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ON THE UNIQUENESS OF GLOBAL CLASSICAL SOLUTIONS OF THE CAUCHY PROBLEM FOR HAMILTON-JACOBI EQUATIONS

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Abstract. We consider the Cauchy problem for Hamilton-Jacobi equations in n-dimensional space ($n \geq 1$) and prove some uniqueness results for classical global solutions. Our method is based on the theory of multivalued mappings and of differential inclusions.

Key words Hamilton-Jacobi equations, Cauchy problems, classical global solutions, differential inclusions.

The theory of nonlinear partial differential equations of first order in general and of Hamilton-Jacobi equations in particular has attracted much interest during the past decades, partly due to its applications in some fields such as classical mechanics, the theory of waves, optimal control, and so on. Through the works of S. H. Benton, J. D. Cole, E. D. Conway, M.G. Crandall, A. Douglis, L. C. Evans, W. H. Fleming, J. Glimm, E. Hopf, S. N. Kruzkov, P. D. Lax, P. L. Lions, V. P. Maslov, O. A. Oleinhik, L. Rozdestvenskii, M. Tsuji and others many fundamental results on global (classical and generalized) solutions of Cauchy problems have been obtained and various kinds of generalized solutions have been introduced. Recently, A. I. Subbotin and others [1] (see the references cited therein) studied global generalized solutions by methods of the theory of differential games.

In this paper we consider the Cauchy problems for Hamilton-Jacobi equations in n-dimensional space ($n \geq 1$) and prove some uniqueness theorems for

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classical global solutions. Our method is based on the theory of multivalued mappings and of differential inclusions.

§1. The uniqueness of classical global solutions

Let T be a positive number, $\Omega_T = (0,T) \times \mathbb{R}^n = \{(t,x) \in \mathbb{R}^{n+1} | 0 < t < T\}$. We consider the Cauchy problem for Hamilton-Jacobi equations:

$$\frac{\partial u(t,x)}{\partial t} + H(t,x,\nabla_x u(t,x)) = 0 \quad \text{in} \quad \Omega_T, \tag{1.1}$$

$$u(0,x) = u_0(x) \quad \text{on} \quad \{t = 0, x \in \mathbb{R}^n\}, \tag{1.2}$$

where $H(t,x,p)$ is a function of $(t,x,p) \in (0,T) \times \mathbb{R}^n \times \mathbb{R}^n$. The vector $p = (p^1, p^2, ..., p^n)$ corresponds to $\nabla_x u = (\partial u/\partial x_1, ..., \partial u/\partial x_n)$ and $u_0(x)$ is a known function. We are interested in the uniqueness of global classical solutions for Cauchy problem (1.1),(1.2).

Definition 1.1. A function $u(t,x)$ in $C^1(\Omega_T) \cap C([0,T) \times \mathbb{R}^n)$ is called a global classical solution of the problem (1.1),(1.2) if and only if $u(t,x)$ satisfies (1.1) in Ω_T and (1.2) on $\{t = 0, x \in \mathbb{R}^n\}$ everywhere.

Further, let us denote by $\|\cdot\|$ and $< \cdot,\cdot >$ the norm and the scalar product in $\mathbb{R}^n$, respectively.

Theorem 1.1. *Suppose that there exists a number $N \geq 0$ such that for all $p_1, p_2 \in \mathbb{R}^n$,*

$$|H(t,x,p_1) - H(t,x,p_2)| \leq N(1 + \|x\|)\|p_1 - p_2\|. \tag{1.3}$$

If $u_1(t,x)$ and $u_2(t,x)$ are global classical solutions of the problem (1.1),(1.2), then $u_1(t,x) = u_2(t,x)$ in Ω_T.

Remark 1.1. The condition (1.3) is fulfilled if, for example, Hamiltonian $H(t,x,p)$ is differentiable with respect to p and

$$\sup_{\substack{(t,x)\in\Omega_T \\ p\in\mathbb{R}^n}} \frac{\|\nabla_p H(t,x,p)\|}{1 + \|x\|} < \infty.$$

THEOREM 1.2. *Suppose that $H(t,x,p)/(1+\|x\|)$ is locally Lipschitz continuous with respect to p. If $u_1(t,x)$ and $u_2(t,x)$ are global classical solutions of the problem (1.1),(1.2) and*

$$\sup_{(t,x)\in\Omega_T} \|\nabla_x u_i(t,x)\| < \infty, \quad i = 1,2,$$

then $u_1(t,x) = u_2(t,x)$ in Ω_T.

From Theorem 1.2 we get the following

COROLLARY 1.1. *Suppose that $H(t,x,p)$ is locally Lipschitz continuous with respect to p and it is independent of x. If $u_1(t,x)$ and $u_2(t,x)$ are global classical solutions of the problem (1.1),(1.2) and*

$$\sup_{(t,x)\in\Omega_T} \|\nabla_x u_i(t,x)\| < \infty, i = 1,2,$$

then $u_1(t,x) = u_2(t,x)$ in Ω_T.

REMARK 1.2. The condition of Corollary 1.1 is satisfied if, for example, Hamiltonian $H(t,p)$ is differentiable with respect to p.

We will give the proof of Theorems 1.1 and 1.2 in §3. Our proof is based on the main lemma in §2.

§2. The main lemma

LEMMA 2.1 (Main Lemma). *Let $w(t,x)$ be a function in $C^1(\Omega_T)\cap C([0,T)\times \mathbb{R}^n)$, $w(0,x) = 0$ on $\{t = 0, x\in \mathbb{R}^n\}$. Suppose that there exists a nonegative constant N such that for any $(t,x)\in\Omega_T$:*

$$\left|\frac{\partial w(t,x)}{\partial t}\right| \le N(1+\|x\|)\|\nabla_x w(t,x)\|. \tag{2.1}$$

Then $w(t,x)\equiv 0$ in Ω_T.

PROOF: Let (t_0,x_0) be an arbitrary point in Ω_T. We have to show that $w(t_0,x_0) = 0$. For this we define a multivalued function $F(t,x) : \Omega_T \to$ comp

($\mathbb{R}^n$) in the following way:

$$F(t,x) = \{f \in \mathbb{R}^n | \quad \|f\| \leq N(1+\|x\|), \qquad \frac{\partial w(t,x)}{\partial t} + < f, \nabla_x w(t,x) >= 0\}. \quad (2.2)$$

We now consider the differential inclusion

$$\dot{x}(t) \in F(t, x(t)), \quad (2.3)$$

$$x(t_0) = x_0. \quad (2.4)$$

Let $X(t_0, x_0)$ be the set of absolutely continuous functions $x(\cdot) : [0,T] \longrightarrow \mathbb{R}^n$ which satisfy almost everywhere in $[0,T]$ the differential inclusion (2.3) and the condition (2.4). We are going to show that $X(t_0, x_0)$ is a non-empty compact subset in $C([0,T], \mathbb{R}^n)$. To prove this we need the following result.

THEOREM 2.1 (Theorem П.3, p. 206 in [1]). *Suppose that the multivalued function $F(t,x) : \Omega_T \longrightarrow$ comp $(\mathbb{R}^n)$ satisfies the conditions:*

(i) $F(t,x)$ is a non-empty convex set in $\mathbb{R}^n$ for $(t,x) \in \Omega_T$.

(ii) The function $F(t,x)$ is upper semicontinuous in Ω_T.

(iii) There exists a number $N \geq 0$ such that for all $(t,x) \in \Omega_T$,

$$\sup\{\|f\| \mid f \in F(t,x)\} \leq N(1+\|x\|).$$

Then the set $X(t_0, x_0)$ of solutions of (2.3),(2.4) is a non-empty compact set in $C([0,T], \mathbb{R}^n)$.

We now show that the function $F(t,x)$ defined by (2.2) satisfies all conditions of Theorem 2.1. Indeed we have to verify only (i) and (ii).

First we check that for any $(t,x) \in \Omega_T$, F(t,x) is a non-empty and convex set in $\mathbb{R}^n$. If $\nabla_x w(t,x) = 0$, then by (2.1) it follows that $\partial w(t,x)/\partial t = 0$ and $0 \in F(t,x)$. If $\nabla_x w(t,x) \neq 0$, we put

$$f = -\frac{\partial w(t,x)/\partial t}{\|\nabla_x w(t,x)\|^2} . \nabla_x w(t,x).$$

By virtue of (2.1) we obtain

$$\|f\| = \frac{|\partial w(t,x)/\partial t|}{\|\nabla_x w(t,x)\|} \le N(1+\|x\|).$$

On the other hand,

$$\frac{\partial w(t,x)}{\partial t} + < f, \nabla_x w(t,x) > =$$
$$= \frac{\partial w(t,x)}{\partial t} - - \frac{\partial w(t,x)/\partial t}{\|\nabla_x w(t,x)\|^2} < \nabla_x w(t,x), \nabla_x w(t,x) >= 0.$$

Thus we have shown that $f \in F(t,x)$, i.e. F(t,x) is non-empty. Since the set $\{f | \|f\| \le N(1+\|x\|)\}$ is convex and the expression $\partial w(t,x)/\partial t + < f, \nabla_x w(t,x) >= 0$ is affine with respect to f, F(t,x) is a convex, closed and bounded subset in $\mathbb{R}^n$. Hence F(t,x) is a nonempty compact set in $\mathbb{R}^n$.

To verify (ii) we observe that the multivalued function F(t,x) is bounded in a neighborhood of $(t,x) \in \Omega_T$, i.e. there exist numbers $l > 0, r > 0$ such that

$$\sup\{\|f\| | f \in F(\tau,y), (\tau,y) \in B_l(t) \times B_r(x) \subset \Omega_T\} < \infty \quad \forall (t,x) \in \Omega_T,$$

where $B_l(t)$(resp. $B_r(x)$) is a ball in $\mathbb{R}^1$(resp. $\mathbb{R}^n$) centered in t (resp. x) with radius l (resp. r). In addition, it is easy to see that the function F(t,x) is closed because for any sequence $(t_k, x_k) \in \Omega_T$ $(k = 1,2,...), (t_k, x_k) \longrightarrow (t_0, x_0)$, and for any sequence $f_k \in F(t_k, x_k)$ $(k = 1,2,...), f_k \longrightarrow f_0$, we have $f_0 \in F(t_0, x_0)$. Then F(t,x) is upper semicontinuous in Ω_T.

Thus, we have shown that the function $F(t,x)$ defined by (2.2) satisfies all conditions of Theorem 2.1 . By virtue of this theorem the set $X(t_0,x_0)$ of solutions of (2.3),(2.4) is non-empty and compact in $C([0,T],\mathbb{R}^n)$.

Now let $x(.) \in X(t_0,x_0)$. We consider the function $\varphi(t) \equiv w(t,x(t))$. Since $w(t,x) \in C^1(\Omega_T)$ and $x(t)$ is absolutely continuous on [0,T], we conclude that $\varphi(t)$ is absolutely continuous on $[\epsilon, T-\epsilon]$ for any $\epsilon \in (0, T/2)$. On the other hand,

$$\dot\varphi(t) = \frac{\partial w(t,x(t))}{\partial t} + < \dot x(t), \nabla_x w(t,x(t)) >= 0$$

almost everywhere on $[\epsilon, T-\epsilon]$. Then $\varphi(t)$ is constant on $[\epsilon, T-\epsilon]$. Since ϵ is an arbitrary number and $\varphi(t)$ is continuous at $t=0$, we obtain that $\varphi(t) = \varphi(0) =$

$w(0, x(0)) = 0$ for $t \in [0, T)$. In particular, $\varphi(t_0) = w(t_0, x(t_0)) = w(t_0, x_0) = 0$. The proof of Lemma 2.1 is complete.

§3. Proof of Theorems 1.1 and 1.2

PROOF OF THEOREM 1.1: Assume that the problem (1.1),(1.2) has two global classical solutions $u_1(t,x)$ and $u_2(t,x)$ in $C^1(\Omega_T) \cap C([0,T) \times \mathbb{R}^n)$. We consider the function $u(t,x) = u_1(t,x) - u_2(t,x)$. Then $u(0,x) \equiv 0, x \in \mathbb{R}^n$. Moreover , from condition (1.3) we get

$$\begin{aligned} \left| \frac{\partial u(t,x)}{\partial x} \right| &\leq |H(t,x,\nabla_x u_1(t,x)) - H(t,x,\nabla_x u_2(t,x))| \\ &\leq N(1+\|x\|)\|\nabla_x u_1(t,x) - \nabla_x u_2(t,x)\| \\ &= N(1+\|x\|)\|\nabla_x u(t,x)\|. \end{aligned}$$

Hence $u(t,x) \equiv 0$ in Ω_T by Lemma 2.1 . This proves Theorem 1.1.

PROOF OF THEOREM 1.2: Let $u_1(t,x)$ and $u_2(t,x)$ be global classical solutions of the problem (1.1) and (1.2). We consider the function $u(t,x) = u_1(t,x) - u_2(t,x)$, $u(0,x) \equiv 0, x \in \mathbb{R}^n$. Let

$$k = \max_{i=1,2}\{ \sup_{(t,x)\in\Omega_T} \|\nabla_x u_i(t,x)\|\}.$$

Denote by $\bar{B}_k$ the ball $\bar{B}_k = \{f \in \mathbb{R}^n | \quad \|f\| \leq k\}$. Since $H(t,x,p)/(1+\|x\|)$ is locally Lipschitz continuous with respect to p , we have

$$\left| \frac{H(t,x,p_1)}{1+\|x\|} - \frac{H(t,x,p_2)}{1+\|x\|} \right| \leq L\|p_1 - p_2\|, \quad \forall (t,x) \in \Omega_T, p_1, p_2 \in \bar{B}_k.$$

Then

$$\begin{aligned} \left| \frac{\partial u(t,x)}{\partial t} \right| &\leq |H(t,x,\nabla_x u_1(t,x) - H(t,x,\nabla_x u_2(t,x)| \\ &\leq L(1+\|x\|)\|\nabla_x u(t,x)\|. \end{aligned}$$

Applying Lemma 2.1 to $u(t,x)$ we obtain that $u(t,x) \equiv 0$ in Ω_T, which proves Theorem 1.2.

The uniqueness of global classical solutions of the Cauchy problems for general partial differential equations of first order will be considered in a forthcoming paper by the method used here.

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