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ON ASSOCIATED PRIMES OF MODULES OVER A NON-COMMUTATIVE GOREINSTEIN RING

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Introduction

In [3], Björk introduced the notion of non-commutative Goreinstein rings and studied them in some details. This generalized many classical rings of differential operators. By definition, a non-commutative Goreinstein ring is a Noetherian ring R with finite injective dimension and satisfies the following condition: For any finitely generated (left or right) R-module M, any integer k, and any submodule N of $\mathrm{Ext}^k_R(M,R)$, we have $\mathrm{grade}(N) \geq k$. Here $\mathrm{grade}(N)$ is the smallest integer n such that $\mathrm{Ext}^n_R(N,R) \neq 0$. This is called the Auslander condition. While the condition is trivially true for any commutative Goreinstein ring, it is a basic tool to handle the non-commutative case that we are going to discuss here.

Let R be a non-commutative Gorenstein ring, and M a finitely generated R-module. Then there exists a spectral sequence whose E_2 term is the direct sum of certain double Ext-modules and E_∞ is isomorphic to a graded module associated to some filtration on M. The construction of this spectral sequence is due to Roos-Björk-Ischebeck and is well-known. It is the Auslander condition that essentially implies the convergence, and thus provides useful information about the module M. Among other things, it induces on any R-module M a filtration $\mathcal{B}_0(M) \subseteq ... \subseteq \mathcal{B}_\mu(M), \mu = \mathrm{inj.dim}R$ (the $\mathcal{B}$-filtration). This filtration can be characterized by $\mathcal{B}_k(M)$, the largest submodule of M whose grade is greater than $\mu - k$ and readily known in commutative algebra (see for example

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[4]). However, the approach here by means of spectral sequence has many advantages. For example, it allows the use of many methods and techniques from homological algebra. When the $\mathcal{B}$-filtration degenerates, this leads to the notion of modules having pure dimension, which play an important role in our study.

When R is filtered such that the associated graded ring is commutative, we consider R-modules M equipped with good filtrations and their related objects such as the characteristic ideals $J(M)$, the sets of $\mathcal{G}(J(M))$ of their prime divisors. In particular, we study the set $\mathcal{A}_F(M)$ of associated primes of the module $\mathrm{gr}_F M$, which in general depends on the filtration F and is not easy to handle. Here we have evaluated this set by means of filtration-free objects derived from the module M.

In more detail, the contents of the present paper are arranged as follows.

Section 1 is for the study of the $\mathcal{B}$-filtration on an R-module induced by the Roos-Björk-Ischebeck spectral sequence, R being a non-commutative Goreinstein ring. We prove that $\mathcal{B}_k(M)$ is the unique largest submodule of M whose grade is $\geq \mu - k$, $\mu = \mathrm{inj.dim} R$ (Theorem 1.4). This implies that the filtration is canonical in the sense: If $N \subseteq M$, then $\mathcal{B}_k(N) = N \cap \mathcal{B}_k(M)$ (Corollary 1.5). From this we can recapture the ubiquitous description of modules having pure dimension (Proposition 1.6) which is contained in [3]. As a remark, we point out that the map $M \to \mathcal{B}_k(M)$, with $k = \mu - \mathrm{grade}(M) - 1$, could be viewed as a kind of radical in the category of left (or right) R-modules with respect to the class of modules having pure dimension (Remark 1.7).

In Section 2, we assume that R is filtered such that the associated graded ring is commutative and study many objects defined on good filtrations on an R-module M. In particular, we study the dependence of the associated primes of the module $\mathrm{gr}_F M$ on the good filtration F. Our result is that we always have

$$\mathrm{Ass}(\mathrm{gr}_F M) \supseteq \cup_k \mathcal{G}(J(\mathrm{Ext}^k_R(\mathrm{Ext}^k_R(M, R), R))),$$

where the latter object is filtration-free (Theorem 2.8). We also give a sufficient condition so that the equality holds (Theorem 2.11). This condition is satisfied in the special case when R itself is commutative, and we thus obtained a decomposition formula for the set of associated primes of a module over a commutative Goreinstein ring (Corollary 2.13), which generalizes a result of Grothendieck in [**6**] for complete local rings.

Convention: In this paper, a module can either mean a left or a right one, and is always finitely generated.

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1. The $\mathcal{B}$-filtration

Let R be a non-commutative Goreinstein ring. Recall [**3**] that R is a Noetherian ring with finite injective dimension and satisfies the following condition: For any R-module M, any integer k, and any submodule N of $\mathrm{Ext}^k_R(M,R)$ we have $\mathrm{grade}(N) \geq k$. Here by definition

$$\mathrm{grade}(N) = min\{n|\mathrm{Ext}^n_R(N,R) \neq 0\}.$$

This condition was first proposed by Auslander, and for the definition of a non-commutative ring here we follow Björk [**3**]. Observe that in the commutative case, the last requirement is not necessary.

Examples of non-commutative Goreinstein rings can be seen in abundance: quasi- Frobenius rings, Weyl algebras, stalks of the rings $\mathcal{D}_X$, $\mathcal{E}_X$ of differential operators and microdifferential operators. In fact, it was a generalization of these rings that leads to the notion of non-commutative Goreinstein rings.

The basic property of a non-commutative Goreinstein ring R is that for every R-module M, there is a convergent spectral sequence with

$$E_2^{p,-q} = \mathrm{Ext}^p_R(\mathrm{Ext}^q_R(M,R),R).$$

The term $E_\infty^n = \oplus_{p-q=n} E_\infty^{p,-q}$ vanishes for all $n \neq 0$, while E_∞^0 is isomorphic to some associated graded module of the module M. If $\mu = \text{inj.dim}R$, all the indexes range in $\{0,1,...,\mu\}$, and we can interprete the above phenomenon as that M has a filtration

$$(1.1) \qquad \mathcal{B}_0(M) \subseteq \mathcal{B}_1(M) \subseteq ... \subseteq \mathcal{B}_\mu(M) = M$$

with $\mathcal{B}_{\mu-k}(M)/\mathcal{B}_{\mu-k-1}(M) = E_\infty^{k,-k}$ for $k = 0, 1, ..., \mu$.

We refer to (1.1) as the $\mathcal{B}$-filtration of M (conventionally, $\mathcal{B}_{-1}(M) = 0$).

The convergence of the above spectral sequence has many consequences, among them we wish to recall two important facts in the following

PROPOSITION 1.2. *[3,§1]. Let M be an R-module and let $\{E_r\}_{r\geq 2}$ be the spectral sequence as above. Then for any integer $k \geq 0$,*
(i) grade$(E_2^{\text{grade}(M)+k+1,-\text{grade}(M)}) \geq \text{grade}(M) + k + 2$;
(ii) There is an exact sequence

$$0 \to \mathcal{B}_{\mu-k}(M)/\mathcal{B}_{\mu-k-1}(M) \to E_2^{k,-k} \to S_k \to 0,$$

where S_k is some R-module with grade$(S_k) \geq k+2$.

PROOF: The $\mathcal{B}$-filtration can be used to study modules over a non-commutative Goreinstein ring. For example, it is closely related to the Ext-modules, and for filtered modules, to certain associated primes as we shall see in §2. It turns out that this filtration can be described in another way which we are going to discuss here. First of all we mention some elementary properties of grade. Recall that R denotes a fixed non-commutative Goreinstein ring with inj.dim$R = \mu$.

PROPOSITION 1.3. *(i) If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of R-modules, then grade$(M) = min\{\text{grade}(M'), \text{grade}(M'')\}$.*
(ii) If N_1, N_2 are submodules of an R-module M and suppose that grade$(N_1) \geq k$, grade$(N_2) \geq k$ for some integer k, then grade$(N_1 + N_2) \geq k$.
(iii) grade$(\mathcal{B}_{\mu-k}(M)) \geq k$ for any R-module M and any integer k.

PROOF: (i) follows from the long exact Ext-sequence and the definition of grade.

(ii) follows from (i) and the exactness of the following sequences

$$0 \to N_1 \to N_1 + N_2 \to (N_1 + N_2)/N_1 \to 0$$

$$0 \to N_1 \cap N_2 \to N_2 \to N_2/(N_1 \cap N_2) \simeq (N_1 + N_2)/N_1 \to 0.$$

(iii) is obtained by induction on k, using (i) and Proposition 1.2 (ii).

By Proposition 1.3.(ii), given an R-module M and an integer k, the set of all submodules of M whose grade $\geq k$ contains a unique largest element, and we denote this submodule by $G_{\mu-k}(M)$. Clearly we have

$$G_0(M) \subseteq ... \subseteq G_\mu(M) = M.$$

THEOREM 1.4. *$\mathcal{B}_k(M) = G_k(M)$ for $k = 0, 1, ..., \mu$.*

PROOF: Having $\mathcal{B}_\mu(M) = G_\mu(M)(= M)$, we use descending induction on k and assume that $\mathcal{B}_k(M) = G_k(M)$. If $\mathcal{B}_{k-1}(M) = \mathcal{B}_k(M)$, we have $\mathcal{B}_{k-1}(M) = G_k(M) \supseteq G_{k-1}(M)$, i.e. $\mathcal{B}_{k-1}(M) = G_{k-1}(M)$ since we always have $\mathcal{B}_{k-1}(M) \subseteq G_{k-1}(M)$. Suppose that $\mathcal{B}_{k-1}(M) \neq \mathcal{B}_k(M)$. If $O \neq P := G_{k-1}(M)/\mathcal{B}_{k-1}(M) \subseteq \mathcal{B}_k(M)/\mathcal{B}_{k-1}(M) \subseteq M' = \mathrm{Ext}_R^{\mu-k}(\mathrm{Ext}_R^{\mu-k}(M, R), R)$, consider the exact sequence

$$0 \to P \to M' \to M'/P \to 0.$$

Put grade$(P) = h$, the long Ext-sequence for this contains

$$\mathrm{Ext}_R^h(M', R) \to \mathrm{Ext}_R^h(P, R) \to \mathrm{Ext}_R^{h+1}(M'/P, R).$$

If $h > \mu - k$, Proposition 1.2.(i) implies grade$(\mathrm{Ext}_R^h(M', R)) > h + 1$, also grade$(\mathrm{Ext}_R^{h+1}(M'/P, R) \geq h + 1$, which show that grade$(\mathrm{Ext}_R^h(P, R)) \geq h + 1$ by Proposition 1.3.(i). This is impossible, since we can easily show that if grade$(P) = h$, then grade$(\mathrm{Ext}_R^h(P, R)) = h$. Hence grade$(P) = \mu - k$. Now the sequence

$$0 \to \mathcal{B}_{k-1}(M) \to G_{k-1}(M) \to P \to 0$$

implies $\text{grade}(G_{k-1}(M)) = \mu - k$ by Proposition 1.3.(i) again. This contradicts the definition of $G_{k-1}(M)$. Therefore $P = 0$, i.e. $\mathcal{B}_{k-1}(M) = G_{k-1}(M)$ and Theorem 1.4 follows.

The following corollary is immediate from Theorem 1.4.

COROLLARY 1.5. *The $\mathcal{B}$-filtration is canonical, i.e. if N is any submodule of M, then for all k,*

$$\mathcal{B}_k(N) = N \cap \mathcal{B}_k(M).$$

Now we describe a class of R-modules for which the $\mathcal{B}$-filtration degenerates. Precisely, we say that a module M has pure dimension if each term $\mathcal{B}_k(M)$ is either 0 or M. (For example, simple modules satisfy this condition). This property can be characterized in several ways.

PROPOSITION 1.6. *For an R-module M, the following conditions are equivalent:*
(i) M has pure dimension.
(ii) $\text{grade}(N) = \text{grade}(M)$ for every $0 \neq N \subseteq M$.

PROOF: Observe that $G_\mu(M) = \dots = G_{\mu-\text{grade}(M)}(M) = M$ and if $M \neq 0$, $G_{\mu-\text{grade}(M)-1}(M) \neq M$. So by Corollary 1.5, M has pure dimension iff $\mathcal{B}_{\mu-\text{grade}(M)-1}(M) = 0$.
(i) $\Rightarrow$ (ii). If M has pure dimension, then by Proposition 1.5, so has N and $\mathcal{B}_{\mu-\text{grade}(M)-1}(N) = 0$. Hence $\mu - \text{grade}(M) - 1 \leq \mu - \text{grade}(N) - 1$, i.e. $\text{grade}(N) \leq \text{grade}(M)$, and this means $\text{grade}(N) = \text{grade}(M)$.
(ii) $\Rightarrow$ (i). If $\text{grade}(N) = \text{grade}(M)$ for every $0 \neq N \subseteq M$, then $M = G_\mu(M) = \dots = G_{\mu-\text{grade}(M)}(M)$ and $G_{\mu-\text{grade}(M)-1}(M) = \dots = G_0(M) = 0$, so by Corollary 1.5, M has pure dimension.

We can easily see from Proposition 1.6 that if M has pure dimension, so does any of its submodules. Also, every R-module M with $\text{grade}(M) = \mu$ must have pure dimension. This class of modules is called *holonomic*, and has nice properties closely related to duality theory. For example, each of the functors $M \to M^* := \text{Ext}^\mu_R(M, R)$ from the category of left (resp., right) holonomic

R-modules to the category of right (resp., left) holonomic ones is exact and satisfies $M^{**} = M$. See [3] for a discussion about this.

In fact, using Propositions 1.2, 1.3, and 1.6 we can construct new modules that have pure dimension from a given R-module M. Among these are the modules

$\mathrm{Ext}_R^{\mathrm{grade}(M)}(M)$, $\mathrm{Ext}_R^k(\mathrm{Ext}_R^k(M,R),R)$, and $\mathcal{B}_k(M)/\mathcal{B}_{k-1}(M), k = 0,1,2,\ldots$

REMMARK 1.7: Put $\mathcal{R}(M) = \mathcal{B}_{\mu-\mathrm{grade}(M)-1}(M)$, we always have that $M/\mathcal{R}(M)$ is an R-module having pure dimension. Furthermore, $\mathcal{R}(M/\mathcal{R}(M)) = 0$. So the map $M \to \mathcal{R}(M)$ could be thought of as a *radical* in the category of R-modules with respect to the purity property. The object $\mathcal{R}(M)$ is characterized as follows. It is the smallest submodule of M such that $M/\mathcal{R}(M)$ has pure dimension and has grade $=$ grade(M).

2. Filtered modules and associated primes

In this section, we consider a filtered ring $(R, \mathcal{F}R)$, i.e. a non-commutative ring R together with a family of additive subgroups $\mathcal{F}R = \{R_n \mid n \in \mathbb{Z}\}$ such that $R_n \subseteq R_{n+1}$, $R_n.R_m \subseteq R_{n+m}$ for all m, n. Furthermore, we assume that the filtration $\mathcal{F}R$ is Artin-Rees, i.e. for any R-module M and any good filtration FM on M, FM is separated and the induced filtration on every submodule of M is again good.

First we recall a well-known fact concerning the Gorensteinness mentioned in §1.

THEOREM 2.1 (Roos-Björk). *Under the assumption that $\mathcal{F}R$ is Artin-Rees, if the associated graded ring $gr_{\mathcal{F}}R$ is Gorenstein, then R is Gorenstein and for any R-module M and any good filtration F on M, we have grade$(M) =$ grade$(gr_F M)$.*

Suppose $gr_{\mathcal{F}}R$ is commutative. Then every R-module M is associated to an ideal $J(M)$ of $gr_{\mathcal{F}}R$, called the characteristic ideal of M. Indeed, take any good filtration F on M and take $J(M) = \sqrt{\mathrm{Ann}(\mathrm{gr}_F M)}$, and this does not depend on the chosen good filtration F. For any ideal I in $gr_{\mathcal{F}}R$, denote by $\mathcal{G}(I)$ the set of minimal prime divisors of I. Recall that a radical ideal I is called equidimensional if for every $\wp \in \mathcal{G}(I)$, we have $\mathrm{ht}\wp = \mathrm{ht}I$.

For any R-module M and any good filtration F on M, put

$$\mathcal{A}_F(M) := \mathrm{Ass}_{\mathrm{gr}_{\mathcal{F}}R}(gr_F M),$$

the set of associated primes of $gr_F M$, which is a (graded, finitely generated) $gr_{\mathcal{F}}R$-module. The aim of this section is to study the family $\mathcal{A}_F(M)$, where F ranges over all good filtrations on M. In order to handle the case, we need to impose more hypothesis on the filtered ring $(R, \mathcal{F}R)$ as follows:

STANDING HYPOTHESIS 2.2. *From now on, R denotes a filtered ring such that the filtration $\mathcal{F}R$ is Artin-Rees and the associated graded ring $gr_{\mathcal{F}}R$ is commutative Gorenstein.*

By Theorem 2.1, R is then a non-commutative Gorenstein ring in the sense of §1.

EXAMPLES 2.3.: The Weyl Algebra $A_n(k)$ (k is a field of characteristic zero), the stalks of sheaves $\mathcal{D}_X$, $\mathcal{E}_X$ of rings of diffenrential operators on a (complex) variety X, have filtrations satisfying Hypothesis 2.2.

Let M be an R-module. From the Gorensteinness of $gr_{\mathcal{F}}R$ and the last part of Theorem 2.1, we see that grade$(M) = \mathrm{ht}J(M)$, and this equality will be used frequently hereafter.

Our study of the family $\mathcal{A}_F(M)$ bases on the following theorem, due to Gabber [5] and Björk [3].

Proposition 2.4. *Let F be a good filtration on the R-module M. Then there is a spectral sequence realized by a sequence of complexes of modules over $gr_{\mathcal{F}}R$ with*

$$E_1^n = Ext^n_{gr_{\mathcal{F}}R}(gr_F M, gr_{\mathcal{F}}R) \Rightarrow Ext^n_R(M,R).$$

If $\wp$ is a minimal prime over $J(M)$, let us consider the localized spectral sequence $(E_r^{\cdot})_\wp$. Since grade$(grM_\wp)$=ht$\wp$=dim$(grR)_\wp$= inj.dim$(grR)_\wp$, we see that all the modules $(E_1^n)_\wp = \mathrm{Ext}^n_{grR}(grM, grR)_\wp \cong \mathrm{Ext}^n_{grR_\wp}(grM_\wp, grR_\wp)$ $= 0$ for all $n \neq \mathrm{ht}\wp$, while $(E_1^{\mathrm{ht}\wp})_\wp \neq 0$. Passing to the limit we find that there exists a good filtration on $\mathrm{Ext}^{\mathrm{ht}\wp}_R(M,R)$ such that gr $\mathrm{Ext}(^{\mathrm{ht}\wp}_R(M,R)_\wp \cong$ $(E_\infty^{\mathrm{ht}\wp})_\wp = (E_1^{\mathrm{ht}\wp})_\wp \neq 0$, hence $\mathrm{Ext}^{\mathrm{ht}\wp}_R(M,R) \neq 0$. Since grade(grExt$^{\mathrm{ht}\wp}_R(M,R)$) $=$ grade(Ext$^{\mathrm{ht}\wp}_R(M,R)$) $\geq \mathrm{ht}\wp$, this also means that $\wp \in \mathcal{G}(J(\mathrm{Ext}^{\mathrm{ht}\wp}_R(M,R)))$. So we obtain the following

Proposition 2.5. *If $\wp \in \mathcal{G}(J(M))$, then $\wp \in \mathcal{G}(J(Ext^{\mathrm{ht}\wp}_R(M,R)))$.*

Proposition 2.5 has many useful consequences.

Corollary 2.6. *If $\wp \in \mathcal{G}(J(M))$, then $Ext^{\mathrm{ht}\wp}_R(Ext^{\mathrm{ht}\wp}_R(M,R)) \neq 0$.*

Proof: Apply Proposition 2.5 twice, first for the module M, and then for $\mathrm{Ext}^{\mathrm{ht}\wp}_R(M,R)$, and observe that an R-module $N = 0$ iff $\mathcal{G}(J(N)) = \emptyset$.

Clearly, for any good filtration F on the R-module M, we have $\mathcal{G}(J(M))$ $\subseteq \mathcal{A}_F(M)$. We shall see that this can be refined. First we use the notation

$$L_k(M) = \mathrm{Ext}^k_R(\mathrm{Ext}^k_R(M,R),R),$$

and define a subset of $Spec(gr_{\mathcal{F}}R)$

$$\mathcal{E}(M) = \cup_k \mathcal{G}(J(L_k(M)))).$$

(If $k <$ grade(M) or $k > \mu$, we have $L_k(M) = 0$, so the union can be taken over grade$(M) \leq k \leq \mu$).

Obviously, $\mathcal{E}(M)$ does not depend on F, and we have the following immediate consequence of Proposition 2.5.

COROLLARY 2.7. $\mathcal{G}(J(M)) \subseteq \mathcal{E}(M)$.

At this stage, we can announce the main result of this section. Recall that we have put $\mathcal{A}_F(M) := \mathrm{Ass}_{\mathrm{gr}_{\mathcal{F}} R}(gr_F M)$.

THEOREM 2.8. *Let M be an R-module, then for any good filtration F,*

$$\mathcal{E}(M) \subseteq \mathcal{A}_F(M).$$

PROOF: Let $\wp \in \mathcal{E}(M)$, so $\wp \in \mathcal{G}(L_k(M))$ for some k. Put $N = \mathrm{Ext}_R^k(M, R)$, we have $\wp \in \mathcal{G}(\mathrm{Ext}_R^{\mathrm{ht}\wp}(\mathrm{Ext}_R^k(N, R), R))$ by Proposition 2.5. If $\mathrm{ht}\wp > k$, Proposition 1.2.(i) of Section 1 implies $\mathrm{grade}(\mathrm{Ext}_R^{\mathrm{ht}\wp}(\mathrm{Ext}_R^k(N, R), R)) > \mathrm{ht}\wp$, a contradiction, since $\mathrm{ht}\wp \geq \mathrm{grade}(\mathrm{Ext}_R^{\mathrm{ht}\wp}(\mathrm{Ext}_R^k(N, R), R))$. Thus we must have $\mathrm{ht}\wp = k$. Consider the exact sequence in Proposition 1.2.(ii)

$$0 \to \mathcal{B}_{\mu-k}(M)/\mathcal{B}_{\mu-k-1}(M) \to L_k(M) \to S_k \to 0.$$

Take any good filtration on $L_k(M)$ and endow the first and the last terms of the above sequence with the induced and the quotient filtration, respectively. We see that

$$J(L_k(M)) = J(\mathcal{B}_{\mu-k}(M)/\mathcal{B}_{\mu-k-1}(M)) \cap J(S_k).$$

Therefore

$$\wp \supseteq J(\mathcal{B}_{\mu-k}(M)/\mathcal{B}_{\mu-k-1}(M)),$$

or

$$\wp \supseteq J(S_k).$$

The latter inclusion is impossible, since $\mathrm{grade}(S_k) > k$.

Thus

$$\wp \supseteq J(\mathcal{B}_{\mu-k}(M)/\mathcal{B}_{\mu-k-1}(M)).$$

Now

$$\text{grade}(\mathcal{B}_{\mu-k}(M)/\mathcal{B}_{\mu-k-1}(M))) \geq \text{grade}(\mathcal{B}_{\mu-k}(M)) \geq k = \text{ht}\wp,$$

and so $\wp$ is a minimal prime over $J(\mathcal{B}_{\mu-k}(M)/\mathcal{B}_{\mu-k-1}(M))$. In particular,

$$\wp \in \text{Supp}(\text{gr}(\mathcal{B}_{\mu-k}(M)/\mathcal{B}_{\mu-k-1}(M))),$$

where on each $\mathcal{B}_k(M)$ the good filtration is the one induced from that of M. Then we get

$$(gr\mathcal{B}_{\mu-k}(M))_\wp/(gr\mathcal{B}_{\mu-k-1}(M))_\wp = \text{gr}(\mathcal{B}_{\mu-k}(M)/\mathcal{B}_{\mu-k-1}(M))_\wp \neq 0.$$

This implies in particular

$$(\text{gr}\mathcal{B}_{\mu-k}(M))_\wp \neq 0.$$

Together with the inequality

$$\text{grade}(\text{gr}\mathcal{B}_{\mu-k}(M)) \geq k = \text{ht}\wp,$$

this shows that $\wp$ is minimal over $J(\mathcal{B}_{\mu-k}(M))$, hence

$$\wp \in \text{Ass}(\text{gr}\mathcal{B}_{\mu-k}(M)) \subseteq \text{Ass}(\text{gr}_F M) = \mathcal{A}_F(M).$$

We have actually shown in the proof of Theorem 2.8 that $J(L_k(M))$ is equidimensional, and $\mathcal{G}(J(L_k(M))) \subseteq \mathcal{G}(J(\mathcal{B}_{\mu-k}(M)/\mathcal{B}_{\mu-k-1}(M))$ for all k. So if M has pure dimension, i.e. each quotient $(\mathcal{B}_{\mu-k}(M)/\mathcal{B}_{\mu-k-1}(M)), k \neq \text{grade}(M)$, is zero, then $\mathcal{E}(M)$ reduces to a single term $\mathcal{G}(J(L_{\text{grade}(M)}(M)))$. Combined with Corollary 2.7, this gives

COROLLARY 2.9 (Kashiwara-Gabber-Björk). *If the R-module M has pure dimension, then $J(M)$ is equidimensional.*

Concerning the inclusion $\mathcal{E}(M) \subseteq \mathcal{A}_F(M)$, we notice that this is in general strict. For example, when M is filtered with a filtration FM such that

M has pure dimension but $gr_F M$ has not, then $\mathcal{E}(M) \neq \mathcal{A}_F(M)$. Indeed, in this case $\mathcal{E}(M)$ only contains primes of height grade(M), while the latter does not. This can be seen using the following characterization of purity in commutative case (see [4]).

PROPOSITION 2.10. *Let M be a module over a commutative ring R. Then M has pure dimension iff all its associated primes have the same height*

PROOF: Suppose M has pure dimension and let $\wp \in \mathrm{Ass}_R(M)$. Then there is an $N \subseteq M$ such that $N \cong R/\wp$, and we have $\mathrm{ht}\wp = \mathrm{grade}(N) = \mathrm{grade}(M)$ by Proposition 1.6.

Conversely, if all associated primes of M have the same height, say h, then for every submodule $N \subseteq M$ we have $\mathrm{grade}(N) = \mathrm{ht}\ \mathrm{Ann}(N) = inf\{\mathrm{ht}\wp | \wp \in \mathrm{Ass}(N)\} = h$, since $\mathrm{Ass}(N) \subseteq \mathrm{Ass}(M)$. Now by Proposition 1.6 again, M has pure dimension.

It is natural to ask when equality holds in Theorem 2.8. In the remainder of this paper, we will give a sufficient condition for this. So let us fix an R-module M and consider a good filtration F on it. Put $\mathcal{M} = gr_F M$, the associated graded $gr_{\mathcal{F}} R$-module.

THEOREM 2.11. *Suppose that the good filtration F satisfies the following condition: If $\mathcal{N} \neq 0$ is any homogeneous cyclic submodule of $\mathcal{M}$, there exists a submodule $0 \neq N \subseteq M$ such that with the induced filtration we have $grN \subseteq \mathcal{N}$. Then*

$$\mathcal{E}(M) = \mathcal{A}_F(M).$$

PROOF: The proof of Theorem 2.11 bases on Proposition 1.4, Theorem 2.10 and the following lemma whose proof is easy.

LEMMA 2.12. *Let R be as in Theorem 2.10. Given an exact sequence of R-modules*

$$0 \to M' \to M \to M'' \to 0,$$

with M having pure dimension and with grade(M'') $\geq$ grade(M) + 1, then

$$\mathcal{G}(J(M)) = \mathcal{G}(J(N)).$$

When the ring R is commutative, we consider the trivial filtration $\mathcal{F}R$: $F_nR = 0$ with $n < 0$ and $F_nR = R$ otherwise. In this case, $gr_{\mathcal{F}}R \cong R$ (as nongraded rings) and for any R-module M we can take a good filtration F on M such that $gr_F M \cong M$. For example, take $F_nM = \sum_i F_nR.u_i$, where $\{u_k\}$ is a finite set of generators for M. Clearly the assumption of Theorem 2.11 is satisfied, hence $\mathcal{E}(M) = \mathcal{A}_F(M)$, i.e.

$$\cup_k \mathcal{G}(L_k(M)) = \mathrm{Ass}(\mathrm{gr}_F M) = \mathrm{Ass}(M).$$

Note that $\mathcal{G}(L_k(M)) = \mathrm{Ass}(L_k(M))$. Then we obtain the following corollary, which was first proved by A. Grothendieck for a complete local ring (see [**6**, Proposition 6.6]).

COROLLARY 2.13. *Let R be a commutative Gorenstein ring. Then for any R-module M, we have*

$$\cup_k Ass(L_k(M)) = Ass(M),$$

where each term $Ass(L_k(M))$ contains only primes of height k.

REFERENCES

[1]. M.F.Atiyah and I.G.MacDonald, *Introduction to Commutative Algebra*, Addison-Wesley 1969.

[2]. J.E.Björk, *Rings of differential operators*, North Holland, Amsterdam, 1979.

[3]. J.E.Björk, *Filtered noetherian rings*, Math. Surveys and monographs, A. M. S. **24** (1987), 59-97.

[4]. H.D.Duan and E.K.Ekstrom, *Purity and equidimensionality of modules for a regular ring*, Report, Univ. of Stockholm **18** (1986).

[5]. O. Gabber, *Equidimensionalité de la variété charactéristique*, Sém. sur les alg. envelop-pantes, Univ. P. et M. Curie (1982).
[6]. A. Grothendieck, *Local cohomology*, Lect. Notes in Math. **41** (1967).
[7]. H. Matsumura, *Commutative ring theory*, Cambridge Univ. Press (1986).

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