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ON THE ALMOST SURE CONVERGENCE OF TWO-PARAMETER MARTINGALES AND THE STRONG LAW OF LARGE NUMBERS IN BANACH SPACES

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Abstract. Let (Ω, F, P) be a probability space, $N^2 = N \times N$ denote the set of parameters with the partial order defined by $(m_1, n_1) \leq (m_2, n_2)$ if and only if $m_1 \leq m_2$ and $n_1 \leq n_2 (m_1, n_1, M-2, n_2 \in N)$. Let (F_{mn}) be an increasing family of sub-δ- fields of F satisfying the usual condition (F_4) and (M_{mn}, F_{mn}) a two-parameter martingale taking values in a Banach space $(B, \|\cdot\|)$. In this paper we investigate the interrelation between geometric properties of Banach spaces and Martingale convergence theorems. Moreover we also study Marcinkiewicz-Zygmund's type strong law of large numbers for two-parameter Banach-valued martingales and the integrability of two-parameter Banach-valued martingale maximal functions.

1. Introduction

The interrelations between the L^p-convergence Martingale Theorem $(1 \leq p < \infty)$ (cf. [20], Definition 1.6 for more informations) and the geometric properties of Banach spaces have been established by Chatterji [4], [5], Pisier [16], Woyczynski [20], [21]. A natural question should be raised is how to check these results for two-parameter martingales ? In the second section of this paper we prove similar results for two-parameter martingales which we also call Chatterji's theorem and Assouad-Pisier's theorem. Our techniques are based on classical results of [10], [16], [20], [22] for one-parameters and of [1], [3], [6], [18] for two-parameter martingales. Further, in the third section, we deal with the Marcinkiewicz-Zygmund's type strong law of large numbers for two-parameter Banach-valued martingales. The obtained results are extensions of several results in [7], [10], [17], [18]. Finally, the integrability of two-parameter Banach-valued martingale maximal functions is discussed in the fourth section.

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2. Definition and Preliminaries

The considered set of parameter will be $N \times N$ (N^2 for short) with the partial ordering defined as $(i,j) \leq (m,n)$ if and only if $i \leq m,\ j \leq n$. Let $z_1 < z_2, z_1, z_2 \in N^2$, then (z_1, z_2) denotes the rectangle $\{z \in N^2 : z_1 \leq z \leq z_2\}$. Suppose that f is a mapping from N^2 into a Banach space B with the norm $\|\cdot\|$. The increment of f on the rectangle (z_1, z_2), $z_1 = (m_1, n_1)$, $z_2 = (m_2, n_2)$ will be $f(z_1, z_2) = f(z_2) - f(m_1 + 1, n_1) - f(m_1, n_2 + 1) + f(z_1)$.

Let (Ω, F, P) be a probability space and $\{F_{mn}\}$ an increasing family of sub-σ-fields of F such that $f = \bigvee_{(m,n)\in N^2} F_{mn}$. Throughout this paper, (F_{mn}) is assumed to satisfy the usual conditions (F_4), i.e. F_m and F_n are conditionally independent for given F_{mn}, where $F_m^1 = \bigvee_{n\in N} F_{mn}$, $F_n^2 = \bigvee_{m\in N} F_{mn}$. Note that the condition (F_4) means that for each $z = (m,n)$ and each integrability element X,

$$E(x \mid F_{mn}) = E(X \mid F_m^1 \mid F_n^2) = E(X \mid F_n^2 \mid F_m^1).$$

A sequence X_{ij} in L_B^1(L^1 for short) is said to be adapted to (F_{mn}) if each X_{mn} is F_{mn}-measurable.

Suppose that $M = (M_{mn})$ is integrability (in the sence of Bochner integrale) F_{mn} -adapted. Then

(1) M is a martingale (strong martingale) if $E(M_m \mid F_m) = M_m$, for any $(m_1, n_1) \geq (m_2, n_2)$,

(2) M is a weak martingale if $E(M((m_2, n_2), (m_1, n_1)) \mid F_{m_2 n_2}) = 0$.

Let M_{mn} be a two-parameter B-valued martingale w.r.t. F_{mn}. For the given M_{mn}, let X_{ij} be one of its increments, i.e.

$$\Delta X_{mn} = M_{mn} - M_{m-1,n} - M_{m,n-1} + M_{m-1,n-1}.$$

In what follows we shall assume $M_{mn} = 0$ if m or n is zero. Note that under this assumption , a two-parameter martingale (M_{mn}, F_{mn}) can be written as

$$(M_{mn} = \sum_{i=1}^{m} \sum_{j=1}^{n} X_{ij}, F_{mn}).$$

A sequence (X_{ij}) is said to be domainated by a positive real random variable $X_0((X_{ij}) < X_0$ for short) if for all $t > 0, P(\|X_{ij}\| > t) \leq P(X_0 > t)$.

A Banach space B is said to be p-smoothable , $1 \leq p \leq 2$, if (possibly after equivalent renorming)

$$\begin{aligned} q_E(t) &= \sup\{\frac{\|x+ty\| + \|x-ty\|}{2} - 1, \quad \|x\| = \|y\| = 1\} \\ &= 0(t^p) \quad \text{as} \quad t \longrightarrow 0, \end{aligned}$$

and superreflexive if B is p-smoothable for some $p > 1$ (cf. [20] for more information).

Throughout the present paper, C_p will be a constant depending only on p, which may be different from one formula to another. In the same way, C will denote an arbitrary constant.

We now present some results and some inequalities which are similar to the corresponding results in the one-parameter case and which will be used very often later on.

LEMMA 2.1 (Kleskov, [10], Lemma 3). *Let $\alpha_1, \ldots, \alpha_k$ be real numbers, $t_1, \ldots, t_k > 0$. Set $\alpha = \max_i \alpha_i$, $\pi = \max_i\{\alpha_i, t_i\}$, $h = card\{i : \alpha_i t_i = \pi\}$, $r = card\{i : \alpha_i = 0\}$. Suppose that $f(x) = \sum n_1^{\alpha_1 - 1} \ldots n_k^{\alpha_k - 1}$. Then*
(i) If $\alpha \leq 0$, then $f(x) = 0((\log^+ x)^r)$
(ii) If $\alpha > 0$, then $f(x) = 0(x^\pi (\log^+ x)^{h+r-1})$, where $\log^+ x = \log^+ x \vee 0, x \in \mathbb{R}^+$.

LEMMA 2.2 (Doob's inequalities, [3], [10]). *Put $M^* = \sup_{(m,n)} \|M_{mn}\|$. Then*
(i) $P(M^* \geq C) \leq C_p \sup_{(m,n)} E \| M_{mn} \|^p$ for any $C > 0$ and $p > 1$.
(ii) $P(M^* \geq C) \leq C_p \sup_{(m,n)} E \| M_{mn} \|^p$ for any $p > 1$.

LEMMA 2.3 (Assouad-Pisier's inequality). *Let B be a p-smoothable Banach space , $1 \leq p \leq 2, (M_{ij}, F_{ij})$ a two-parameter B-valued martingale with increments $(X_{ij}, 1 \leq i \leq n, 1 \leq j \leq m)$. Then there is a constant C_p such that*

$$E \| M_{mn} \|^p \leq C_p \sum_{i=1}^{m} \sum_{j=1}^{n} \| X_{ij} \|^p .$$

PROOF: Put $D_i^n = \sum_{j=1}^{n} X_{ij}, i = 1, 2, \dots, n$. Notice that (D_i^n, F_{in}) is an one-parameter B-valued martingale difference sequence because
$D_i^n = (M_{in} - M_{i-1,n}), n = 1, 2, \dots$, and by the condition (F_4). So we can use Assouad-Pisier's inequality (cf. Corollary 2.3 [20]) for the one-parameter martingale difference sequence $D_i^n, F_i^1)$:

$$\begin{aligned} E \| \sum_{i=1}^{m} \sum_{j=1}^{n} X_{ij} \|^p &= E \| \sum_{i=1}^{m} D_i^n \|^p \\ &= C_p \sum_{i=1}^{m} E \| D_i^n \|^p \\ &= C_p \sum_{i=1}^{m} E \| \sum_{j=1}^{n} X_{ij} \|^p \\ &\leq C_p \sum_{i=1}^{m} \sum_{j=1}^{n} E \| X_{ij} \|^p . \end{aligned}$$

The last inequality follows from Assouad-Pisier's inequality once more because for fixed i, (X_{ij}) is an one-parameter martingale difference sequence w.r.t. (F_j^2).

Using Lemma 2.2 and 2.3 we get the following

COROLLARY 2.1. *Suppose that B is p-smoothable , $1 < p \leq 2$, and $M^* = \sup_{(m,n)} \|M_{mn}\|$.*
(i) For $C > 0$, we have

$$P(M^* \geq C) \leq C_p \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} E \| X_{ij} \|^p .$$

(ii) For $1 < p \leq 2$, we have

$$E(M^*)^P \leq C_p \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} E \| X_{ij} \|^p .$$

Suppose now that (b_{mn}) is a sequence of two-parameter constants satisfying $b_{mn} \longrightarrow \infty$ as $(m,n) \longrightarrow \infty$ and

$$\Delta b_{mn} = b_{mn} - b_{m-1,n} - b_{m,n-1} + b_{m-1,n-1} \geq 0.$$

The following lemma is obtained by a result of Smythe (cf. Theorem 1.1 in [16], also see [10]).

LEMMA 2.4 (Hajek-Renyi's inequality). *Let B be a p-smoothable Banach space, $1 < p \leq 2$, (b_{mn}) a sequence of two-parameter constants such that $\Delta b_{mn} \geq 0$ and $b_{mn} \longrightarrow 0$ as $(m,n) \longrightarrow \infty$. Suppose that (M_{mn}) is a two-parameter B-valued martingale with its increments $(X_{ij}), 1 \leq i \leq m,\ 1 \leq j \leq n$. Then for any $c > 0$*

$$P(\max_{(m,n)} \| \frac{M_{mn}}{b_{mn}} \| \geq c) \leq C_p \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{E \parallel X_{ij} \parallel^p}{b_{ij}^p}.$$

Remind that throughout this paper $(m,n) \longrightarrow \infty$ is understood as $\min(m,n) \longrightarrow \infty$.

DEFINITION 2.1. We say that the $L\log^+ L$ -Martingale Convergence Theorem holds in a Banach space B ($L\log^+ L-MCT$ for short) if for each two-parameter B-valued martingale (M_{mn}, F_{mn}) satisfying the condition $\sup_{(m,n)} E(\|M_{mn}\| \log^+ \|M_{mn}\|) < \infty$, there exists an element $M_{\infty\infty} \in L^1$ such that $M_{mn} \longrightarrow M_{\infty\infty}$ a.s. . We say that the L_p-Martingale Convergence Theorem holds in a Banach space B (MCTp for short), $p > 1$, if for each two-parameter B-valued martingale (M_{mn}, F_{mn}) satisfying the condition $\sup_{(m,n)} E \parallel M_{mn} \parallel^p < \infty$, there is an element $M_{\infty\infty} \in L^p$ such that $M_{mn} \longrightarrow M_{\infty\infty}$ a.s. and in L^p.

We now turn to the investigation of interrelations between the $MCTp$, $p > 1$, and the $L\log^+ L - MCT$ and the Random-Nikodym property (RNP) of Banach spaces. The following result is also named Chatterji's theorem.

THEOREM 2.1. *For a Banach space B, the following properties are equivalent:*

(RNP) *B has the Random-Nikodym property;*

(MCTp) *the MCTp holds in B;*

$(L\log^+ L)$ *the $(L\log^+ L)$–MCT holds in B.*

PROOF: We shall prove the following implications

$(MCTp) \Leftrightarrow (RNP) \Leftrightarrow (L\log^+ L)$.

Note that $(RNP) \longrightarrow (MCTp)$ can be proved like in the proof of Theorem 1.1, [20] for the one-parameter case (see also [14], [19]). The case $(RNP) \longrightarrow (L\log^+ L)$ is carried out by using the methods of [4], [8], [14] (which has been proved in [19], Lemma 1).

We now prove $(MCTp) \longrightarrow (RNP)$. Suppose first that $(\Omega^1, F^1, F_i^1, P^1)$ and $(\Omega^1, G^2, G_i^2, P^2)$ are two filtrations such that $F^1 = \bigvee_{i\in N} F_i^1$, $G^2 = \bigvee_{j\in N} G_j^2$, and $F_i^1, i \in N)$, $(G_j^2, j \in N)$ are mutually independent . E, E^1, E^2 denote the expectations taking values on $(\Omega^1 \otimes \Omega^2, P^1 \otimes P^2)$, $(\Omega^1, P^1), (\Omega^2, P^2)$ respectively. Let us consider a fixed one-parameter (Y_i, F_i^1) on (Ω^1, F^1, P^1) such that $\sup_i E^1|Y_i|^p$ is finite and has non-zero limit, say Y_∞ a.s. and in L^p, and an arbitrary one-parameter B-valued martingale (M_j, G_j^2) on (Ω^2, G^2, P^2). Put $M_{ij} = Y_i M_j$, $i, j = 1, 2, \ldots$.Then (M_{ij}) defines a two-parameter B-valued martingale on $(\Omega^1 \otimes \Omega^2, F^1 \otimes G^2, P^1 \otimes P^2)$ and adaptes to δ-fields $(F_i^1 \otimes G_j^2)$. Clearly, $E \parallel M_{ij} \parallel^p = E^1|Y_i|^p \times E^2 \parallel M_j \parallel^p$ because of the mutual independence of $(F_i^1, i \in N)$ and $(G_j^2, j \in N)$. Suppose now that $(MCTp)$ holds in B and hence on the set

$$H = \{(M_{ij}) = (Y_i M_j)) : \sup_i E^1|Y_i|^p < \infty, Y_\infty > 0$$

$$\text{a.s. and} \quad \sup_j E^2 \parallel M_j \parallel^p < \infty\}.$$

Suppose further $H \ni M_{ij} \longrightarrow g \in B$ a.s. and in L^P . We easily see that $g = Y_\infty$, where $M \in B$ is G_∞^2 - measurable .

Clearly,

$$E(g|F_i^1 \otimes G_j^2) = E^1(Y_\infty|F_i^1)E^2(M|G_j^2),$$

which implies that $M_n = E^2(M|G_n^2), \longrightarrow E^2(M|G_\infty^2) = m$ a.s. and in L^p. By Chatterji's theorem (cf. Theorem 1.1 [18]), we get $(MCTp) \longrightarrow (RNP)$. Similar, we can also prove $(L\log^+ L) \longrightarrow (RNP)$.

Now we are in a position to extend a famous result of Assouad-Pisier (cf. [20]) for two-parameter Banach-valued martingales.

THEOREM 2.2. *If B is a separable Banach space and $1 \leq p \leq 2$, then the following three assertions are equivalent:*
(i) B is isomorphic to a p-smoothable Banach space .
(ii) There exists a constant C_p such that for any two-parameter B-valued martingale $(M_{mn} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} X_{ij}, F_{mn})$,

$$\sup_{(m,n)} E \parallel M_{mn} \parallel^p \leq C_p \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} E \parallel X_{ij} \parallel^p .$$

(iii) For any two-parameter B-valued martingale $(M'_{mn} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{X_{ij}}{ij}, F_{mn})$ satisfying

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{E \parallel X_{ij} \parallel^p}{(ij)^P} < +\infty,$$

M'_{mn} converges a.s. as $(m,n) \longrightarrow \infty$.

PROOF: Note that $(i) \longrightarrow (ii)$ is the conclusion of Lemma 2.3, and $(i) \longrightarrow (iii)$ is a consequence of Theorem 2.1. To prove $(ii) \longrightarrow (i)$ we again use the symbols and the arguments of Theorem 2.1. Suppose that (Y_i, F_i^1) is a fixed one-parameter such that $\sum_{i=1}^{\infty} E^1|\Delta Y_i|^p$ is finite, $\Delta Y_i = Y_i - Y_{i-1}$ and $Y_0 = 0$ and (M_j, G_j^2) is an arbitrary one-parameter B-valued martingale with its increments $\Delta M_j = M_j - M_{j-1}, M_0 = 0$. The two-sequences of σ- fields $(F_i^1, i \in N)$ and $(G_j^2, j \in N)$ are also assumed to be mutually independent. Suppose further that (ii) holds for the two-parameter B-valued martingale $(Y_j M_j, F_i^1 \otimes G_j^2)$, i.e. there exists a constant C_p such that

$$\sup_{(m,n)} E \parallel Y_m M_n \parallel^p = \sup_m E^1 |Y_m|^p \sup_n E^2 \parallel M_n \parallel^p$$
$$\leq C_p \left(\sum_{i=1}^{\infty} E^1 |\Delta Y_i|^p \right) \cdot \left(\sum_{j=1}^{\infty} E^2 |\Delta M_j|^p \right).$$

This inequality implies

$$\sup_n E^2 \parallel M_n \parallel^p \leq C_p \sum_{j=1}^{\infty} E^2 \parallel \Delta M_j \parallel^p .$$

Hence in view of Assouad-Pisier's inequality (cf. [19]) we get the proof for $(ii) \longrightarrow (i)$.

The proof of the implication $(iii) \longrightarrow (i)$ is analogous to that of $(MCTp) \longrightarrow (RNP)$ in Theorem 2.1. We quote it here for the sake of completeness. Suppose that a fixed one-parameter real-valued (Y_i, F_i^1) satisfying $\sup_m \sum_{i=1}^{m} \frac{E^1 |\Delta Y_i|^p}{i^p} < \infty$ and $\sum_{i=1}^{m} \frac{\Delta Y_i}{i}$ converges a.s. to a non-zero limit. It is easy to see that

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{E \parallel \Delta M_{ij} \parallel^p}{(ij)^p} = \left(\sum_{i=1}^{\infty} \frac{\parallel \Delta Y - i \parallel^p}{i^p} \right) \times \left(\sum_{j=1}^{\infty} \frac{E^2 \parallel \Delta M_j \parallel^p}{j^p} \right) < \infty,$$

which implies that $(\sum_{j=1}^{\infty} \frac{|\Delta M_j|^p}{j^p})$ is finite. On the other hand, like in the proof of Theorem 2.1, we have

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \frac{\Delta M_{ij}}{ij} = (\sum_{i=1}^{m} \frac{\Delta Y_i}{i})(\sum_{j=1}^{n} \frac{\Delta M_j}{j}) \longrightarrow g \quad \text{a.s.}$$

Hence the martingale $(\sum_{j=1}^{n} \frac{\Delta M_j}{j})$ converges a.s. in B. So $(\frac{1}{n}) \sum_{j=1}^{n} \Delta M_j$ converges to zero by Kronecker's lemma. The conclusion follows by applying again Assouad-Pisier's theorem.

3. Marcinkiewicz-Zygmund's type strong law of large numbers for two-parameter Banach-valued martingales

In this section we extend Marcinkiewicz-Zygmund's type strong law of large numbers to two-parameter martingales in Banach spaces. For special results on the line, see Smythe ([17], [18]), Klesov ([9], [10]), Gut ([7]), Moricz ([15]).

Let us now begin with some auxiliary lemmas

LEMMA 3.1. *Let $1 < p < 2$ and (Y_{ij}) be two-parameter real-valued sequence of random variables such that $(Y_{ij}) < X_0 \in L^p$. Then, if $Y'_{ij} = Y_{ij} I(|Y_{ij}| \le (ij)^{\frac{1}{p}})$ and $r > p$, we have*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{E\|Y'_{ij}\|^r}{(ij)^{\frac{r}{p}}} < \infty,$$

where $I(A)$ denotes the indicator function of the set A.

PROOF: The proof can be obtained by a straightforward computation as follows:

$$\begin{aligned}
\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{E\|Y_{ij}\|^r}{(ij)^{\frac{r}{p}}} &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} (ij)^{-\frac{r}{p}} \int_0^{(ij)^{\frac{1}{p}}} x^r dP(|Y_{ij}| \le x) \\
&\le \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} r(ij)^{-\frac{r}{p}} \int_0^{(ij)^{\frac{1}{p}}} x^{r-1} P(X_0 > x) dx \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} r \int_0^1 y^{r-1} P(X_0 > y(ij)^{\frac{1}{p}}) dy \\
&= r \int_0^1 y^{r-p-1} dy \left(\sum \frac{1}{i^p}\right) E X_0^p \\
&\le C \frac{r}{r-p} \times E X_0^p < \infty.
\end{aligned}$$

LEMMA 3.2. *Let (Y_{ij}) be a two-parameter real-valued sequence of random variables such that $(Y_{ij}) < X_0$.*
(a) If $X_0 \in L(\log^+ L)^2$ and $Y''_{ij} = Y_{ij}I(|Y_{ij}| > ij)$, then

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{E|Y''_{ij}|}{ij} < \infty,$$

(b) If $X_0 \in L^p(\log^+ L), 1 \le q < p \le 2$, and $Y''_{ij} = Y_{ij}I(|Y_{ij}| > (ij)^{\frac{1}{p}})$, then

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{E|Y_{ij}|}{(ij)^{\frac{q}{p}}} < \infty,$$

where $L(\log^+ L)^2 = \{f : E|f|\log^2|f| < \infty\}$,
$L^p(\log^+ L) = \{f : E|f|^p \log^+|f| < \infty\}$.

PROOF: We first observe that

$$\begin{aligned}
\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{E|Y''_{ij}|^p}{(ij)^{\frac{q}{p}}} &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(ij)^{-\frac{q}{p}} \int_{(ij)^{\frac{1}{p}}}^{\infty} x^{q-1}dP(|Y_{ij}| \le x) \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} q(ij)^{-\frac{q}{p}} \int_{(ij)^{\frac{1}{p}}}^{\infty} x^{q-1}P(|Y_{ij}| > x)dx \\
&\le q\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(ij)^{-\frac{q}{p}} \int_{(ij)^{\frac{1}{p}}}^{\infty} x^{q-1}P(X_0 > x)dx \\
&= q\int_1^{\infty} x^{q-1}P(X_0 > x) \sum_{(ij)^{\frac{1}{p}} \le x} (ij)^{-\frac{q}{p}} dx.
\end{aligned}$$

The assertion (a) is proved if we take $p = q = 1$ and apply Lemma 2.1 (ii) to the last inequality. To obtain the assertion (b) we use Lemma 2.1 (ii) with $\pi = q$, $h = 2, r = 0$.

Using the above observation we can establish the main result of this paper.

THEOREM 3.1. *Let B be p-smoothable, $1 < p \le 2$, and* $(M_{mn}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} X_{ij}, F_{mn})$ *a two-parameter B-valued martingale such that* $(X_{ij}) < X_0$. *Then*
(a) If $X_0 \in L(\log^+ L)^2$, *then* $\frac{M_{mn}}{mn} \longrightarrow 0$ a.s. *as* $(m,n) \longrightarrow \infty$,
(b) If $X_0 \in L^p(\log^+ L), 1 < q < p \le 2$, *then* $\frac{M_{mn}}{(m,n)^{\frac{1}{p}}} \longrightarrow 0$ a.s. *as* $(m,n) \longrightarrow \infty$.

PROOF: Let $Y_{ij} = X_{ij} I(\|X_{ij}\| \le ij), \quad Z_{ij} = X_{ij} - Y_{ij}, \quad i,j = 1,2,...,$ and

$$N_{m,n} = \sum_{i=1}^{m}\sum_{j=1}^{n}[Y_{ij} - E(Y_{ij} \mid F_{i-1,j-1})].$$

Note that $(N_{m,n}, F_{mn})$ is also a two-parameter B-valued martingale. Then we have

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{E[Y_{ij} - E(Y_{ij} \mid F_{i-1,j-1})]^p}{(ij)^p}$$

$$\le C_p \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{E \parallel Y_{ij} \parallel^p}{(ij)^p}$$

$$= C_p \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(ij)^{-p}\int_0^{ij} x^{p-1}P(\|X_{ij}\| > x)dx$$

$$\le C_p \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(ij)^{-p}\int_0^{ij} x^{p-1}P(X_0 > x)dx$$

$$= C_p \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_0^{1} y^{p-1}P(X_0 > y(ij))dy$$

$$\le C_p E X_0 \log^+ X_0 \int_0^1 y^{p-1}dy < \infty$$

because $X_0 \in L(\log^+ L)^2$. From Hajeck-Renny's inequality, Lemma 2.4, it follows that for every $C > 0$,

$$P(\max_{(m,n)>(ij)} \|\frac{N_{m,n}}{mn}\| \ge C) \le C_p \sum_{m=i}^{\infty}\sum_{n=j}^{\infty}\frac{E \parallel Y_{ij} \parallel^p}{(nm)^p} \longrightarrow 0$$

as $(ij) \longrightarrow \infty$. This means that the assumption $X_0 \in L(\log^+ L)^2$ implies the following condition

$$\frac{1}{mn} N_{mn} \longrightarrow 0 \quad \text{a.s. as} \quad (m,n) \longrightarrow \infty. \tag{3.1}$$

Next, put $K(j) = \text{card}\{(m,n) : mn = j\}, \quad j \in N$. It is known that $X_0 \in L\log^+ L$ if and only if

$$\sum_{i=1}^{\infty} K(j)P(X_0 > j) < \infty$$

(see [7], [10], [17]). It follows by a routine application of Borel-Cantelli's lemma that (M_{mn}) and (N_{mn}) are equivalent. By (3.1) to show (a) we have to prove that

$$\frac{1}{mn} \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} E(\|Z_{ij}\| \mid F_{i-1,j-1}) \longrightarrow 0 \quad \text{a.s. as} \quad (m,n) \longrightarrow \infty. \tag{3.2}$$

But, if $X_0 \in L(\log^+ L)$, we have, by Lemma 3.2 (a),

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{E\|Z_{ij}\|}{ij} < \infty,$$

which together with Kronecker's lemma (for positive double series) yieds (3.2).

Similarly to the proof of (a), we define

$$Y_{ij} = X_{ij} I(\|X_{ij}\| \le (ij)^{\frac{1}{q}}),$$
$$Z_{ij} = X_{ij} I(X_{ij} > (ij)^{\frac{1}{q}}) = X_{ij} - Y_{ij}.$$

Note that in view of Lemma 3.1,

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{E \parallel Y_{ij} \parallel^p}{(ij)^{\frac{p}{q}}} < \infty, \tag{3.3}$$

and, by Lemma 3.2 (b), if $X_0 \in L^p(\log^+ L), q > 1$, then

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{E\|Y_{ij}\|}{(ij)^{\frac{1}{q}}} < \infty. \tag{3.4}$$

Notice again that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} P(X_{ij} \neq Y_{ij}) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} P(\|Y_{ij}\| > (ij)^{\frac{1}{q}})$$
$$\leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty})P(X_0 > (ij)^{\frac{1}{q}})$$
$$= EX_0^q(\sum_{i=1}^{\infty}\frac{1}{i^q}) < \infty, \tag{3.5}$$

because $X_0 \in L^q \log^+ L$ and $1 < q < 2$. Then using (3.3), (3.4) and (3.5) and repeating the arguments in the proof for (a) we obtain (b).

COROLLARY 3.1. *Let B be p-smoothable, $1 < p \leq 2$ and $(M_{mn} = \sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}, F_{mn})$ a two-parameter B-valued martingale such that $X_{ij} < X_0$. Let q_1, q_2 be two real numbers and $q = \max(q_1, q_2)$. If $1 < q < p \leq 2$, then $\frac{M_{mn}}{m} \longrightarrow 0$ a.s. as $(m, n) \longrightarrow \infty$, provided $X_0 \in L^q(\log^+ L)$.*

COROLLARY 3.2. *let B be isomorphic to a Hilbert space and $(M_{mn} = \sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}, F_{mn})$ a two-parameter B-valued martingale such that $(X_{ij}) < X_0$. Then*
(a) If $X_0 \in L(\log^+ L)^2$, then $\frac{M_{mn}}{mn} \longrightarrow 0$ a.s. as $(m, n) \longrightarrow \infty$;
(b) If $X_0 \in L^p(\log^+ L)$, $1 < p < 2$, then $\frac{M_{mn}}{(mn)^{\frac{1}{p}}} \longrightarrow 0$ a.s. as $(m, n) \longrightarrow \infty$.

PROOF: Since B is isomorphic to a Hilbert space, there is a positive δ such that B is $(p + \delta)$- smoothable for any $1 < p < 2$. Therefore, the conclusions follow from Theorem 3.1.

REMARK: Setting $B = R$, the real line. From Corollary 3.2 we get the Marcinkiewicz-Zygmund's type strong law of large numbers for two-parameter martingales. In the case $1 < p < 2$, the sufficient conditions of Theorem 3.2 [7] (Gut, 1976), Theorem 2 [17] (Smythe, 1973), Theorem 2(a) [9] (Klesov, 1980) are consequences of this result.

4. The integrability of maximal functions

Let $M^{(p)}(\omega) = \sup_{(m,n)} \frac{\|M_{mn}\|}{(mn)^{\frac{1}{p}}}$, $1 < p < 2$. The result below is an extention of Theorem 3 [23] (Woyczynski, 1981) for two-parameter Banach-valued martingales.

THEOREM 4.1. *Let* $(M_{mn} = \sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}, F_{mn})$ *be a two-parameter martingale with values in Banach space* B . *Then*
(a) If $(X_{ij} < X_0 \in L(\log^+ L)^2)$ *and* B *is superreflexive, then* $M(1) \in L^1$.
(b) If $(X_{ij} < X_0 \in L^q(\log^+ L))$, $1 < q < p < 2$, *and* B *is r-smoothable for* $r > p$, *then* $M(p) \in L^q$.

To prove this theorem we need the following lemma (see [9] or [16]).

LEMMA 4.1. *Let* (b_{mn}) *be a two-parameter real-valued sequence of constants satisfying the conditions of Lemma 2.4 and* (X_{mn}) *a two-parameter* B*-valued sequence . Suppose that* $S_{mn} = \sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}$. *Then there exists a constant* C *(=4) satisfying*

$$\max_{(m,n)} \left\| \frac{S_{mn}}{b_{mn}} \right\| \leq C \max_{(u,v)\leq(m,n)} \left\| \sum_{i=1}^{u}\sum_{j=1}^{v} \frac{X_{ij}}{b_{ij}} \right\|.$$

PROOF: Put $T_{mn} = \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{X_{ij}}{b_{ij}}$. then

$$S_{mn} = \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{X_{ij}}{b_{ij}} = \sum_{i=1}^{m}\sum_{j=1}^{n}(b_{ij}) \sum_{u=i}^{m}\sum_{v=j}^{n}(T_{uv})$$
$$= \sum_{i=1}^{m}\sum_{j=1}^{n}(b_{ij}) \sum_{u=i}^{m}\sum_{v=j}^{n} \frac{x_{uv}}{b_{uv}}$$

Since $\sum_{i=1}^{m}\sum_{j=1}^{n} \frac{b_{ij}}{b_{mn}} = 1$, we have

$$\left\| \frac{S_{mn}}{b_{mn}} \right\| \leq \max_{(ij)\leq(m,n)} \left\| \sum_{u=i}^{m}\sum_{v=j}^{n} \frac{x_{uv}}{b_{uv}} \right\|.$$

Thus,

$$\max_{(m,n)} \|S_{mn} b_{mn}\| \le \max_{(ij)\le(m,n)} \max_{(m,n)} \left\| \sum_{u=i}^{m} \sum_{v=j}^{n} \frac{x_{uv}}{b_{uv}} \right\|.$$

Now the sum in the right hand side is equal to the two-dimensional difference of partial sums $\left(\left\| \sum_{i=1}^{u} \sum_{j=1}^{v} \frac{X_{ij}}{b_{ij}} \right\| \right)$ taken over four vertices of the rectrangle $(i,j) \le (u,v) \le (m,n)$. Therefore we have

$$\max_{(m,n)} \left\| \frac{S_{mn}}{b_{mn}} \right\| \le 4 \max_{(u,v)\le(m,n)} \left\| \sum_{i=1}^{u} \sum_{j=1}^{v} \frac{X_{ij}}{b_{ij}} \right\|.$$

PROOF OF THEOREM 4.1: From the conclusion of Lemma 4.1 we get

$$M^{(p)} \le C \left(\sup_{(u,v)\le(m,n)} \left\| \sum_{i=1}^{u} \sum_{j=1}^{v} \frac{\Delta'_{ij}}{ij} \right\| \right) + \sup_{(u,v)\le(m,n)} \left\| \sum_{i=1}^{u} \sum_{j=1}^{v} \frac{\Delta''_{ij}}{ij} \right\|,$$

where

$$\Delta'_{ij} = Y_{ij} - E(Y_{ij} \mid F_{i-1,j-1}),$$

$$\Delta''_{ij} = Y_{ij} - E(Y_{ij} \mid F_{i-1,j-1}),$$

and (Y_{ij}) and (Z_{ij}) are defined as in the proof of Theorem 3.1 (b).

In case (a), B is superreflexive and hence is r-smoothable for $r > 1$. So by Davis' type inequality for two-parameter Banach-valued martingales (see Corollary 2.1 (ii)), we get

$$E \sup_{(u,v)\le(m,n)} \left\| \sum_{i=1}^{u} \sum_{j=1}^{v} \frac{\Delta'_{ij}}{ij} \right\| \le C_r \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{E\|\Delta'_{ij}\|^r}{(ij)^r}.$$

The latter series is finite by Lemma 3.1. Further,

$$E \sup_{(u,v)\le(m,n)} \left\| \sum_{i=1}^{u} \sum_{j=1}^{v} \frac{\Delta'''_{ij}}{ij} \right\| < \infty$$

by Lemma 3.2 (a) because

$$E \sup_{(u,v)\leq(m,n)} \left\| \sum_{i=1}^{u}\sum_{j=1}^{v} \frac{\Delta''_{ij}}{ij} \right\| \leq C \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{E\|Z_{ij}\|}{ij}.$$

Thus, $EM^{(1)} < \infty$.

In case (b), with $1 \leq q < p < r \leq 2$ and B being r-smoothable, again by Corollary 2.1 (ii), we have

$$\left(E \sup_{(m,n)} \left\| \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{\Delta''_{ij}}{(ij)^{\frac{1}{p}}} \right\| \right)^{\frac{r}{q}} \leq C_r \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{E\|Y_{ij}\|^r}{(ij)^{\frac{r}{p}}},$$

which is finite by Lemma 3.1. Since r-smoothability implies q-smoothability for $q < r$ and by Corollary 2.1 we again have

$$E\left(\sup_{(ij)\leq(m,n)} \left\| \sum_{i=1}^{u}\sum_{j=1}^{v} \frac{\Delta''_{ij}}{(ij)^{\frac{1}{p}}} \right\|^q \right) \leq C_q \left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{E\|Z_{ij}\|^q}{(ij)^{\frac{p}{q}}} \right),$$

where the latter series converges a.s. in view of Lemma 3.2. (b).
Thus, $E(M^{(p)})^q < \infty$.

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