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ON THE LAWS OF LARGE NUMBERS FOR MARTINGALE DIFFERENCES IN VON NEUMANN ALGEBRAS

NGUYEN VAN QUANG AND NGUYEN DUY TIEN

Abstract. The aim of this paper is to investigate Laws of Large Numbers in von Neumann algebras with emphasis on the bilateral almost uniform convergence and the convergence in measure of weighted averages of martingale differences. The stability of quadratic forms in martingale differences is also studied.

1. Introduction and Notations

Recently a great deal of works has been done in order to generalize various strong limit theorems to the von Neumann algebras context (c.f. [2], [5], [7]). Following this direction of research, we study the bilateral almost uniform convergence and the convergence in measure of weighted averages and quadratic forms in martingale differences in von Neumann algebras. The results of our paper are related to those of [2], [3], [7], [8], [9].

We start with some definitions and notations.

Let H be a Hilbert space, $\mathcal{B}(H)$ the set of all bounded linear operators on H and $\mathcal{A} \subseteq \mathcal{B}(H)$ a finite von Neumann algebra. Let τ be a faithful normal trace on $\mathcal{A}$. We denote by $\widetilde{\mathcal{A}}$ the algebra of measurable operators in Segal's sence (c.f. [10]). For every fixed real number $r \geq 1$, one can define the Banach spaces $L^r(\mathcal{A}, \tau)$ of (possibly unbounded) operators on H as the noncommutative analogue of the Lebesgue space in [12]. Note that if $\mathcal{B}$ is a von Neumann subalgebra of $\mathcal{A}$ and τ_1 is the restriction of τ to $\mathcal{B}$ ($\tau_1 = \tau/\mathcal{B}$), then $L^r(\mathcal{B}, \tau_1) \subset L^r(\mathcal{A}, \tau)$ for all $r \geq 1$. Umegaki [11] has defined the conditional expectation $E^{\mathcal{B}} : L^1(\mathcal{A}, \tau) \to L^1(\mathcal{B}, \tau_1)$ by :

$$\tau(xy) = \tau((E^{\mathcal{B}}x)y) \quad x \in \mathcal{A},\ y \in \mathcal{B}. \tag{1.1}$$

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Note that $E^{\mathcal{B}}$ is a positive linear mapping of norm one and uniquely defined by (1.1) (c.f. [3], [11] for more details).

Let $(\mathcal{B}_n)$, $n \in \mathbb{N}$, be an increasing sequence of von Neumann subalgebras of $\mathcal{A}$.

DEFINITION 1.1: a) A sequence $(x_n) \subset L^1(\mathcal{A}, \tau)$ is said to be an L^1-martingale if for all $n \in \mathbb{N}$, (i) $x_n \in L^1(\mathcal{B}_n, \tau/\mathcal{B}_n)$ and (ii) $E^{\mathcal{B}_n} x_{n+1} = x_n$.

b) (x_n) is an L^1-martingale difference sequence (L^1-m.d. for short) if it satisfies the condition (i) and $E^{\mathcal{B}_n} x_{n+1} = 0$.

For the general theory of von Neumann algebras and the martingale theory in von Neumann algebras, the reader is referred to [3], [5].

DEFINITION 1.2: (Toeplitz matrix) An array (a_{nk}) of real numbers is said to be a Toeplitz matrix if the following conditions are satisfied :

(i) $\lim_{n\to\infty} a_{nk} = 0$ for each $k \geq 1$.

(ii) $\sum_{k=1}^{n} |a_{nk}| \leq M$ for each $n \geq 1$ and some $M > 0$.

If p is a projection in $\mathcal{A}$, $p^{\perp}$ denotes $(I - p)$ where I is the identity operator. Denote by 1_A the indicator function of the set A.

LEMMA 1.1. (Prop. 3 [1]). *Let (x_n) be an L^1-martingale. If $\sup_n \tau(|x_n|) < \infty$, then there exists an element $x \in L^1(\mathcal{A}, \tau)$ such that $x_n \to x$ b.a.u. (bilaterally almost uniformly), i.e. for each $\epsilon > 0$ there exists a projection $p \in \mathcal{A}$ such that $\tau(p^{\perp}) < \epsilon$, $p(x_n - x)p \in \mathcal{A}$ $(\forall n)$ and $\|p(x_n - x)p\| \to 0$ as $n \to \infty$.*

LEMMA 1.2. (Kronecker's Lemma [2]). *Let (x_n) be a sequence in $\widetilde{\mathcal{A}}$ and (b_n) a sequence of real numbers such that $b_n \uparrow \infty$. Put $S_n = \sum_{k=1}^{n} x_k$. If S_n converges b.a.u. then $b_n^{-1} \sum_{k=1}^{n} b_k x_k$ converges b.a.u. to zero.*

This paper is organized as follows : The convergence of weighted sums is studied in Section 2, where the main result is Theorem 2.2. Section 3 deals with the stability of quadratic forms in von Neumann algebras.

2. Convergence of weighted sums

Throughout this section the following assumptions are made : (x_n) is an L^1-martingale difference sequence (L^1-m.d. for short); (a_n), (A_n) are two sequences of real numbers such that $a_n > 0$, $A_n > 0$, $A_n \uparrow \infty$ and $a_n/A_n \to 0$ as $n \to \infty$. Let $S_n = \sum_{k=1}^n a_k x_k$, $n = 1, 2, ..,$ denote the partial weighted sums.

PROPOSITION 2.1. *Let (x_n) be an L^1-m.d. If $\sum_{n=1}^\infty (\frac{a_n}{A_n})^2 \tau(|x_n|^2) < +\infty$, then $S_n/A_n \to 0$ b.a.u.*

PROOF: Note that if (x_n) is an L^1-m.d., the sequence $(\sum_{k=1}^n (a_k/A_k)x_k), n \in \mathbf{N}$, is an L^1-martingale. Thus, by Kronecker's Lemma and Lemma 1.1, it suffices to show that $\sup_n \tau(|\sum_{k=1}^n (a_k/A_k)x_k|) < \infty$. Observe that for all $i, j \in N$, $i > j$,

$$\tau(x_i^* x_j) = \tau((E^{\mathcal{B}_j} x_i^*)x_j) = \tau((E^{\mathcal{B}_j} x_i)^* x_j) = \tau((E^{\mathcal{B}_j}(E^{\mathcal{B}_{i-1}} x_i))^* x_j) = 0.$$

This means that (x_n) is a sequence of pairwise orthogonal elements from $L^2(\mathcal{A}, \tau)$. Consequently,

$$\begin{aligned}\tau(|\sum_{k=1}^n (a_k/A_k)x_k|) &\le [\tau(|\sum_{k=1}^n (a_k/A_k)x_k|^2)]^{1/2} \\ &= [\sum_{k=1}^n (a_k/A_k)^2 \tau(|x_k|^2)]^{1/2} \\ &\le [\sum_{k=1}^\infty (a_k/A_k)^2 \tau(|x_k|^2)]^{1/2} < \infty.\end{aligned}$$

In what follows, $\int f(x)|dF(x)|$ denotes the integral $\int f(x)d(-F(x))$, where F is a non-increasing function. Note that if $H_i(x)$, $i = 1, 2$, are non-increasing functions such that $H_i(x) \to 0$ as $x \to \infty$ and $H_1(x) \ge H_2(x)$ for all $x \ge 0$ and if $f(x)$ is a nondecreasing positive function in x for all $x \ge 0$, then $\int_a^\infty f(x)|dH_1(x)| \ge \int_a^\infty f(x)|dH_2(x)|$ for all $a \ge 0$.

THEOREM 2.2. *Let (x_n) be an L^1-m.d. satisfying the following conditions:*

$$F(\lambda) = \sup_n \tau(e_{[\lambda,\infty)}(|x_n|)) \to 0 \quad as \quad \lambda \to \infty, \tag{2.0}$$

$$\int_0^\infty \lambda^2 \int_{y \ge \lambda} y^{-3} N(y) dy |dF(\lambda)| < \infty, \tag{2.1}$$

$$\int_0^\infty \lambda \int_{0<y<\lambda} y^{-2} N(y) dy |dF(\lambda)| < \infty, \tag{2.2}$$

where $N(y) = \text{card}\,\{n : (A_n/a_n) \leq y\} = \sum_{n=1}^\infty 1_{\{y: y \geq A_n/a_n\}}$. Then $S_n/A_n \to 0$ b.a.u.

PROOF: . By Kronecker's Lemma it is enough to show that the L^1-martingale $\sum_{k=1}^n (a_k/A_k)x_k$ converges b.a.u. We show that

$$\sup_n \tau(|\sum_{k=1}^n (a_k/A_k)x_k|) < \infty. \tag{2.3}$$

Put

$$y_k = (a_k/A_k)x_k e_{[0,A_k/a_k]}(|x_k|),$$
$$z_k = (a_k/A_k)x_k e_{(A_k/a_k,\infty)}(|x_k|).$$

It is easy to see that

$$E^{\mathcal{B}_{k-1}} z_k = -E^{\mathcal{B}_{k-1}} y_k. \tag{2.4}$$

Using the triangle inequality and (2.4) we get

$$\tau(|\sum_{k=1}^n (a_k/A_k)x_k|) = \tau(|\sum_{k=1}^n (y_k + z_k)|) \tag{2.5}$$
$$\leq \tau(|\sum_{k=1}^n (y_k - E^{\mathcal{B}_{k-1}} y_k)|) + \tau(|\sum_{k=1}^n (z_k - E^{\mathcal{B}_{k-1}} z_k)|).$$

On the other hand, since $(y_k - E^{\mathcal{B}_{k-1}} y_k)$ is a sequence of pairwise orthogonal elements from $L^2(\mathcal{A}, \tau)$ and $E^{\mathcal{B}_{k-1}}(\cdot)$ are the orthogonal projections in $L^2(\mathcal{A}, \tau)$, we have

$$\tau(|\sum_{k=1}^n (y_k - E^{\mathcal{B}_{k-1}} y_k)|^2) = \sum_{k=1}^n \tau(|y_k - E^{\mathcal{B}_{k-1}} y_k)|^2)$$
$$= \sum_{k=1}^n (\tau(|y_k|^2) - \tau(|E^{\mathcal{B}_{k-1}} y_k)|^2)) \leq \sum_{k=1}^n (\tau(|y_k|^2)$$
$$= \sum_{k=1}^n (a_k/A_k)^2 \int_{[0,A_k/a_k]} \lambda^2 d\tau(e_{[0,\lambda)}(|x_k|))$$

$$
\begin{aligned}
&= \sum_{k=1}^{n} (a_k/A_k)^2 \int_{[0,A_k/a_k]} \lambda^2 |d\tau(e_{[\lambda,\infty)}(|x_k|))| \\
&\leq \sum_{k=1}^{\infty} (a_k/A_k)^2 \int_{[0,A_k/a_k]} \lambda^2 |dF(\lambda)| \\
&= \int_0^\infty \lambda^2 \sum_{\{k: A_k/a_k \geq \lambda\}} (a_k/A_k)^2 |dF(\lambda)| \\
&= \int_0^\infty \lambda^2 \int_{y \geq \lambda} y^{-3} N(y) dy |dF(\lambda)| < \infty.
\end{aligned}
$$

Thus,

$$
\sup_n \tau(|\sum_{k=1}^{n}(y_k - E^{\mathcal{B}_{k-1}} y_k)|) \leq \sup_n \tau[(|\sum_{k=1}^{n}(y_k - E^{\mathcal{B}_{k-1}} y_k)|^2)]^{1/2} < \infty \tag{2.6}
$$

We now estimate the second term of (2.5). Note that $E^{\mathcal{B}_{k-1}}(\cdot)$ are linear mappings of norm one in $L^1(\mathcal{A}, \tau)$. Then we obtain

$$
\begin{aligned}
\tau(|\sum_{k=1}^{n}(z_k - E^{\mathcal{B}_{k-1}} z_k)|) &\leq 2 \sum_{k=1}^{n} \tau(|z_k|) \qquad (2.7) \\
&= 2 \sum_{k=1}^{n} (a_k/A_k) \int_{\lambda > A_k/a_k} \lambda |d\tau(e_{[\lambda,\infty)}(|x_k|))| \\
&\leq 2 \sum_{k=1}^{\infty} (a_k/A_k) \int_{\lambda > A_k/a_k} \lambda |dF(\lambda)| \\
&= 2 \int_0^\infty \lambda \{ \sum_{\{k: 0 < A_k/a_k < \lambda\}} (a_k/A_k) \} |dF(\lambda)| \\
&= 2 \int_0^\infty \lambda \int_{0<y<\lambda} y^{-2} N(y) dy |dF(\lambda)| < \infty.
\end{aligned}
$$

Combining (2.6) and (2.7) we obtain (2.3). By Lemma 1.1 the proof is complete.

Recall that a sequence $(x_n) \subset \widetilde{\mathcal{A}}$ is said to be dominated in tail measure by a positive measurable operator $x \in \widetilde{\mathcal{A}}$ ($(x_n) < x$ for short) if there exists a positive constant C such that for all $t > 0$ and all $n \in \mathbf{N}$

$$
\tau(e_{[t,\infty)}(|x_n|)) \leq C\tau(e_{[t,\infty)}(x)).
$$

We now present some consequences of Theorem 2.1. The proofs are similar to those for some corollaries of Section 3 [8] (by using Theorem 2.2).

COROLLARY 2.3. . *If $1 < r < 2$ and (x_n) is an L^1-m.d. such that $(x_n) < x$ with $\tau(x^r) < \infty$, then $n^{-1/r}\sum_{k=1}^n x_k \to 0$ b.a.u.*

COROLLARY 2.4. . *Let $1 \leq r \leq 2$, $a_n > 0$, $(a_n) \in \ell_\infty$ and $A_n = (\sum_{k=1}^n a_k^r)^{(1/r)}$,
$A_n \uparrow \infty$ as $n \to \infty$. If (x_n) is an L^1-m.d. such that $(x_n) < x$ with $\tau(x^r \log^+ x) < \infty$, then $S_n/A_n \to 0$ b.a.u.*

Note that if $r = 1$, we get the Strong Law of Large Numbers for martingale differences in von Neumann algebras.

We end this section with a result on the convergence in L^1 and in measure of weighted sums.

THEOREM 2.5. *Suppose that (a_{nk}) is a Toeplitz matrix of real numbers, (x_n) is an L^1-m.d. such that $(x_n) < x$. If*

i) $\max_{1\leq k\leq n} a_{nk} \to 0$ *as* $n \to \infty$,

ii) $\lim_{T\to\infty} \tau(xe_{(T,\infty)}(x)) = 0$,

then $S_n = \sum_{k=1}^n a_{nk}x_k \to 0$ in $L^1(\mathcal{A}, \tau)$ and in measure.

PROOF: Since the convergence in $L^1(\mathcal{A}, \tau)$ implies the convergence in measure, we have to prove the conclusion in $L^1(\mathcal{A}, \tau)$ only. Put

$$x_{nk} = a_{nk}x_k e_{[0,a_{nk}^{-1}]}(|x_k|),$$
$$\overline{S}_n = \sum_{k=1}^n (x_{nk} - E^{\mathcal{B}_{k-1}}x_{nk}).$$

We now can suppose $a_{nk} \neq 0$ for all n and k. For n large enough, by the assumption we have

$$\tau(|\overline{S}_n|^2) = \sum_{k=1}^n [\tau(|x_{nk}|^2) - \tau(|E^{\mathcal{B}_{k-1}}x_{nk}|^2)] \tag{2.8}$$

$$
\begin{aligned}
&\leq \sum_{k=1}^{n} \tau(|x_{nk}|^2) \\
&\leq 2 \sum_{k=1}^{n} |a_{nk}|^2 \int_{\lambda \leq |a_{nk}|^{-1}} \lambda \tau(e_{[\lambda,\infty)}(|x|)) d\lambda \\
&\leq 2C \sum_{k=1}^{n} |a_{nk}| (\frac{1}{|a_{nk}|^{-1}}) \int_{\lambda \leq |a_{nk}|^{-1}} \lambda \tau(e_{[\lambda,\infty)}(|x_{nk}|)) d\lambda \\
&= 2C\epsilon \sum_{k=1}^{n} |a_{nk} \leq 2CM\epsilon,
\end{aligned}
$$

where M is a positive constant by the property of the Toeplitz matrix. The latter inequality follows from the fact that if $T\tau(e_{(T,\infty)}(x)) \to 0$ as $T \to \infty$, then

$$
T^{-1} \int_{0 \leq \lambda \leq T} \lambda \tau(e_{(\lambda,\infty)}(x)) d\lambda \to 0.
$$

From (2.8) we get

$$
\tau(|\overline{S}_n|) \leq (\tau(|\overline{S}_n|^2)^{1/2} \to 0 \quad as \quad n \to \infty. \tag{2.9}
$$

Consequently, since $E^{\mathcal{B}_{k-1}} x_{nk} = -E^{\mathcal{B}_{k-1}}(x_k - x_{nk})$ for all $k \geq 1$, we have for n large enough

$$
\begin{aligned}
\tau(|\sum_{k=1}^{n} E^{\mathcal{B}_{k-1}} x_{nk}|) =& \tau(|\sum_{k=1}^{n} E^{\mathcal{B}_{k-1}}(a_{nk} x_k e_{(|a_{nk}|^{-1},\infty)}(|x_k|)) \\
&\leq \sum_{k=1}^{n} \tau |a_{nk} x_k e_{(|a_{nk}|^{-1},\infty)}(|x_k|) \\
&= \sum_{k=1}^{n} |a_{nk}| \int_{\lambda > |a_{nk}|^{-1}} \lambda d\tau(e_{(0,\lambda)}(|x_k|)) \\
&= \sum_{k=1}^{n} |a_{nk}| \int_{\lambda > |a_{nk}|^{-1}} \lambda |d\tau(e_{[\lambda,\infty)}(|x_k|))| \\
&\leq C \sum_{k=1}^{n} |a_{nk}| \int_{\lambda > |a_{nk}|^{-1}} \lambda |d\tau(e_{[\lambda,\infty)}(x)| \\
&= C \sum_{k=1}^{n} |a_{nk}| \tau(x e_{[|a_{nk}|^{-1},\infty)}(x)) \leq CM\epsilon.
\end{aligned}
$$

Thus $\tau(|\sum_{k=1}^{n} E^{\mathcal{B}_{k-1}} x_{nk}|) \to 0$ as $n \to \infty$. This together with (2.9) implies the conclusion.

3. Convergence of quadratic forms

Let (a_{ij}) and (A_n) be sequences of real numbers such that $a_{ii} = 0$ for all $i = 1, 2, \ldots,$ $0 < A_n \uparrow \infty$ and $\max\limits_{1 \le i,j \le n} (a_{ij}/A_n) \to 0$ as $n \to \infty$. Suppose that (x_n) is an L^1-m.d.. In this section we shall study the bilateral almost uniformly convergence of quadratic forms

$$Q_n = \sum_{i,j=1}^{n} a_{ij} x_i^* x_j. \tag{3.1}$$

THEOREM 3.1. *Suppose that the following conditions are satisfied*

$$G(\lambda) = \sup_{i,j} \tau(e_{[\lambda,\infty)}(|x_i^* x_j|)) \to 0 \quad \text{as} \quad \lambda \to \infty. \tag{3.0}$$

$$\int_0^\infty \lambda^2 \int_{y \ge \lambda} y^{-3} M(y) dy |dG(\lambda)| < \infty. \tag{3.1}$$

$$\int_0^\infty \lambda \int_{0 < y \le \lambda} y^{-2} M(y) dy |dG(\lambda)| < \infty, \tag{3.2}$$

where $M(y) = \text{card}\{(i,j) : i > j,\ A_i/|a_{ij}| \le y\} + \text{card}\{(i,j) : i < j,\ A_i/|a_{ij}| \le y\} \overset{df}{=} M^1(y) + M^2(y)$.

Then $Q_n/A_n \to 0$ *b.a.u.*

PROOF: Note that

$$Q_n/A_n = A_n^{-1}\Big(\sum_{i<j=2}^{n} a_{ij} x_i^* x_j + \sum_{2=i>j}^{n} a_{ij} x_i^* x_j\Big).$$

It is sufficient to show that

$$Q_n^{(1)}/A_n = A_n^{-1} \sum_{i<j=2}^{n} a_{ij} x_i^* x_j \to 0 \quad \text{b.a.u.} \tag{3.4}$$

To prove (3.4), by Kronecker's Lemma, we need only to show that $\sum_{i<j=2}^{n} (a_{ij}/A_j) x_i^* x_j$ converges b.a.u. Note that $(\sum_{i<j=2}^{n} (a_{ij}/A_j) x_i^* x_j)\ 1 \le$

$i < j \leq n,\ n = 1, 2, \ldots$ is an L^1-martingale. Then it remains to prove that

$$\sup_k \tau(|\sum_{i<j\leq k} (a_{ij}/A_j)x_i^* x_j|) < \infty. \tag{3.5}$$

Put for $1 \leq i < j \leq k,\ k = 1, 2\ldots,$

$$\begin{aligned} y_{ij} &= (a_{ij}/A_j)x_i^* x_j \\ z_{ij} &= y_{ij} e_{[0,1]}(|y_{ij}|) \\ t_{ij} &= y_{ij} e_{[1,\infty)}(|y_{ij}|) = y_{ij} - z_{ij}. \end{aligned}$$

Note that

$$E^{\mathcal{B}_{j-1}} z_{ij} = -E^{\mathcal{B}_{j-1}} t_{ij}. \tag{3.6}$$

By the triangle inequality and (3.6) we have

$$\begin{aligned} \tau(|\sum_{i<j\leq k} (a_{ij}/A_j)x_i^* x_j|) =& \tau(|\sum_{i<j\leq k} (z_{ij} + t_{ij})|) \\ \leq& \tau(|\sum_{i<j\leq k} (z_{ij} - E^{\mathcal{B}_{j-1}} z_{ij})|) + \\ &+ \tau(|\sum_{i<j\leq k} (t_{ij} - E^{\mathcal{B}_{j-1}} t_{ij})|) \\ \leq& (\sum_{i<j\leq k} \tau(|z_{ij}|^2))^{1/2} + 2 \sum_{i<j\leq k} \tau(|t_{ij}|). \end{aligned}$$

Using a similar argument as in Theorem 2.2, we obtain that

$$\begin{aligned} \sup_k (\sum_{i<j\leq k} \tau(|z_{ij}|^2))^{1/2} &\leq (\int_0^\infty \lambda^2 \int_{y\geq\lambda} y^{-3} M^1(y)dy\ |dG(\lambda)|)^{1/2} \\ &\leq \int_0^\infty \lambda^2 \int_{\lambda\geq y} y^{-3} M(y)dy\ |dG(\lambda)| < \infty. \end{aligned} \tag{3.7}$$

Similarly,

$$\sup_k (\sum_{i<j\leq k} \tau(|t_{ij}|)) \leq (\int_0^\infty \lambda \int_{0<y<\lambda} y^{-2} M(y)dy\ |dG(y)| < \infty. \tag{3.8}$$

Combining (3.7) and (3.8) we get (3.4). The proof is completed.

Now we investigate the convergence in $L^1(\mathcal{A},\tau)$ and in measure of quadratic forms in martingale differences. The following result is proved by the same techniques as in Theorem 2.5 and Theorem 3.1, so we omit the proof.

THEOREM 3.2. *Let $(a_{i,j,n},\ i,j=1,2,...,n,\ n=1,2,...)$ be a set of real numbers such that $a_{i,i,n}=0$ for all $i,n=1,2,...,$ $\sum_{i,j=1}^{n}|a_{i,j,n}|<M$ for some positive constant M and $\sup\limits_{i,j\le n} a_{i,j,n}\to 0$ as $n\to\infty$. Let (x_n) be an L^1-m.d. such that*

$$(x_i^*x_j)_{i,j=1,2,...,n}<x,\quad n=1,2,....$$

Suppose that $\lim\limits_{T\to\infty}\tau(xe_{(T,\infty)}(x))=0$. Then $Q_n=\sum_{i,j=1}^{n}a_{i,j,n}x_i^*x_j\to 0$ in $L^1(\mathcal{A},\tau)$ and in measure.

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REFERENCES

1. C. Barnett, *Supermartingale on semifinite Von-Neumann algebras I*, London Math. Soc. **24** (1981), 175–181.
2. C.J.K. Batty, *The strong laws of large numbers for states and traces of a W^*-algebra*, Z. Wahrscheinlichkeitstheorie verw. Gebiete **48** (1979), 177–191.
3. I. Cuculescu, *Supermartingales on W^*-algebras, Rev. Rumanie Math. Pures Appl.* **14** (1969), 759–773.
4. N. Dang Ngoc, *Pointwise convergence of martingales in von Neumann algebras*, Israel J. Math. **34** (1979), 273–280.
5. R. Jajte, *Strong limit theorems in non-commutative probability*, Lect. Notes in Math. **1110** (1985).
6. E. Henze, *Strong laws of large numbers for orthogonal sequences in von Neumann algebras*, Lect. Notes in Math. **1391** (1987), 112–124.
7. A. Luczak, *Laws of large numbers in von Neumann algebras and related results*, Studia Math., T. LXXXI (1985), 231–243.
8. Nguyen Van Hung and Nguyen Duy Tien, *On the convergence of weighted sums of martingale differences*, Lect. Notes in Math. **1391** (1987), 293–307.
9. Nguyen Van Hung and Nguyen Duy Tien, *On stability of quadratic forms in martingale differences in Banach spaces*, Proceedings of the COSMEX-89 Meeting, World Scientific (1990), 148–158.
10. I.E. Segal, *A non-commutative extension of abstract integration*, Ann. Math. **57** (1953), 401–457.
11. H. Umegaki, *Conditional expectations in operator algebras II*, Tohoku Math. J. **8** (1956), 86–100.
12. P.J. Yesdom, *Non-commutative L^p-spaces*, Proc. Cambridge Phil. Soc. **77** (1975), 91-102.

DEPARTMENT OF MATHEMATICS
UNIVERSITY, HANOI, VIETNAM