1.3.d] B is a random operator. It is easy to check that B is an SpS random operator. Next, let $M = [(\theta_n)]$ be the closed subspace of $L_0(\Omega)$ spanned by the sequence (θ_n). It is well known that (M, ρ) is isometric to ℓ_p (see [3]). Then if A has a representation B of the form (3.4), [A] embeds isometrically into ℓ_p under the mapping $\sum_{k=1}^{n}(Ax_k, y_k) \to \sum_{k=1}^{n}(bx_k, y_k)$.

THEOREM 3.4. *Suppose that A is an SpS random operator ($1 < p < 2$) such that $[A]$ may be isometrically embedded into ℓ_p. Then there exists a bounded sequence (b_n) in $L(X, Y)$ satisfying*

1) For each $x \in X$, the series $\sum_{n=1}^{\infty} \theta_n b_n x$ converges a.s. in Y.

2) The SpS random operator B from X into Y given by

$$Bx = \sum_{n=1}^{\infty} \theta_n b_n x$$

is a representation of A.

PROOF: Because ℓ_p embeds into some $L_p(S, \sum, \mu)$ it follows that there exist sequences (e_n) in $L_p(S, \sum, \mu)$, (g_n) in $L'_p(S, \sum, \mu)$ and a mapping $I : [A] \to L_p(S, \sum, \mu)$ such that

$$Ih = \sum_{n=1}^{\infty}(Ih, g_n)e_n$$

and

$$\| Ih \|^p = [\rho(h)]^p = \sum_{n=1}^{\infty} | (Ih, g_n) |^p \tag{3.5}$$

Put $T = I \circ \hat{A} : X \otimes Y' \to L_p(S, \sum, \mu)$. From (3.2) and (3.5) we get

$$E\exp\{i\hat{A}u\} = \exp\{- \| Tu \|^p\} = \exp\{- \sum_{n=1}^{\infty} | (Tu, g_n) |^p\}.$$

As we have shown in the proof of Theorem 3.2, there are an SpS random measure M on $(S,\sum,\mu)$ and a linear continuous operator G from X into $\mathcal{L}_Y(M)$ such that $T(x\otimes y)(\cdot)=(Gx(\cdot),y)$ μ- a. s. . For each fixed $g\in L'_p(S,\sum,\mu)$ consider the linear mapping $Vg : X\to Y$ given by

$$Vg(x)=\int g(t)Gx(t)d\mu(t) \tag{3.7}$$

The Bochner integral (3.7) exists since $\int \| Gx(t)\|^p\, d\mu(t)<\infty$. From Theorem 2.1 we obtain

$$\begin{aligned}\| Vg(x)\| &\le \| g\| \,[\int \| Gx(t)\|^p\, d\mu(t)]^{1/p}\\ &\le C\| g\|\| Gx\|_{\mathcal{L}_Y(M)}\le C\| g\|\| G\|\| x\|,\end{aligned}$$

which showns that $Vg\in L(X,Y)$ and $\| Vg\|\le C\| g\|\| G\|$. Put $b_n=Vg_n$. We have a bounded sequence (b_n) in $L(X,Y)$ and

$$(b_nx,y)=(Vg_n(x),y)=\int (g_n(t)Gx(t),y)d\mu(t)$$

$$=\int T(x\otimes y)(t)g_n(t)d\mu(t)=(T(x\otimes y),g_n).$$

Consequently, putting $u=x\otimes y$, from (3.6) it follows that

$$E\exp\{i(Ax,y)\}=\exp\{-\sum_{n=1}^{\infty}|(T(x\otimes y),g_n)|^p\}=\exp\{-\sum_{n=1}^{\infty}|(b_nx,y)|^p\}. \tag{3.8}$$

Now we are ready to prove that the series $\sum_{n=1}^{\infty}\theta_nb_nx$ converges a.s. in Y for each $x\in X$. The ch.f. of the partial sum $\sum_{n=1}^{N}\theta_nb_nx$ is equal to $\exp\{-\sum_{n=1}^{N}|(b_nx,y)|^p\}$ and it converges to $E\exp\{i(Ax,y)\}$ as $N\to\infty$ by (3.8).. Thus $\sum_{n=1}^{\infty}\theta_nb_nx$ is convergent a.s. in Y by ItoNisio's Theorem.

It remains to check that the SpS random operator B given by $Bx=\sum_{n=1}^{\infty}\theta_nb_nx$ is a representation of A. Indeed, for every finite sequence

$\{x_k, y_k)\}_1^n \subset X \times Y'$ and $(t_k) \subset R$, we have

$$E\exp\{i\sum_1^n t_k(Bx_k, y_k)\} = E\exp \quad i\{\sum_{i=1}^{\infty}\sum_{k=1}^{n} t_k(b_i x_k, y_k)\theta_i\} =$$

$$\exp\{-\sum_{i=1}^{\infty} \mid \sum_{k=1}^{n} t_k(T(x_k \otimes y_k), g_i) \mid^p\} = \exp\{-\sum_{i=1}^{\infty} \mid (Tu, g_i) \mid^p\} =$$

$$E\exp\{i\hat{A}u\} = E\exp\{i\sum_{k=1}^{n} t_k(Ax_k, y_k)\},$$

where u stands for $\sum_{k=1}^{n} t_k(x_k \otimes y_k)$.

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