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ON THE INVERSE SOURCE PROBLEM FOR THE WAVE OPERATOR

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Introduction

In this paper we study the following inverse source problem for the wave operator

$$P_n := \frac{\partial^2}{\partial t^2} - a^2\Delta_n, a > 0,$$

where Δ_n is the Laplace operator, on the closed strip

$$\overline{G}_T := \{(x,t) \in R^{n+1} : x \in R^n, 0 \leq t \leq T\}, T > 0.$$

Given any distribution $v \in \mathcal{D}'(R^{n+1})$ with supp $v \subseteq R^n \times (t > T)$, satisfying the wave equation

$$P_n v(x,t) = 0, t > T, \text{in } \mathcal{D}'(R^n \times (t > T)),$$

find a (source) distribution ν with supp $\nu \subseteq \overline{G}_T$ such that the wave potential $E_n * \nu$, the convolution of the fundamental solution E_n of P_n and ν, satisfies the condition

$$E_n * \nu(x,t) = v(x,t), t > T.$$

In Section 1 we shall present some results and remarks about the generalized Cauchy problem (see [5], [6]). Section 2 deals with the solvability, the structure of the set of solutions, and the stability of the proposed problem (cf. [4] for the case of the heat conduction operator).

We use the terminologies of [5], [6].

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1. Remarks about the generalized Cauchy problem

Due to Wladimirow ([5], [6]) the classical Cauchy problem:

$$u(x,t) \in C^2(R^n \times (t>0)) \cap C^1(R^n \times (t>0)), \tag{1}$$

$$P_n u(x,t) = f(x,t), t>0, \tag{2}$$

$$\frac{\partial u(x,t)}{\partial t} = u_1(x) \in C(R^n), \tag{3}$$

$$u(x,0) = u_0(x) \in C(R^n), \tag{4}$$

where f, u_1, u_0 are given, may be generalized to the so-called generalized Cauchy problem of finding a distribution $u \in \mathcal{D}'(R^{n+1})$ which satisfies the equation

$$P_n u(x,t) = f(x,t) + u_0(x) \times \delta'(t) + u_1(x) \times \delta(t), \text{in } \mathcal{D}'(R^{n+1}), \tag{5}$$

where $f \in \mathcal{D}'(R^{n+1}, R^n \times (t \geq 0)), u_0 \in \mathcal{D}'(R^n)$, and $u_1 \in \mathcal{D}'(R^n)$. For arbitrary set $A \subset R^{n+1}$ $\mathcal{D}'(R^{n+1}, A)$ denotes the class of all distributions g in $\mathcal{D}'(R^{n+1})$ with supp $g \subseteq A$.

Put

$$F(x,t) = f(x,t) + u_0(x) \times \delta'(t) + u_1(x) \times \delta(t).$$

Since E_n is the fundamental solution of P_n and $F \in \mathcal{D}'(R^{n+1}, R^n \times (t \geq 0))$, the convolution $E_n * F$ always exists in $\mathcal{D}'(R^{n+1})$. The generalized Cauchy problem (5) has a unique solution, namely (see [5])

$$u = E_n * F,$$

which continuously depends on F with respect to the weak topology in $\mathcal{D}'(R^{n+1})$. Hence, the generalized Cauchy problem (5) is well-posed.

Throughout the paper we consider the cases $n = 1, 2, 3$. At first let us study some properties of the convolution $E_n * F$. The fundamental solutions E_n for $n = 1, 2, 3$ have the explicite form (see [5]):

$$E_1(x,t) = \frac{1}{2}\theta(at - |x|), n = 1,$$

$$E_2(x,t) = \frac{\theta(at - |x|)}{2\pi a\sqrt{a^2t^2 - |x^2|}}, n = 2,$$

$$E_3(x,t) = \frac{\theta(t)}{4\pi a^2t^2}\delta_{S_{at(x)}} \equiv \frac{\theta(t)}{2\pi a}\delta(a^2t^2 - |x|^2), n = 3,$$

where θ is the Heaviside function.

Note that the fundamental solutions $E_n, n = 1,2,3$, are locally integrable and have the support on the closed cone

$$\overline{\Gamma^+} := \{(x,t) : a^2t^2 \leq |x|^2\}.$$

Further, for $t \to +0$ we have (see [5], p. 170)

$$E_n(x,t) \to 0, \frac{\partial E_n(x,t)}{\partial t} \to \delta(x), \frac{\partial^2 E_n(x,t)}{\partial t^2} \to 0 \quad \text{in} \quad \mathcal{D}'(R^n) \tag{6}$$

If $f(x,t) \in \mathcal{D}'(R^{n+1}, \overline{G}_T)$, then $E_n * f$ exists in $\mathcal{D}'(R^{n+1})$. If

$$E_n * f \in C^2(R^n \times (t > 0)) \cap C^1(R^n \times (t \geq 0)),$$

we say that f belongs to the class $S_n(\overline{G}_T)$. This class if nonempty (see [5]). For the potential $E_n * f$ we have

$$P_n(E_n * f) = (P_nE_n) * f = \delta * f = f.$$

Particularly, if $f \in C^2(\overline{G}_T)$ for $n = 2,3$, and $f \in C^1(\overline{G}_T)$ for $n = 1$, then we obtain

$$V_n := E_n * f \in C^2(R^n \times t \geq 0)),$$
$$P_nV_n(x,t) = 0 \quad \text{in} \quad R^n \times (t > T),$$
$$V_n|_{t=0} = 0, \frac{\partial V_n}{\partial t}|_{t=0} = 0.$$

If $F(x,t)$ takes the form $F(x,t) = u_0(x) \times \delta'(t)$ or $F(x,t) = u_1(x) \times \delta(t)$ with $u_0, u_1 \in \mathcal{D}'(R^n)$, we call the corresponding convolutions $E_n * F$ surface potentials and denote them by

$$V_n^1(x,t) := E_n(x,t) * [u_0(x) \times \delta'(t)],$$
$$V_n^0(x,t) := E_n(x,t) * [u_1(x) \times \delta(t)].$$

We say that a distribution $u(x,t) \in \mathcal{D}'(R^{n+1})$ belongs to the class $C^p(t \geq 0)^*, p = 1,2,...,\infty$, if $(u,\varphi) \in C^p(t \geq 0)$ for all $\varphi \in \mathcal{D}(R^n)$ (cf. [5], p. 170), i.e.

$$C^p(t \geq 0)^* := \{u \in \mathcal{D}'(R^{n+1}) : (u(x,t), \varphi(x)) \in C^p(t \geq 0), \varphi \in \mathcal{D}(R^n)\}.$$

A function $u_0(x) \in C(R^n)$ is said to belong to the class S_0^n of type zero, i.e. $u_0 \in S_0^n$, if the corresponding surface potential

$$E_n(x,t) * [u_0(x) \times \delta'(t)] \in C^2(R^n \times (t > 0)) \cap C^1(R^n \times (t \geq 0)).$$

Similarly, a function $u_1(x) \in C(R^n)$ is said to belong to the class $S_1^n, u_1 \in S_1^n$, if

$$E_n(x,t) * [u_1(x) \times \delta(t)] \in C^2(R^n \times (t > 0)) \cap C^1(R^n \times (t \geq 0)).$$

It is well known that the convolution has the translation property, i.e. for $f, g \in \mathcal{D}'(R^n)$ we have (see [5])

$$f(x+h) * g(x) = f(x) * g(x+h) = (f * g)(x+h), \forall h \in R^n, \tag{7}$$

where

$$(f(x+h), \varphi(x)) := (f(x), \varphi(x-h)), \forall \varphi \in \mathcal{D}(R^n). \tag{8}$$

In addition, for the fundamental solution $E_n(x,t)$ (see [5], p. 173) we have

$$E_n(x,t) * u(x) \times \delta(t) = E_n(x,t) * u(x) \quad \text{in} \quad \mathcal{D}'(R^{n+1}), \tag{9}$$

where $E_n(x,t) * u(x)$ is defined by

$$(E_n(x,t) * u(x), \varphi(x,t)) := (E_n(x,t) \times u(y), \eta(a^2t^2 - |x|^2)\varphi(x+y,t)), \quad \varphi \in \mathcal{D}(R^{n+1}), \tag{10}$$

and $\eta(\tau)$ is any function of the class $C^\infty(R^1)$ vanishing for $-\tau < -\delta$ and equal 1 for $\tau > -\epsilon$(δ, ϵ are arbitrary numbers with $\delta > \epsilon > 0$). In view of (7)-(10) we have

$$E_n(x,t) * u(x) \times \delta(t-T) = (E_n * u \times \delta)(x, t-T) = (E_n * u)(x, t-T), \tag{11}$$

where $(E_n * u)(x, t-T) = E_n(x, t-T) * u(x)$ is defined by

$$
\begin{aligned}
(E_n(x,t-T)*u(x),\varphi(x,t)) &:= ((E_n*u)(x,t-T),\varphi(x,t))\\
&= ((E_n*u)(x,t),\varphi(x,t+T))\\
&= (E_n(x,t)\times u(y),\eta(a^2t^2-|x|^2)\varphi(x+y,t+T),\varphi\in\mathcal{D}(R^{n+1})
\end{aligned}
\tag{12}
$$

This means that

$$
E_n(x,t)*u(x)\times\delta(t-T)=E_n(x,t-T)*u(x)\quad\text{in}\quad\mathcal{D}'(R^{n+1}). \tag{13}
$$

It is well known that

$$
D^\alpha(f(x)*g(x))=D^\alpha f(x)*g(x)=f(x)*D^\alpha g(x),
$$

where $\alpha=(\alpha_1,\cdots,\alpha_n)$ is any multi-integer and $f,g\in\mathcal{D}'(R^n)$. This together with (13) implies

$$
\begin{aligned}
E_n(x,t)*u(x)\times\delta'(t-T) &= \frac{\partial}{\partial t}(E_n(x,t)*u(x)\times\delta(t-T))\\
&= \frac{\partial}{\partial t}(E_n(x,t-T)*u(x))=\frac{\partial}{\partial t}E_n(x,t-T)*u(x).
\end{aligned}
\tag{14}
$$

So we have proved

LEMMA 1. *The following equalities hold:*

$$
\begin{aligned}
&E_n(x,t)*u(x)\times\delta(t-T)=E_n(x,t-T)*u(x)\quad\text{in}\quad\mathcal{D}'(R^{n+1}),\\
&E_n(x,t)*u(x)\times\delta'(t-T)=\frac{\partial}{\partial t}E_n(x,t-T)*u(x)\quad\text{in}\quad\mathcal{D}'(R^{n+1}),
\end{aligned}
$$

where $E_n(x,t-T)*u(x)$ is defined by (12).

LEMMA 2. *Let $u_0,u_1\in\mathcal{D}'(R^n)$. Then the following statements holds:*

(i) The convolution

$$
u(x,t):=E_n(x,t)*[u_0(x)\times\delta'(t-T)+u_1(x)\times\delta(t-T)] \tag{15}
$$

is the unique solution of the operator equation

$$
P_nu(x,t)=u_0(x)\times\delta'(t-T)+u_1(x)\times\delta(t-T)\quad\text{in}\quad\mathcal{D}'(R^{n+1}).
$$

Furthermore, it has the properties:

$$u(x,t) \in C^\infty(t > T)^*, \tag{16}$$

$$P_n u(x,t) = 0 \quad in \quad \mathcal{D}'(R^n \times (t > T)), \tag{17}$$

$$u(x,t) \to u_0(x), \quad \text{as} \quad t \to +T, \quad in \quad \mathcal{D}'(R^n), \tag{18}$$

$$\frac{\partial u(x,t)}{\partial t} \to u_1(x), \quad \text{as} \quad t \to +T, \quad in \quad \mathcal{D}'(R^n). \tag{19}$$

The generalized Cauchy problem of finding the distribution $u \in \mathcal{D}'(R^{n+1})$, satisfying (17)–(19), is well-posed.

(ii) Let $u_0 \in S_0^n, u_1 \in S_1^n$. Then the distribution u of (15) has the following properties:

$$u(x,t) \in C^2(R^n \times (t > T)) \cap C^1(R^n \times (t \geq T)), \tag{20}$$

$$P_n u(x,t) = 0 \quad in \quad R^n \times (t > T), \tag{21}$$

$$u(x,t) \to u_0(x), \quad \text{as} \quad t \to +T, \tag{22}$$

$$\frac{\partial u(x,t)}{\partial t} \to u_1(x), \quad \text{as} \quad t \to +T. \tag{23}$$

The classical Cauchy problem of finding the function u of the class of (20), satisfying (21)–(23), is well-posed.

PROOF: (i) Since $E_n(x,t) * u(x) \in C^\infty(t > 0)^*$ (see [5], [6]), we obtain

$$E_n(x,t-T) * u(x) \in C^\infty(t > 0)^*.$$

By virtue of Lemma 1 we have (16). Because

$$\text{Supp}[u_0(x) \times \delta'(t-T) + u_1(x) \times \delta(t-T)] \subseteq R^n \times (t = T)$$

we obtain

$$\begin{aligned} P_n u(x,t) &= P_n E_n(x,t) * [u_0(x) \times \delta'(t-T) + u_1(x) \times \delta(t-T)] \\ &= u_0(x) \times \delta'(t-T) + u_1(x) \times \delta(t-T) \\ &= 0 \quad \text{in} \quad \mathcal{D}'(R^n, R^n \times (t \neq 0)). \end{aligned}$$

Consequently,

$$P_n u(x,t) = 0 \quad \text{in} \quad \mathcal{D}'(R^n \times (t > T)).$$

In view of (6), (11) and (14) we get

$$\begin{aligned} u(x,t) &= E_n(x,t) * [u_0(x) \times \delta'(t-T) + u_1(x) \times \delta(t-T)] \\ &= E_n'(x,t-T) * u_0(x) + E_n(x,t-T) * u_1(x) \\ &\to u_0(x), \quad \text{as} \quad t \to +T, \quad \text{in} \quad \mathcal{D}'(R^n), \end{aligned}$$

and

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} &= E_n''(x,t-T) * u_0(x) + E_n'(x,t-T) * u_1(x) \\ &\to u_1(x), \quad \text{as} \quad t \to +T, \quad \text{in} \quad \mathcal{D}'(R^n). \end{aligned}$$

Thus (16)–(19) are proved. Now, let $u_0, u_1 \in \mathcal{D}'(R^n)$ be given. Then the generalized Cauchy problem of finding a distribution $u \in \mathcal{D}'(R^{n+1})$, satisfying (17)–(19), is uniquely solvable and u is expressed by (15). From the continuity of the convolution it follows that the problem considered is well-posed.

(ii) Let $u_0 \in S_0^n, u_1 \in S_1^n$. Then by definition of S_0^n, S_1^n the distribution

$$\begin{aligned} u(x,t) &= E_n(x,t) * [u_0(x) \times \delta'(t-T) + u_1(x) \times \delta(t-T)] \\ &= (E_n' * u_0 + E_n * u_1)(x, t-T) \end{aligned} \tag{24}$$

has the property (20)–(23). Indeed, (see [5], [6]) the distribution

$$u(x.\tau) := (E_n' * u_0 + E_n * u_1)(x,\tau)$$

has the properties

$$u(x,\tau) \in C^2(R^n \times (\tau > 0)) \cap C^1(R^n \times (\tau \geq 0)), \tag{25}$$

$$P_n u(x,\tau) = 0 \quad \text{for} \quad \tau > 0, \tag{26}$$

$$u(x,\tau) \to u_0(x), \quad \text{as} \quad \tau \to +0, \tag{27}$$

$$\frac{\partial u(x,\tau)}{\partial \tau} \to u_1(x), \quad \text{as} \quad \tau \to +0. \tag{28}$$

Setting $\tau = t - T$ in (25)–(28) we get (20)–(23). The classical Cauchy problem (25)–(28) is therefore transformed to the classical Cauchy problem (20)–(23)

which has the unique solution $u(x,t)$ of (24). The stability of the problem (20)-(23) follows from that of the problem (25)–(28) (see [5], p.181). The problem (20)–(23) is thus well-posed.

2. Inverse source problem for the strip $\overline{G}_T$

2.1. *Existence and uniqueness theorems*

We denote

$$\begin{aligned} G_T &:= \{(x,t) : x \in R^n, 0 < t < T\}, T > 0, \\ \overline{G}_T &:= \{(x,t) : x \in R^n, 0 \le t \le T\}, \\ G_1 &:= \{(x,t) : x \in R^n, t > T\}, \\ H &:= \{v \in \mathcal{D}'(G_1) : P_n v = 0 \quad \text{in} \quad \mathcal{D}'(G_1)\}. \end{aligned}$$

The inverse source problem for the wave operator P_n on the closed strip $\overline{G}_T$ with respect to any distribution $v \in H$ is to find a distribution $\nu \in \mathcal{D}'(R^{n+1}, \overline{G}_T)$ satisfying

$$E_n * \nu(x,t) = v(x,t), t > T, \quad \text{in} \quad \mathcal{D}'(G_1).$$

Denote by $L(v)$ the set of all solutions of the considered problem, i.e.

$$L(v) := \{\nu \in \mathcal{D}'(R^{n+1}, \overline{G}_T) : E_n * \nu(x,t) = v(x,t), t > T\}.$$

Then the inverse source problem may be considered as the study of the following multivalued mapping

$$\begin{aligned} L : H &\to 2^{\mathcal{D}'(R^{n+1}, \overline{G}_T)}, \\ v &\mapsto L(v) \in 2^{\mathcal{D}'(R^{n+1}, \overline{G}_T)}. \end{aligned}$$

Let us introduce the following subclasses of H:

$$\begin{aligned} H_0 &:= H \cap \mathcal{D}'(R^{n+1}; \overline{G}_1), \\ H_1 &:= H \cap C^1(t \ge T)^*, \\ H_2 &:= H \cap S^n(R^n \times (t \ge T)), \end{aligned}$$

where

$$S^n(R^n \times (t \geq T)) := \{v \in C^2(R^n \times (t > T)) \cap C^1(R^n \times (t \geq T)) : \\ v(x,T) \in S_0^n, \frac{\partial v(x,t)}{\partial t}|_{t=T} \in S_1^n\}.$$

Then we have

$$H_2 \subset H_1 \subset H_0 \subset H. \tag{29}$$

Further, let us introduce the following subclasses of $\mathcal{D}'(R^{n+1}, R^n \times (t = T))$:

$$A^*(R^n \times (t = T)) := \{\nu = v_0(x) \times \delta'(t - T) + \\ + v_1(x) \times \delta(t - T) : v_0, v_1 \in \mathcal{D}'(R^n)\}, \\ A^{**}(R^n \times (t = T)) := \{\nu \in A^*(R^n \times (t = T)) : v_0 \in S_0^n, v_1 \in S_1^n\}.$$

Then

$$A^{**}(R^n \times (t = T)) \subset A^*(R^n \times (t = T)) \subset \mathcal{D}'(R^{n+1}, R^n \times (t = T)). \tag{30}$$

A relation between (29), (30) is established in the following

THEOREM 1. (Inverse Statements)

(i) $\forall v \in H_0 \quad \exists \nu \in L(v) \cap \mathcal{D}'(R^{n+1}, R^n \times (t = T))$,
(ii) $\forall v \in H_1 \quad \exists! \nu \in L(v) \cap A^*(R^n \times (t = T))$,
(iii) $\forall v \in H_2 \quad \exists! \nu \in L(v) \cap A^{**}(R^n \times (t = T))$.

PROOF: (i) Let $v \in H \cap \mathcal{D}'(R^{n+1}, \overline{G}_1)$. Since supp $v \subset \overline{G}_1$, we have

$$P_n v(x,t) = 0 \quad \text{for} \quad t < T \quad \text{in} \quad \mathcal{D}'(R^{n+1}). \tag{31}$$

On the other hand, by the definition of H we have

$$P_n(v(x,t) = 0 \quad \text{for} \quad t > T \quad \text{in} \quad \mathcal{D}'(R^{n+1}). \tag{32}$$

Combining (31) and (32) we get

$$P_n v(x,t) = 0 \quad \text{for} \quad t \neq T \quad \text{in} \quad \mathcal{D}'(R^{n+1}) \tag{33}$$

which yields supp $P_n v \subseteq R^n \times (t = T)$. By the definition of the fundamental solution we obtain

$$\begin{aligned} v(x,t) &= \delta(x,t) * v(x,t) = (P_n E_n(x,t)) * v(x,t) \\ &= E_n(x,t) * P_n v(x,t) \quad \text{in} \quad \mathcal{D}'(R^{n+1}). \end{aligned}$$

Consequently,

$$v(x,t) = E_n(x,t) * P_n v(x,t), t > T \quad \text{in} \quad \mathcal{D}'(R^{n+1}).$$

That means $P_n v \in L(v)$. In view of (33),

$$\nu = P_n v \in L(v) \cap \mathcal{D}'(R^{n+1}, R^n \times (t = T)).$$

(ii) Let $v \in H_1 = H \cap C^1(t \geq T)^*$. Then $v \in \mathcal{D}'(R^{n+1}) \subset \mathcal{D}'(R^n \times (t > T))$, and $(v(x,t), \varphi(x)) \in C^1(t \geq T) \quad \forall \varphi \in \mathcal{D}(R^n)$. From this there exist $v_0(x)$, and $v_1(x)$ in $\mathcal{D}'(R^n)$ such that

$$\begin{aligned} (v_0(x), \varphi(x)) &= \lim_{t \to +T} (v(x,t), \varphi(x)), \varphi \in \mathcal{D}(R^n), \\ (v_1(x), \varphi(x)) &= \lim_{t \to +T} \frac{\partial}{\partial t} (v(x,t), \varphi(x)), \varphi \in \mathcal{D}(R^n). \end{aligned}$$

Then we have

$$v(x,t) \to v_0(x), \quad \text{as} \quad t \to +T, \quad \text{in} \quad \mathcal{D}'(R^n), \tag{34}$$

$$\frac{\partial v(x,t)}{\partial t} \to v_1(x), \quad \text{as} \quad t \to +T, \quad \text{in} \quad \mathcal{D}'(R^n). \tag{35}$$

The potential

$$v(x,t) = E_n(x,t) * [v_0(x) \times \delta'(t-T) + v_1(x) \times \delta(t-T)] \tag{36}$$

is the unique solution of the generalized Cauchy problem (see Lemma 2) for the wave equation $P_n v(x,t) = 0$ for $t > T$ with the initial conditions (34), (35). Hence

$$\nu := v_0(x) \times \delta'(t-T) + v_1(x) \times \delta(t-T) \in L(v) \cap A^*(R^n \times (t = T)).$$

Now suppose that there are two distributions ν_1, ν_2 of the class $L(v) \cap A^*(R^n \times (t = T))$, i.e. there exist $v_1^i, v_0^i \in \mathcal{D}'(R^n), i = 1, 2$, such that

$$\nu_i(x,t) = v_0^i(x) \times \delta'(t-T) + v_1^i(x) \times \delta(t-T), i = 1, 2, \tag{37}$$

$$\begin{aligned} E_n * \nu_i(x,t) &= E_n * \nu(x,t) = \\ &= v(x,t), t > T, \text{ in } \mathcal{D}'(R^{n+1}), i = 1, 2. \end{aligned} \tag{38}$$

Substituting ν_i of (37), by virtue of Lemma 2 (i) and (34), (35), we obtain

$$\begin{aligned} E_n * \nu_i(x,t) &= E_n(x,t) * [v_0^i(x) \times \delta'(t-T) + v_1^i(x) \times \delta(t-T)] \\ &\to v_0(x) = v_0^i(x), \text{ as } t \to +T, \text{ in } \mathcal{D}'(R^n), \ i = 1, 2. \end{aligned}$$

This together with (37) implies

$$\nu_1 = \nu_2 = v_0(x) \times \delta'(t-T) + v_1(x) \times \delta(t-T) = \nu.$$

(iii) The proof of this part is analogous to that of (ii).

REMARK 1: Fixing T we may consider the inverse source problem on the closed domain

$$\overline{G_{T_0,T}} := \{(x,t) : x \in R^n, T_0 \le t \le T\}, 0 \le T_0 < T,$$

as the study of the solution set

$$L_{T_0}(v) := \{\nu \in \mathcal{D}'(R^{n+1}, \overline{G_{T_0,T}}) : E * \nu(x,t) = v(x,t), t > T, \text{ in } \mathcal{D}'(G_1)\}$$

with $v \in H$. In this case we have the same results as in Theorem 1.

REMARK 2: (i) Let $v_0, v_1 \in \mathcal{D}'(R^n), 0 \le T_1 \le T_2$. Then the distribution

$$v(x,t) = E_n(x,t) * [v_0(x) \times \delta'(t-T) + v_1(x) \times \delta(t-T_1)] \tag{39}$$

for $t \ge T_2$ is the unique solution of the generalized Cauchy problem:

$$u(x,t) \in C^\infty(t > T)^*, \tag{40}$$

$$P_n u(x,t) = 0, t > T_2, \quad \text{in} \quad \mathcal{D}'(R^{n+1}), \tag{41}$$

$$u(x,t) \to v(x,T_2), \quad \text{as} \quad t \to +T_2, \quad \text{in} \quad \mathcal{D}'(R^n), \tag{42}$$

$$\frac{\partial u(x,t)}{\partial t} \to \frac{\partial v(x,t)}{\partial t}\Big|_{t=T_2}, \quad \text{as} \quad t \to +T_2, \quad \text{in} \quad \mathcal{D}'(R^n). \tag{43}$$

(ii) Let $v_0 \in S_0^n, v_1 \in S_1^n, 0 \le T_1 \le T_2$. Then the distribution $v(x,t)$ of (39) is the unique solution of the following classical Cauchy problem for $t \ge T_2$:

$$u \in C^2(R^n \times (t > T_2)) \cap C^1(R^n \times (t \ge T_2)), \tag{44}$$

$$P_n u(x,t) = 0, t > T_2, \tag{45}$$

$$u(x,T_2) = v(x,T_2) \in S_0^2, \tag{46}$$

$$\frac{\partial u(x,T_2)}{\partial t} = \frac{\partial v(x,T_2)}{\partial t} \in S_1^2. \tag{47}$$

Now we introduce the following classes:

$$A^*(G_T) := \{\nu = v_0(x) \times \delta'(t-T_1) + v_1(x) \times \delta(t-T_2) : v_0, v_1 \in \mathcal{D}'(R^n)$$
$$0 \le T_1,\ T_2 < T\},$$
$$A^{**}(G_T) := \{\nu \in A^*(G_T) : v_0 \in S_0^n, v_1 \in S_1^n\}.$$

denoting by $E_n * \nu|_{G_1}$, as usual, the restriction of $E_n * \nu$ on G_1 we have the following

THEOREM 2. (Direct Statement) *The following statements holds:*

(i) $\pi_1 : A^*(G_T) \to H_1 \subset \mathcal{D}'(G_1)$

$\nu \mapsto \pi_1\nu := E_n * \nu|_{G_1} \in H_1;$

(ii) $\pi_2 : A^{**}(G_T) \to H_2 \subset \mathcal{D}'(G_1)$

$\nu \mapsto \pi_2\nu := E_n * \nu|_{G_1} \in H_2;$

(iii) $\pi_3 : \mathcal{S}(G_T) \to \mathcal{S}'(G_1)$

$\nu \mapsto \pi_3\nu := E_n * \nu|_{G_1} \in H_2 \cap \mathcal{S}'(G_1).$

PROOF: (i) Let $\nu \in A^*(G_T)$, i.e. there exist T_1, T_2 with $0 \le T_1, T_2 < T$ and $v_0, v_1 \in \mathcal{D}'(R^n)$ such that

$$\nu(x,t) = v_0(x) \times \delta'(t-T_1) + v_1(x) \times \delta(t-T).$$

From Lemma 2 it follows that $E_n * \nu \in C^\infty(t > \max(T_1,T_2))^*$. Since $T_1 < T, T_2 < T$, we get $E_n * \nu \in C^1(t \ge T)^*$. In addition, because

$$P_n(E_n * \nu(x,t)) = v_0(x) \times \delta'(t-T_1) + v_1(x) \times \delta(t-T_2) = 0 \text{ in } \mathcal{D}'(G_1),$$

we obtain $E_n * \nu|_{G_1} \in H_1$.

(ii) Let $\nu \in A^{**}(G_T)$ with

$\nu(x,t) = v_0(x) \times \delta'(t-T) + v_1(x) \times \delta(t-T), 0 \leq T_1, T_2 < T, v_0 \in S_0^n, v_1 \in S_1^n$.

From Remark 2 (ii) it follows that

$$E_n * \nu(x,T) \in S_0^n,$$
$$\frac{\partial}{\partial t} E_n * \nu(x,T) \in S_1^n.$$

By definition we have $E_n * \nu(x,t)|_{G_1} \in H_2 = H \cap S(t \geq T)$.

(iii) Let $\nu \in \mathcal{S}(G_T)$. Since $E_n \in \mathcal{S}'(R^{n+1})$, the convolution $E_n * \nu$ exists in θ_M (see [6], p. 98). Therefore $E_n * \nu|_{G_1} \in H_2$. Hence the theorem is proved.

Corollary. (Sweeping-Out Principle)

(i) $\forall \nu \in A^*(G_T) \quad \exists! \nu' \in L'(\nu) \cap A^*(R^n \times (t=T))$,
(ii) $\forall \nu \in A^{**}(G_T) \quad \exists! \nu' \in L'(\nu) \cap A^{**}(R^n \times (t=T))$,
(iii) $\forall \nu \in \mathcal{S}(G_T) \quad \exists! \nu' \in L'(\nu) \cap A^{**}(R^n \times (t=T))$,

where $L'(\nu) := L(E_n * \nu|_{G_1})$.

Proof: (i) Let $\nu \in A^*(G_T)$, i.e. there exist $v_0, v_1 \in \mathcal{D}'(R^n)$ and T_1, T_2 with $0 \leq T_1, T_2 < T$ such that

$$\nu = v_0(x) \times \delta'(t-T_1) + v_1(x) \times \delta(t-T_2).$$

By Theorem 2 we have $E_n * \nu|_{G_1} \in H_1$. By Theorem 1 there exists a uniquely determined distribution $\nu' \in L(E_n * \nu|_{G_1}) \cap A^*(R^n \times (t=T))$ or $\nu' \in L'(\nu) \cap A^*(R^n \times (t=T))$.

(ii), (iii). The proof of these parts is analogous to that of (i).

2.2. Structure of the solution set $L(v)$

As in the case of the heat conduction operator (see [4]) we introduce the set

$$E(\overline{G}_T) := \{\nu \in \mathcal{D}'(R^{n+1}, \overline{G}_T) : E_n * \nu \text{ exists in } \mathcal{D}'(R^{n+1})\}.$$

We have $E(\overline{G}_T) = \mathcal{D}'(R^{n+1}, \overline{G}_T)$ (see Section 1). Let us consider the mapping

$$\pi : E(\overline{G}_T) \to \mathcal{D}'(G_1)$$
$$\nu \mapsto \pi\nu := E_n * \nu|_{G_1},$$

and denote Im $\pi := \pi(E(\overline{G}_T))$. It is obvious that

$$\pi(E(\overline{G}_T)) \subseteq H \subset \mathcal{D}'(G_1),$$
$$\mathcal{D}'(R^{n+1}, \overline{G}_1) \subset \mathcal{D}'(R^{n+1}) \subset \mathcal{D}'(G_1).$$

From this we obtain

$$\pi(E(\overline{G}_T)) \cap \mathcal{D}'(R^{n+1}, \overline{G}_1) \subseteq H_0 = H \cap \mathcal{D}'(R^{n+1}, \overline{G}_1) \subset \mathcal{D}'(R^{n+1}) \subset \mathcal{D}'(G_1). \tag{48}$$

Let $v_0 \in H_0$. Then by Theorem 1 (i) there exists an element $\nu \in E(\overline{G}_T) = \mathcal{D}'(R^{n+1}, \overline{G}_T)$ such that $\pi\nu = E_n * \nu|_{G_1} = v_0$. This means $v_0 = \pi\nu \in \pi(E(\overline{G}_T)) \cap \mathcal{D}'(R^{n+1}, \overline{G}_1)$. Consequently, $H_0 \subseteq \pi(E(\overline{G}_T)) \cap \mathcal{D}'(R^{n+1}, \overline{G}_1)$. This together with (48) implies

$$H_0 = \text{ Im } \pi \cap \mathcal{D}'(R^{n+1}, \overline{G}_T).$$

Using this we consider the chain $H_2 \subset H_1 \subset H_0 \subset H$ and the corresponding so-called multivalued mappings L, L_0, L_1, L_2:

$$\begin{array}{ccccccc} H_2 & \subset & H_1 & \subset & H_0 & \subset & H \\ \downarrow L_2 & & \downarrow L_1 & & \downarrow L_0 & & \downarrow L \\ 2^{\mathcal{D}'(R^{n+1}, \overline{G}_T)} & & 2^{\mathcal{D}'(R^{n+1}, \overline{G}_T)} & & 2^{\mathcal{D}'(R^{n+1}, \overline{G}_T)} & & 2^{\mathcal{D}'(R^{n+1}, \overline{G}_T)} \end{array}$$

where $L_i(v) := L(v), i = 0, 1, 2.$

For each element $v \in H$ it is not known whether the solution set $L(v)$ is empty or not. As we have shown (see Theorem 1 (i)), for each element $v \in H_0$ there exists at least an element $\nu \in L_0(v)$. So it is reasonable to study the set

$$\text{Im } L_0 := \{L_0(v) : v \in H_0\}.$$

We introduce in Im L_0 the additive operation $\oplus$ and the multiplication with real number λ as follows:

$$L_0(v_1) \oplus L_0(v_2) := \{\nu \in \mathcal{D}'(R^{n+1}, \overline{G}_T) : E_n * \nu(x,t) = (v_1 + v_2)(x,t), t > T\}, \tag{49}$$

$$\lambda L_0(v) := \{\nu \in \mathcal{D}'(R^{n+1}, \overline{G}_T) : E_n * \nu(x,t) = \lambda v(x,t), t > T\}, \tag{50}$$

where $v_1, v_2, v \in H_0$. It is easy to verify that the following properties hold:

- Commutativity: $L_0(v_1) \oplus L_0(v_2) = L_0(v_2) \oplus L_0(v_1)$, (51)
- Associativity: $[L_0(v_1) \oplus L_0(v_2)] \oplus L_0(v_3) = L_0(v_1) \oplus [L_0(v_2) \oplus L_0(v_3)]$, (52)
- Homogeneity: $\lambda L_0(v) = L_0(\lambda v)$, (53)
- Distributivity: $\alpha L_0(v_1) \oplus \beta L_0(v_2) = L_0(\alpha v_1 + \beta v_2)$. (54)

There is the zero element

$$L_0(0) = \{\nu \in \mathcal{D}'(R^{n+1}, \overline{G}_T) : E_n * \nu(x,t) = 0,\ t > T\},$$

with the properties

$$L_0(v) \oplus L_0(0) = L_0(v) \quad \forall v \in H_0,$$
$$\lambda L_0(0) = L_0(0) \quad \forall \lambda \in R.$$

Note that the zero element $L_0(0)$ is usually called the null effect. We have Ker $\pi = L_0(0)$. A multivalued mapping

$$F : X \to 2^{\mathcal{D}'(R^{n+1}, \overline{G}_T)}$$

is said to be weakly closed and convex (cf. [3]) if the set $F(x)$ is convex and closed with respect to the weak topology of $\mathcal{D}'(R^{n+1})$ for every $x \in X$.

THEOREM 3. *The mappings L_0, L_1, L_2 are convex and weakly closed.*

PROOF: The convexity of L_0, L_1, L_2 is easy to verify. It remains to show their weak closedness. Let v be an arbitrary element of H_0 and $\{\nu_i\}, i = 1, 2, \ldots,$ any sequence of $L_0(v)$ which weakly converges to v_0 in $\mathcal{D}'(R^{n+1})$. We shall verify that $v_0 \in L_0(v)$. Indeed, by the definition of $\nu_i \in \mathcal{D}'(R^{n+1}, \overline{G}_T)$ we obtain

$$(\nu_0, \varphi) = \lim_{i \to \infty} (\nu_i, \varphi) = 0 \quad \forall \varphi \in \mathcal{D}(R^{n+1} \setminus \overline{G}_T). \tag{55}$$

Hence $\nu_0 \in \mathcal{D}'(R^{n+1}, \overline{G}_T)$. On the other hand, from the assumption

$$E_n * \nu_i(x,t) = v(x,t), t > 0, \text{ in } \mathcal{D}'(R^{n+1}), i = 1, 2, \ldots,$$

and the continuity of the convolution it follows that

$$E_n * \nu_0(x,t) = v(x,t), t > 0, \text{ in } \mathcal{D}'(R^{n+1}).$$

This together with (55) implies $\nu_0 \in L_0(v)$. Analogously, from the fact that

$$L_i(v) = L_0(v) = L(v) \qquad \forall v \in H_i, i = 1, 2,$$

it follows that L_1, L_2 are weakly closed. The proof is complete.

2.3. Stability

THEOREM 4. *The following transformations*

$$\begin{aligned} T_1 : H_1 \to \ & \mathit{Im}\ T_1 \subset A^*(R^n \times (t = T)), \\ & v \mapsto T_1(v) := \nu \in L(v) \cap A^*(R^n \times (t = T)), \\ T_2 : H_2 \to \ & \mathit{Im}\ T_2 \subset A^{**}(R^n \times (t = T)), \\ & v \mapsto T_2(v) := \nu \in L(v) \cap A^{**}(R^n \times (t = T)). \end{aligned}$$

are isomorphic and homeomorphic.

PROOF: By Theorem 1, for each $v \in H_1 = H \cap C^1(t \geq T)^*$ there exists a uniquely determined distribution $\nu \in L(v) \cap A^*(R^n \times (t = T))$. So the transformation T_1 is well-defined. Conversely, let $\nu \in \text{Im } T_1$, i.e. there is an element $v \in H_1$ such that $T_1(v) = \nu$. That means there exists the inverse

$$\begin{aligned} T_1^{-1} :\ & \text{Im } T_1 \to H_1, \\ & \nu \mapsto T_1^{-1}(\nu) = E_n * \nu|_{G_1} = v \text{ in } \mathcal{D}'(G_1). \end{aligned}$$

The transformation T_1 is therefore an isomorphism. Because of the weak continuity of the convolution it is also an homeomorphism. The similar statement about T_2 is proved analogously.

There is a problem of coupling several fields such as Newtonian potentials and wave potentials (see [1], [2]), which we will discuss in other separate paper.

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