ACTA MATHEMATICA VIETNAMICA
Volume 17, Number 1, 1992

HOLOMORPHIC EXTENSION SPACES AND FINITE PROPER HOLOMORPHIC SURJECTIONS

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Introduction

The extension of holomorphic maps has been investigated by several authors. Recent results can be found in the works of B. Shiffman, A. Hirschowitz, G. Dloussky, S. M. Ivashkovich,.... In the present paper, based on ideas of G. Dloussky [1], we shall prove the invariance of holomorphic extendibility under finite proper holomorphic projections (Theorem 2.1).

1. Holomorphic extension spaces

1.1. Let X be a complex space. We say that X is a holomorphic extension space (shortly, HES) if the following two conditions are satisfied :

(H) Every holomorphic map from a spreaded domain D over a Stein manifold to X can be holomorphically extended to $^\wedge D$, the envelope of holomorphy of D.

(R) Every holomorphic map from $Z \setminus S$, where Z is a normal complex space and S is an analytic set in Z of codimension ≥ 2, to X can be holomorphically extended to Z.

The case where (H) (resp. (R)) is satisfied is called a Hartogs (Riemann) holomorphic extension space.

By the Docquier-Grauert theorem [2] as in [13] we have

1.2. Proposition. *Let X be a complex space . Then the following are equivalent:*

Received August 28, 1991

(i) *X is a Hartogs holomorphic extension space .*

(ii) *Every holomorphic map from a Hartogs domain* $H_n(r), n \geq 2, 0 < r < 1$ *can be extended holomorphically to* Δ^n.

(iii) *The spreaded domain associated to the sheaf of germs of holomorphic maps on a Stein manifold with values in* X *is Stein.*

Here, by $H_n(r)$ we denote

$$H_n(r) = \{z \in \Delta^n : |z_i| < r, 1 \leq i \leq n-1; |z_n| < 1\} \bigcup \{z \in \Delta^n : |z_n| > 1-r; |z_i| < 1, 1 \leq i \leq n-1\}.$$

1.3. EXAMPLES.

1) The following are Hartogs holomorphic extension spaces:

(a) Every Stein space [4].

(b) Every locally pseudoconvex spreaded domain not containing a compact analytic set of positive dimension over a homogeneous projective manifold [6].

(c) Every complex space satisfying the disc condition , in particular, every complex manifold whose universal cover manifold has a complete Hermitian metric with non-positive holomorphic curvature and every complete hyperbolic space [13].

(d) Every holomorphic convex Kahler manifold not containing a rational curve [7].

2) Every Stein space is a Riemann holomorphic extension space [3], whereas every hyperbolic projective manifold is not a Riemann holomorphic extension space [11].

3) We now construct a Riemann holomorphic extension space where every holomorphic function is constant.

Consider the three vectors $e_1 = (1,0), e_2 = (0,1)$ and $e_3 = (a+i, b)$ in $\mathbb{C}^2$, where $a \in \mathbb{R}$ and $b = c + id,\ c, d \in \mathbb{R}$, d is irrational. Then e_1, e_2, e_3 are linearly independent over $\mathbb{R}$. Let G denote the subgroup of $\mathbb{C}^2$ generated by e_1, e_2, e_3. Then $X = \mathbb{C}^2/G$ is a non-compact complex manifold .

(i) First we check that X is a (H, C)-group,i.e., every holomorphic function on X is constant . For $m = (m_1, m_2, m_3) \in \mathbb{Z}^3$, put

$$K_m = m_1(a + i) + m_2(c + id) - m_3.$$

Then

$$|K_m|^2 = (m_1 a + m_2 c - m_3)^2 + (m_1 + m_2 d)^2.$$

We have to prove that

$$|K_m| > 0 \quad \text{for} \quad m \neq 0.$$

For the contrary suppose that there exists $m \in \mathbb{Z}^3, m \neq 0$ such that $K_m = 0$. This yields the following equalities

$$\begin{cases} m_1 a + m_2 c - m_3 = 0 \\ \qquad m_1 + m_2 d = 0. \end{cases}$$

Since d is irrational, from the second equality we have $m_1 = m_2 = 0$. Hence, from the first equality we get $m_3 = 0$. This is impossible because of the relation $m \neq 0$.

(ii) Since X is analytically homeomorphic to $(\mathbb{R}/\mathbb{Z})^3 \times \mathbb{R}$ [8], $H_2(X, \mathbb{Q}) = \mathbb{Q}^3$ and it is generated by $S_{ij} = \eta(\mathbb{R}e_i + \mathbb{R}e_j), 1 \leq i < j \leq 3$, where η is the canonical map from $\mathbb{C}^2$ onto X.

We prove that X does not contain a compact curve. Indeed, suppose that there is a compact curve C in X , and let $\nu : \tilde{C} \longrightarrow C$ be the normalization of C. Then

$$C \cong \sum_{1 \leq i < j \leq 3} a_{ij} S_{ij}, \quad a_{ij} \in \mathbb{Q}.$$

Consider the differential form $1/2i(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2)$ on $\mathbb{C}^2$. Obviously, this form is an invariant of G. It induces a differential form ω on X. If we consider z_1, z_2 as functions in a neighbourhood of a point $x \in C$, then in a neighbourhood of $\nu^{-1}(x)$, the form $\nu^*\omega$ is equal to

$$\frac{1}{2i}\left(\left|\frac{d\nu^*(z_1)}{dt}\right|^2 + \left|\frac{d\nu^*(z_2)}{dt}\right|^2\right)dt \wedge d\bar{t} > 0,$$

where t is a local coordinate at $\nu^{-1}(x)$.

Thus $\int_G \omega > 0$ and C is not homologous to zero. This yields $\sum |a_{ij}| > 0$. Now consider the differential form on C inducing by $dz_1 \wedge dz_2, \eta$. By Stoke's formula,

$$\int_C \eta = \sum a_{ij} \int_{S_{ij}} \eta.$$

It is easy to see that if $e_i = (\alpha_i, \beta_i)$, then

$$\int_{S_{ij}} \eta = \alpha_i \beta_j - \alpha_j \beta_i.$$

On the other hand, $\int_C \eta = 0$. Indeed, similarly as above, we can show that

$$\nu^*(\eta) = \left(\frac{d\nu^*(z_1)}{dt}\right)\left(\frac{d\nu^*(z_2)}{dt}\right) dt \wedge dt = 0.$$

Hence

$$\begin{cases} a_{12} + ca_{13} - aa_{23} = 0 \\ \qquad da_{13} - a_{23} = 0. \end{cases}$$

Since d is irrational and $a_{13}, a_{23} \in \mathbb{Q}$, it follows that $a_{13} = a_{23} = 0$ and hence $a_{12} = 0$, a contradiction.

(iii) Finally, we show that X is a Riemann holomorphic extension space. Given $f : Z \backslash S \longrightarrow X$ a holomorphic map , where Z is a normal complex space and S an analytic set in Z of codimension ≥ 2. By Kählerian property of X [14], $\overline{\Gamma}_f$, where Γ_f denotes the graph of f, is an analytic set in $Z \times X$. Let φ be a plurisubharmonic exhaustion function on X (Such a function exists by the pseudoconvexity of X [9]). Then φf is plurisubharmonic on Z [6] and hence it is semicontinuous on Z. This implies that the canonical projection $p : \overline{\Gamma}_f \longrightarrow X$ is proper. Thus $\overline{\Gamma}_f$ defines a mermorphic extension $\hat{} f$ of f. Since X is not compact dim $\hat{} f(z) \leq 1$ for $z \in Z$. By (ii), dim $\hat{} f(z) = 0$. Thus, by the normality of Z, it follows that $\hat{} f$ is holomorphic .

2. Finite proper holomorphic surjections and holomorphic extension spaces

The aim of this section is to prove the invariance of holomorphic extendibility under finite proper holomorphic surjections.

2.1. THEOREM. *Let θ be a finite proper holomorphic surjective map from a complex space X onto a complex space Y . Then X is a HES if and only if so is Y.*

For the proof we need the following lemmas.

2.2. LEMMA([5] and [15]). *Let Z and W be complex manifolds and A an analytic set in Z of codimension ≥ 1. Assume that $\varphi : W \longrightarrow Z\backslash A$ is an unbranched covering map. Then there exists a commutative diagram:*

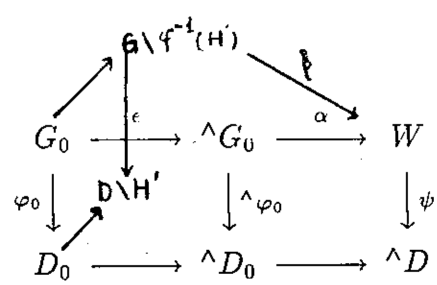
$$\begin{array}{ccc} W & \xrightarrow{\tilde{e}} & \widetilde{W} \\ {\scriptstyle\varphi}\downarrow & & \uparrow{\scriptstyle\tilde{\varphi}} \\ Z\backslash A & \xrightarrow{e} & Z \end{array}$$

where $(\tilde{W}, \tilde{\varphi}, Z)$ is a branched covering map and $\tilde{e}$ is an open embedding.

2.3. LEMMA. *Let $\varphi : G \longrightarrow D$ be a branched covering map , where G is a normal complex space and D is a spreaded domain over a Stein manifold , such that the points of D are separated by holomorphic functions on D. Assume that H is the branch locus of φ and*

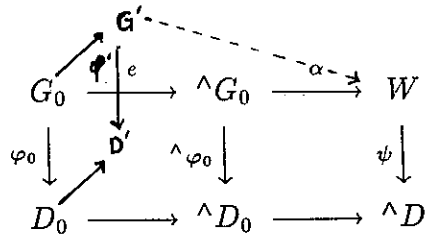
$$D_0 = D\backslash H, G_0 = G\backslash\varphi^{-1}(H),\ \varphi_0 = \varphi\big|_{G_0}.$$

Then there exist an analytic set H' in D contained in H with $\hat{}(D\backslash H') = \hat{}D$ and a commutative diagram of normal complex spaces



where $^\wedge\varphi_0$ *is an unbranched covering map* , $\psi,\beta : G\backslash\varphi^{-1}(H') \longrightarrow Im\beta$ *are branched covering maps* , α *is an open embedding, and* $\beta^{-1}(\alpha e(G_0)) = G_0$.

PROOF: Since G and D are normal, either H is a hypersurface in D or $H = \emptyset$. The case where $H = \emptyset$ is trivial, therefore we may assume that H is a hypersurface in D. Then there exists an analytic set $^\wedge H$ in $^\wedge D$ such that $^\wedge D_0 = {}^\wedge D \backslash {}^\wedge H$ [1]. Observe that $^\wedge H \cap D \subseteq H$. We write $H = ({}^\wedge H \cap D) \cup H'$, where H' is an analytic set in D such that $^\wedge(D\backslash H') = {}^\wedge D$. By [10], the map $^\wedge\varphi_0 : {}^\wedge G_0 \longrightarrow {}^\wedge D_0$ is an unbranched covering map , and using Lemma 3.2 to $^\wedge\varphi_0$ we can construct a commutative diagram



of normal complex spaces, where $D' = D\backslash H', G' = G\backslash\varphi^{-1}(H')$ and $\varphi' = \varphi\big|_{G'}'$. Moreover, ψ is a branched covering map and α is an open embedding. Put $^\wedge\alpha = \alpha.\ e$. We shall prove that $^\wedge\alpha$ can be extended to a holomorphic map β from G' to W. Since the Stein property is invariant under finite proper holomorphic surjections [2], W is a Stein space.

Thus, by the normality of G' it suffices to check that $^\wedge\alpha$ is locally compact on G', i.e. for each $z \in G'$, there exists a neighbourhood U of z such that $^\wedge\alpha(U \cap G_0)$ is relatively compact in W. Assume that $z_0 \in \varphi^{-1}(H')$ and $\{z_n\} \subset G_0$ converging to z_0. Then

$$\lim \psi^\wedge\alpha(z_n) = \lim \varphi'(z_n) = \varphi'(z_0) \in D' \hookrightarrow {}^\wedge D.$$

Thus, from the property of ψ it follows that $\{^\wedge\alpha(z_n)\}$ is relatively compact in W. This yields the local compactness of $^\wedge\alpha$ on G'.

Let $\beta : G' \longrightarrow W$ be a holomorphic extension of $^\wedge\alpha$. Since φ' and ψ are finite proper maps and D' is contained in $^\wedge D$ as open subset, it is easy to see that $\beta : G' \longrightarrow \mathrm{Im}\beta$ is finite proper . Hence, by the normality of W and by the equality $\dim G' = \dim W, \mathrm{Im}\beta$ is open in W and $\beta : G' \longrightarrow \mathrm{Im}\beta$ is a branched covering map [3]. Finally, if $^\wedge\alpha(z_0) = \beta(z_1)$, where $z_0 \in G_0$ and $z_1 \in G'$, then

$$\varphi'(z_1) = \psi\beta(z_1) = \psi^\wedge\alpha(z_0) = \varphi_0(z_0).$$

This implies that $z_1 \in G_0$. Hence $\beta^{-1}(^\wedge\alpha(G_0)) = G_0$. The proof of Lemma 2.3 is now complete.

2.4. Lemma. *Let X be a holomorphic extension space and Z a normal Stein space. Assume that H is a hypersurface in Z and G is an open subset of Z meeting every irreducible branch of H . Then every holomorphic map f from $(Z\backslash H) \cup G$ to X can be extended holomorphically to Z.*

Proof: Since Z is normal, codim $S(Z) \geq 2$ [3], where $S(Z)$ denotes the singular locus of z. We write, by the Steiness of Z, $S(Z)$ in the form

$$S(Z) = \bigcap\{Z(h) : h \text{ is holomorphic on } Z, h\big|_{S(Z)} = 0$$
$$\text{and} \quad h \neq \text{ const on every irreducible branch of } H\}.$$

It suffices to show that for every such function h, the map $f_h = f\big|_{Z_h\backslash H}$, where $Z_h = Z\backslash Z(h)$, can be holomorphically extended to Z_h. Put $G_h = G\backslash Z(h)$ and $H_h = H \cap Z_h$. Then G_h also meets every irreducible branch of H_h . Consider the holomorphic map $f_h\big|_{(Z_h\backslash H_h)\cup G_h}$. Since $^\wedge((Z_h\backslash H_h) \cup G_h) = Z_h$ [1], $f_h\big|_{Z_h y\backslash H_h)\cup G_h}$ can be extended to a holomorphic $^\wedge f_h$ on Z_h.

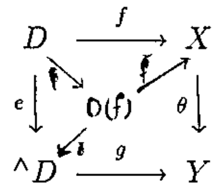
2.5. Lemma. *Let M, N and X be complex spaces and $\theta : M \longrightarrow N$ a branched covering map with N normal. Assume that f is a holomorphic map from M to X which can be factorized holomorphically through a non-empty open subset U of N. Then f is factorized holomorphically through θ .*

Proof: Since the map $\theta \times id : M \times X \longrightarrow N \times X$ is proper, $(\theta \times id)\Gamma_f$ is an analytic set in $N \times X$ [4]. Observe that the canonical projection $p : (\theta \times id)\Gamma_f \longrightarrow N$ is finite proper and hence a branched covering map [3]. From the hypothesis it follows that $p^{-1}(z) = 1$ for $z \in U$. Hence $(\theta \times id)\Gamma_f$ defines a holomorphic g from N to X such that $g\theta = f$.

Proof of Theorem 2.1.

a) Sufficiency.

(i) Given f a holomorphic map from a spreaded domain D over a Stein manifold to X. Consider the following commutative diagram



where g is a holomorphic extension of $\theta.\ f$, $D(f)$ is the domain of the existence of f, and $\tilde{f}$ is the canonical extension of f . Let $z \in {}^\wedge D$. Take two Stein neighbourhoods U and V of z and $g(z)$ respectively such that $g(U) \subseteq V$ and $\theta^{-1}(V)$ is Stein. Then, as in [13] we can check that $\gamma^{-1}(U)$ is p_7-convex, i.e. every holomorphic embedding $h : H_n(r) \longrightarrow \gamma^{-1}(U), n = dimD$, can be extended holomorphically to Δ^n. This yields, by the Docquier-Grauert theorem [2], the Stein property of $\gamma^{-1}(U)$. Hence $D(f)$ is Stein, too. Thus β can be extended to a holomorphic map ${}^\wedge\beta : {}^\wedge D \longrightarrow D(f)$ and $\tilde{f}{}^\wedge\beta$ is a holomorphic extension of f to ${}^\wedge D$.

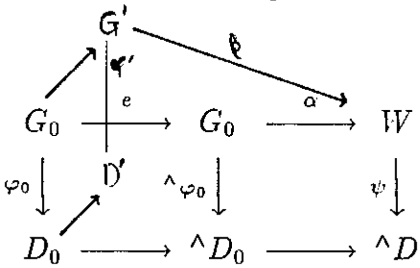
(ii) Let $f : Z\backslash S \longrightarrow X$ be a holomorphic map , where Z is a normal complex space and S is an analytic set in Z of codimension ≥ 2. For each $z \in S$ take two neighbourhoods U and V of z and $g(z)$ respectively as in (i), where g is a holomorphic extension of θf. Then $f(U\backslash S) \subseteq \theta^{-1}(V)$. By Steiness of $\theta^{-1}(V), f\big|_{U\backslash S}$ is holomorphically extended to U.

b) Necessity.

(i) Let f be a holomorphic map from a spreaded domain D over a Stein manifold . By Proposition 1.2 it suffices to consider the case where D is Hartogs domain. Consider the commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\ g\ } & X \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\theta} \\ D & \xrightarrow{\ f\ } & y \end{array}$$

where $G = D \times_Y X, g$ and φ are canonical projections. We may assume that G is normal. Observe that φ is a branched covering map [3]. Let H denote the branch locus of φ . With the notations of Lemma 2.3 we have the following commutative diagram of normal complex spaces:



where $\varphi_0, {}^\wedge\varphi_0$ are unbranched covering maps $\varphi', \psi, \beta : G' \longrightarrow \mathrm{Im}\beta$ are branched covering maps, and $G_0 = \beta^{-1}(\alpha e(G_0))$. Thus $g\big|_{G'}$ can be factorized holomorphically through $\beta : G' \longrightarrow \mathrm{Im}\beta$. Let ${}^\wedge g_0$ be a holomorphic extension of $g\big|_{G_0}$ on ${}^\wedge G_0$ and $\tilde{g}$ a holomorphic map from $\mathrm{Im}\beta$ to X such that $\tilde{g}\beta = g\big|_{G'}$. Define a holomorphic map g_1 from ${}^\wedge G_0 \cup \mathrm{Im}\beta$ to X by

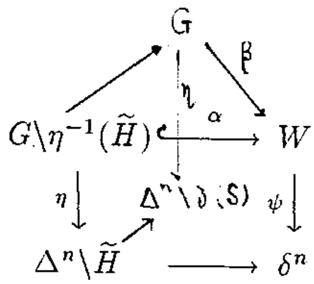
$$g_1 = {}^\wedge g_0 \quad \text{on} \quad {}^\wedge G_0 \quad \text{and} \quad g_1 = \tilde{g} \quad \text{on} \quad \mathrm{Im}\beta.$$

Since ψ is finite proper and every irreducible branch of ${}^\wedge H$ meets D', every irreducible branch of $\psi^{-1}({}^\wedge H)$ also meets $\mathrm{Im}\beta$. Thus, by Lemma 2.4 we have a holomorphic extension g_2 of g_1 on W . By Lemma 2.5, θg_2 is factorized holomorphically through ψ. Hence f is extended to a holomorphic map to ${}^\wedge D$.

(ii) Finally, we show that Y is a Riemann holomorphic extension space . Given $f : Z\backslash S \longrightarrow Y$ a holomorphic map , where Z is a normal complex space and S is an analytic set of codimension ≥ 2. Since the problem is local, we may assume that there exists a branched covering map $\gamma : Z \longrightarrow \Delta^n$. Consider the commutative diagram of normal complex spaces

$$\begin{array}{ccc} G & \xrightarrow{g} & X \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\theta} \\ Z\backslash S & \xrightarrow{f} & Y \end{array}$$

where G is the normalization of $(Z\backslash S) \times_Y X$. Since φ is finite proper and Z is normal, φ is a branched covering map [3]. Let H be the branch locus of φ. From the inequality dim$H$ >dim$S$ (without loss of generality we may assume that H is a hypersurface in $Z\backslash S$) it follows from the Remmert-Stein theorem [4] that $\overline{H}$ is an analytic set in Z. Take a hypersurface $\widetilde{H}$ in Δ^n containing the branch locus of γ and $\gamma(S \cup H)$. Applying Lemma 2.3 to the branched covering map $\eta = \gamma\varphi : G \longrightarrow \Delta^n\backslash\gamma(S)$ we can construct the following commutative diagram



where $\eta, \psi, \beta : G \longrightarrow \text{Im}\beta$ are branched covering maps. Obviously, $\beta^{-1}(\alpha(G\backslash\eta^{-1}(\widetilde{H}))) = G\backslash\eta^{-1}(\widetilde{H})$. Thus g can be factorized holomorphically through $\beta : G \longrightarrow \text{Im}\beta$. Since $W\backslash\psi^{-1}(\gamma(S)) = \text{Im}\beta$, a holomorphic map $\tilde{g}$ from Imβ to X with $\tilde{g}\beta = g$ can be extended to a holomorphic map $^\wedge\tilde{g}$ from W to X. From Lemma 2.5 it follows that $\theta{}^\wedge\tilde{g}$ is factorized holomorphically through ψ. Thus f is extended to a holomorphic map to Z. The proof of the Theorem 2.1 is now complete .

2.6. Remark:

(i) From the proof of Theorem 2.1 it follows that Riemann holomorphic extendibility is invariant under finite proper holomorphic surjections.

(ii) Let X be a projective hyperbolic manifold . Then X is a Hartogs holomorphic extension space . By the projectivity of X there exists a finite holomorphic map from X onto a projective space. Since every projective space does not have the Hartogs holomorphic extension property, we deduce that the Hartogs holomorphic extendibility is not invariant under finite proper holomorphic surjections.

ACKNOWLEDGEMENT. The author would like to thank Prof. Nguyen Van Khue for helpful advice and suggestions that led to improvements of the presentation of the paper.

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