ACTA MATHEMATICA VIETNAMICA
Volume 17, Number 1, 1992

ON A FREE BOUNDARY PROBLEM ARISING FROM MATHEMATICAL MODELS FOR SORPTION OF SWELLING SOLVENTS IN GLASSY POLYMER

PHAN HUU SAN AND NGUYEN DINH TRI

§1. Introduction

In [4] the following problem has been considered : find (T,s,c,C) , $T > 0, s(t) \in C^1[0,T], C(t) \in C^1[0,T], c(x,t) \in C^{2,1}(D_T) \cap C^{1,0}(\bar{D}_T), D_T = \{(x,t) : 0 < x < s(t), 0 < t < T\}$ such that the following equation and conditions are satisfied:

$$c_{xx} - \dot{c}_t = 0 \quad \text{in} \quad D_T, \tag{1.1}$$

$$s(0) = 0, \tag{1.2}$$

$$c(0,t) = C(t),\ C(0) = 1, 0 < t < T, \tag{1.3}$$

$$c_x(0,t) = \dot{C}(t), 0 < t < T, \tag{1.4}$$

$$\dot{s}(t) = f(c(s(t),t)), 0 < t < T, \tag{1.5}$$

$$c_x(s(t),t) = -(q + c(s(t),t))\dot{s}(t), 0 < t < T, \tag{1.6}$$

where $f \in C[0,+\infty) \cap C^1(0,+\infty), f(0) = 0, f'(y) > 0$ for $y > 0$, q is a positive constant.

This problem arises from a model for sorption of swelling solvent in a glassy polymer. This model has been described in [2]. Consider a slab of a glassy polymer in contact with a solvent. We observe that if the solvent concentration exceeds some threshold value q, then the solvent penetrates into the polymer. The solvent is assumed to diffuse in the penetrated zone according to Fick's

Received May 29, 1991; in revised form November 4, 1991

law. Note that $c(x,t)+q$ is the concentration of the swollen region and the glassy region (a free boundary).

The condition (1.6) is the mass balance at the interface $x = s(t)$ and (1.5) is an empirical law connecting the speed of penetration of the solvent with the jump of concentration at the free boundary. The boundary conditions (1.3),(1.4) arise from the mass balance when the polymer is in perfect contact with a (well-stirred) bath in which the (excess of) concentration of the solvent is given by $C(t)$.

Some similar free boundary problems have been investigated e.g. in [1], [5], [8].

In this paper, we consider the penetration of solvent in the nonhomogeneous polymer. So q depends on the space argument, and the law of the penetration of the solvent is given by the following:

$$\dot{s}(t) = f(c(s(t),t), s(t)).$$

However, we shall restrict ourselves to the case $s(0) = b > 0$. Specifically, we will study the following problem:

PROBLEM I. Find (T,s,c,C), $T > 0, s(t) \in C^1[0,T], C(t) \in C^1[0,T], c(x,t) \in C^{2,1}(D_T) \cap C^{1,0}(\bar{D}_T), D_T = \{(x,t) : 0 < x < s(t), 0 < t < T\}$, such that the following equation and conditions are satisfied:

$$c_{xx} - c_t = 0 \quad \text{in} \quad D_T, \tag{1.7}$$

$$s(0) = b > 0, \tag{1.8}$$

$$c(x,0) = h(x), 0 < x < b, \tag{1.9}$$

$$c(0,t) = C(t),\ C(0) = h(0), 0 < t < T, \tag{1.10}$$

$$c_x(0,t) = \dot{C}(t), 0 < t < T, \tag{1.11}$$

$$\dot{s}(t) = f(c(s(t),t), s(t)), 0 < t < T, \tag{1.12}$$

$$c_x(s(t),t) = -(q(s(t)) + c(s(t),t))\dot{s}(t), 0 < t < T, \tag{1.13}$$

where f, g, h are given functions.

We assume that f, g, h satisfy the following conditions:

$$\left.\begin{aligned} &f \in C^1(\Omega), \Omega = \{(c,x) : 0 \le c \le h(0), x \ge 0\} \\ &f(0,x) = 0 \quad \forall x \ge 0; f_c(c,x) > 0, f_x(c,x) > 0, \forall(c,x) \in \Omega \\ &f(h(0),x) \le F \quad \forall x \ge 0, F \text{ is a positive constant} \end{aligned}\right\} \tag{1.14}$$

$$q \in C^1(R^+), q(x) \ge 0 \; \forall x \ge 0, q'(x) \ge 0 \quad \forall x \ge 0 \tag{1.15}$$

$$\left.\begin{aligned} &h \in C^2[0,b], h(x) > 0, h'(x) < 0 \quad \text{in} \quad [0,b], \\ &h'(0) = h''(0), h'(b) = -(q(b) + h(b))f(h(b),b). \end{aligned}\right\} \tag{1.16}$$

From (1.14) it follows that there exists a function Φ such that (1.12) can be rewritten in the following equivalent form:

$$c(s(t),t) = \Phi(\dot{s}(t), s(t)), 0 < t < T. \tag{1.12'}$$

We notice that, if Problem I has a classical solution, then c_{xx} and c_t are continuous up to the boundary $x = 0$ for $t > 0$. Differentiating (1.10), from (1.11) we obtain

$$c_x(0,t) = c_t(0,t) = c_{xx}(0,t), 0 < t < T. \tag{1.17}$$

§2. Auxiliary results

First, we prove some a prior estimates for the solution of Problem I.

PROPOSITION 2.1. *Let (T,s,c,C) be a solution of Problem 1. Then*

$$\dot{s}(t) > 0, \quad 0 \le t \le T, \tag{2.1}$$

$$c_x(x,t) < 0 \quad \text{in} \quad \bar{D}_T, \tag{2.2}$$

$$c(x,t) > 0 \quad \text{in} \quad \bar{D}_T. \tag{2.3}$$

PROOF: It is clear that (2.1)-(2.3) are satisfied for $t = 0$. So there exists $\bar{t} > 0$ such that $\dot{s}(t) > 0, 0 \le t < \bar{t}$. Since $c(s(t),t) = \Phi(\dot{s}(t), s(t)) > 0, 0 \le t < \bar{t}$, it follows that $c_x(s(t),t) < 0, 0 \le t < \bar{t}$. If $c_x(x,t)$ attains its positive maximum on

$x = 0$ at $t = t_0$, from (1.17) we get $c_{xx}(0, t_0) = c_x(0, t_0) > 0$, which contradicts the strong maximum principle. Therefore (2.2) and (2.3) hold in $D_{\bar{t}}$ by the maximum principle.

Suppose that $\dot{s}(\bar{t}) = 0$. Then $c(s(\bar{t}), \bar{t}) = \Phi(\dot{s}(\bar{t}), s(\bar{t})) = \Phi(0, s(\bar{t})) = 0$ so that $c(x,t)$ attains its minimum at $(s(\bar{t}), \bar{t})$ in $\overline{D}_{\bar{t}}$. Because of the strong maximum principle we have $c_x(s(\bar{t}), \bar{t}) < 0$, which contradicts the condition $c_x(s(\bar{t}), \bar{t}) = 0$. Then $\dot{s}(\bar{t}) > 0$ and we can repeat the above argument for any $t \in (\bar{t}, T]$.

COROLLARY 2.1. *Let (T,s,c,C) be a solution of Problem I. Then*

$$c(x,t) \le h(0) \quad in \quad \bar{D}_T, \tag{2.4}$$

$$\dot{s}(t) \le F, \quad 0 \le t \le T. \tag{2.5}$$

PROOF: From $\dot{C}(t) = c_x(0,t) < 0$ it follows that $c(0,t) < C(0) = h(0)$. Since $c(x, 0) = h(x) < h(0)$ and because of (2.2), we get (2.4) by using the maximum principle. (2.5) follows from (1.12) and (1.14).

Now we show that $s \in C^2[0, T]$ if Problem 1 has a solution. We consider the following

PROBLEM II. Let $r \in C^1[0,T], r(0) = b \ge 0, \dot{r}(t) > 0,\ 0 \le t < T, \eta \in C[0,b]$ if $b > 0, \Psi \in C[0,T], \eta(0) = \Psi(0)$, $g \in C^1(R^+)^2, g_y(y,x) < 0, g_x(y,x) < 0, g(x,y) < 0$ for $y > 0, x > 0$. Find a triple (T_0, Z, γ) such that $T_0 > 0, Z \in C^{2,1}(D_{T_0}(r)) \cap C(\bar{D}_{T_0}(r)), \gamma \in C[0,T_0], D_{T_0}(r) = \{(x,t) : 0 < x < r(t), 0 < t < T_0\}$ and the following equation and conditions are satisfied:

$$Z_{xx} - Z_t = 0 \quad \text{in} \quad D_{T_0}(r), \tag{2.6}$$

$$Z(x, 0) = \eta(x), \quad 0 \le x \le b, \tag{2.7}$$

$$Z(0,t) = \Psi(t), \quad 0 \le t \le T_0, \tag{2.8}$$

$$Z_x(r(t),t) + \dot{r}(t)Z(r(t),t) = g_x(\gamma(t), r(t))\dot{r}(t) +$$
$$+ g_y(\gamma(t), r(t))[\dot{r}(t)g(\gamma(t), r(t)) + Z(r(t),t)],$$

$$0 < t < T_0, \tag{2.9}$$

$$\gamma(t) = \gamma_0 + \int_0^t [\dot{r}(\tau) g(\gamma(\tau), r(\tau)) +$$

$$+ Z(r(\tau), \tau)] d\tau, 0 < t < T_0, \tag{2.10}$$

where γ_0 is a given positive constant.

PROPOSITION 2.2. *Problem II has at least one solution with T_0 depending on γ_0 and $\sup |\dot{r}|$. Moreover $\gamma \in C^1[0, T_0]$.*

PROOF: For any $T > 0$, we put

$$B(T) = \{\gamma \in C[0, T] : \|\gamma - \gamma_0\| \leq \frac{\gamma_0}{2}\}$$

and define $\mathcal{F}$ on $B(T)$ as follows:

$$(\mathcal{F}\gamma)(t) = \gamma_0 + \int_0^t \{\dot{r}(\tau) g(\gamma(\tau), r(\tau)) + Z(r(\tau), \tau)\} d\tau,$$

where $Z(x, t)$ is the unique continuous solution of (2.6)-(2.9) corresponding to the given function γ (the existence and uniqueness of Z can be proved as in [6]). We shall prove that $\mathcal{F}(B(T)) \subset B(T)$ for a convenient T.

Fix $T_1 > 0$ and let

$$\Omega_1 = \{(y, x) : \frac{\gamma_0}{2} < y < \frac{3\gamma_0}{2}, 0 < x < b + R_1 T_1\},$$

$$\Psi_1 = \sup_{0<t<T_1} |\Psi(t)|,$$

$$R_1 = \sup_{0<t<T_1} |\dot{r}(t)|$$

$$\eta_b = \sup_{0<x<b} |\eta(x)|,$$

$$G = \sup_{(y,x)\in\Omega_1} |g(y, x)|,$$

$$G_1 = \sup_{(y,x)\in\Omega_1} \left| \frac{g_x(y, x)}{g_y(y, x)} \right|.$$

In any $D_t(r), t \leq T_1$, we have

$$|Z(x,t)| \leq \max\{\Psi_1, \eta_b, \sup_{0<\tau<t} |Z(r(\tau),\tau)|\}. \tag{2.11}$$

In order to estimate $\sup |Z(r(\tau),\tau)|$ we assume that Z attains its positive maximum on $x = r(t)$ for some t_0. Then

$$0 < Z_x(r(t_0),t_0) < g_x(\gamma(t_0), r(t_0))\dot{r}(t_0)+$$

$$g_\gamma(\gamma(t_0), r(t_0))[\dot{r}(t_0)g(\gamma(t_0), r(t_0)) + Z(r(t_0),t_0)]. \tag{2.12}$$

Hence $0 < Z(r(t_0),t_0) < GR_1$.

In the same way, supposing that Z attains its negative minimum on $x = r(t)$ for some t_1, we show that

$$0 > Z(r(t_1),t_1) > -G_1R_1 - GR_1.$$

In both cases we have

$$0 < |Z(r(\tau),\tau)| < (G_1 + G)R_1 \tag{2.13}$$

in $D_t(r)$. Coming back to (2.11), we see that

$$0 < |Z(x,\tau)| \leq \max\{\Psi_1, \eta_b, (G_1 + G)R_1\} \equiv Z^*$$

in $D_t(r)$ for any $t \leq T_1$.

Therefore, we get

$$|(\mathcal{F}\gamma)(t) - \gamma_0| \leq t\{R_1G + Z^*\}.$$

If we choose $T_0 = \min\{T_1, \gamma_0/2(R_1G + Z^*)\}$, then $\mathcal{F}(B(T)) \subset B(T)$ for any $T \leq T_0$.

Because $B(T)$ is a closed convex subset of $C[0,T]$ and $\mathcal{F}(B(T)) \subset B(T)$ is precompact, using Schauder's Theorem (see [7], p. 189) we only need to show that $\mathcal{F}$ is continuous in C norm.

Split $\mathcal{F}$ into the sum of $\mathcal{F}_1$ and $\mathcal{F}_2$, where

$$(\mathcal{F}_1\gamma)(t) = \gamma_0 + \int_0^t \dot{r}(\tau)g(\gamma(\tau), r(\tau))d\tau,$$

$$(\mathcal{F}_2\gamma)(t) = \int_0^t Z(r(\tau), \tau)d\tau.$$

Then $\mathcal{F}_1$ is a Lipschitz continuous function because

$$\|\mathcal{F}_1\gamma_1 - \mathcal{F}_1\gamma_2\| \le TG'R_1\|\gamma_1 - \gamma_2\|,$$

where $G' = \sup_{(y,x)\in\Omega_1} |g_y(y,x)|$. Now we prove that $\mathcal{F}_2$ is continuous. For this it suffices to show that the application $\gamma \in B(T) \longrightarrow Z \in C(\bar{D}_T(r))$ is continuous.

Let $\gamma_1, \gamma_2 \in B(T)$ and Z_1, Z_2 be the corressponding solutions of (2.6)-(2.9). Put $W = Z_1 - Z_2$, then

$$W_{xx} - W_t = 0 \quad \text{in} \quad D_T(r),$$
$$W(0,t) = 0, \quad 0 < t < T,$$
$$W(x,0) = 0, \quad 0 < x < b (\text{if } b > 0),$$

$$W_x(r(t),t) + \{\dot{r}(t) - g_y(\gamma_1(t), r(t))\}W(r(t),t) =$$
$$\dot{r}(t)\{g_y(\gamma_1(t), r(t)) - g_y\gamma_2(t), r(t))g(\gamma_2(t), r(t))\}$$
$$+Z_2(r(t),t), \{g_y(\gamma_1(t), r(t)) - g_y(\gamma_2(t), r(t))\}$$
$$+\dot{r}(t)\{g_x(\gamma_1(t), r(t)) - g_x(\gamma_2(t), r(t))\}, 0 < t < T_0.$$

From the latter equation we get

$$|W(r(t),t)| < \frac{R_1}{R_T + H_0}\{|g_y(\gamma_1(t), r(t))g(\gamma_1(t), r(t)) -$$
$$-g_y(\gamma_2(t), r(t))g(\gamma_2(t), r(t))| + |g_x(\gamma_1(t), r(t)) - g_x(\gamma_2(t), r(t))|\} +$$
$$+\frac{Z^*}{R_T + H_0}|g_y(\gamma_1(t), r(t)) - g_y(\gamma_2(t), r(t))|, \ 0 < t < T_0,$$

where $0 < H_0 = \inf_{(y,x)\in\Omega_1} |g_y(y,x)|$, $R_T = \inf_{0<t<T} \dot{r}(t) > 0$. Now the continuity of the application $\gamma \longrightarrow Z$ follows from the uniform continuity of the functions $g_y, g_x, g_y g$ in Ω_1.

COROLLARY 2.2. *Let (T, s, c, C) be a solution of Problem I. Then $s \in C^2[0,T]$.*

PROOF: We consider Problem II and let $\Psi(t) = \dot{C}(t)$, $\eta(x) = h''(x), \gamma_0 = h(b), g(y,x) = -(q(x)+y)f(y,x), r(t) = s(t)$. It is clear that this choice of data verifies all assumptions of Problem II. By Proposition 2.2 there exists a solution (T_0, Z, γ) with $T_0 > 0$.

Define $v(x,t)$ as follows

$$v(x,t) = h(0) + x[\int_0^t Z_x(0,\tau)d\tau + h'(0)]$$

$$+ \int_0^t Z(0,\tau)d\tau + \int_0^x d\xi \int_0^\xi Z(y,t)dy. \tag{2.14}$$

Then v satisfies the following conditions:

$$v_{xx} - v_t = 0 \quad \text{in} \quad D_{T_0}, \tag{2.15}$$

$$v(0,t) = C(t), \quad 0 \le t \le T_0, \tag{2.16}$$

$$v(x,0) = h(x), \quad 0 \le x \le b, \tag{2.17}$$

$$v_x(s(t),t) = g(v(s(t),t), s(t)), \quad 0 \le t \le T_0, \tag{2.18}$$

$$\frac{d}{dt} v(s(t),t) = \dot{\gamma}(t), \quad 0 \le t \le T_0, \tag{2.19}$$

$$v_x(s(t),t) = g(v(s(t),t), s(t)), \quad 0 \le t \le 0. \tag{2.20}$$

(To get (2.18) we use Stoke's theorem by integrating the heat equation for Z in D_{T_0}). Since the solution of (2.15)-(2.17), (2.20) is unique (see e.g. [6]), $v(x,t) \equiv c(x,t)$ in $\bar{D}_{T_0}$. The continuity of s in $[0, T_0]$ follows from (2.19) and

$$\ddot{s}(t) = f_c(c(s(t),t), s(t)) \frac{d}{dt} c(s(t),t) + f_r(c(s(t),t), s(t))\dot{s}(t).$$

Finally, if $T_0 < T$, using Proposition 2.1 and Corollary 2.1 we can repeat the above argument, starting at T_0 with $\gamma_0 = c(s(T_0), T_0) > 0$ to reach a time T_1 and so on . A careful analysis of the bounds for $Z(x,t)$, (see (2.13)) along with the fact that $c(s(t), t)$ is strictly positive in $[0, T]$ shows that the sequence T_k cannot converge before T.

§3. Local existence

Now we shall prove that Problem I has a unique local solution in a suitable functional class. We define

$$v(x,t) = -e^{-x} \int_x^{s(t)} (c(y,t) + q(y))dy. \tag{3.1}$$

If (T,s,c,C) is a solution of Problem I, then

$$v_{xx} + 2v_x + v - v_t = e^{-x} q'(x) \quad \text{in} \quad D_T, \tag{3.2}$$

$$s(0) = b, \tag{3.3}$$

$$v(x,0) = -e^{-x} \int_x^{s(t)} (h(y) + q(y))dy, 0 < x < b, \tag{3.4}$$

$$v_x(0,t) = a \equiv h(0) + q(0) + \int_0^b (h(y) + q(y))dy, 0 < t < T, \tag{3.5}$$

$$v(s(t), t) = 0, 0 < t < T, \tag{3.6}$$

$$\dot{s}(t) \qquad = f(e^{s(t)} v_x(s(t), t) - q(s(t)), s(t)), 0 < t < T. \tag{3.7}$$

Note that (3.2)-(3.7) is a Stefan-type free boundary problem with nonlinear Stefan condition. In [3], a general theorem for local existence and uniqueness of such a problem has been proved with $s \in H_{1+\beta}[0, T], \beta < \frac{1}{2}$. However, our problem does not fit completely the assumptions of [3] because f may be not Lipschitz continuous at the origin.

We can bypass this difficulty by replacing f by $\tilde{f}$ as follows:

$$\tilde{f}(y,x)=\begin{cases} f(y,x), & 0\le y\le h(0), x\ge 0,\\ \bar{f}(y,x), & y\in(-\infty,0)\cup(h(0),+\infty), x\ge 0,\end{cases}$$

where $\bar{f}$ is any C^1 function such that $\bar{f}(0,x)=0 \quad \forall x\ge 0$, $\bar{f}$ is uniformly Lipschitz continuous in $\mathbb{R}$ with respect to y for $x\ge 0$, and $\bar{f}$ is a C^1 continuation of f outside $[0,h(0)]\times[0,+\infty)$.

We can define a new problem with $\tilde{f}$ instead of f, i.e.

$$\dot{s}(t)=\tilde{f}(e^{s(t)}v_x(s(t),t)-q(s(t)),s(t)). \tag{3.7'}$$

From Theorem 1 of [3] we get the following

PROPOSITION 3.1. *Problem (3.2)-(3.6), (3.7') has a unique solution (T,s,v) with* $s\in H_{1+\alpha}[0,T]$ *and* $v\in C_{1+2\alpha}(\bar{D}_T)$ *for any* $\alpha(0<\alpha<\frac{1}{2})$.

PROPOSITION 3.2. *Problem I with $\tilde{f}$ instead of f has a unique solution which coincides with the solution of the original Problem I.*

PROOF: Define $\bar{c}(x,t)=e^{v(x,t)}(v_x(x,b)+v(x,t)-q(x))$. Then (1.7)-(1.13) are satisfied with $\bar{C}(t)=v_x(0,t)+v(0,t)-q(0)$. Notice that we also have the following estimates for $\tilde{f}$:

$$0=\tilde{f}(0,r)\le\tilde{f}(\bar{c}(x,t),r)\le\tilde{f}(h(0),r),\quad \forall r\ge 0.$$

The uniqueness of the solution implies that $\bar{c}\equiv c$ in D_T. Finally the regularisation result of Corollary 2.2 implies that the solution is unique in the class $s\in C^2[0,T]$.

Now we can summarize the obtained results as follows.

THEOREM 3.1. *Problem I admits a unique solution (T,s,c,C) with* $s\in C^2[0,T]$.

§4. Uniqueness

First we shall prove the following:

LEMMA 4.1. *Let $(T_1, s_1, v_1), (T_2, s_2, v_2)$ be two solutions of the following equations*

$$v_{ixx} - v_{it} = q'(x) \quad in \quad D_{T_i}, \tag{4.1}$$

$$s_i(0) = b_i, \tag{4.2}$$

$$v_i(x,0) = h_i(x), \quad 0 < x < b_i, \tag{4.3}$$

$$v_{ix}(0,t) = v_i(0,t) + a_i(t), \quad 0 < t < T_i, \tag{4.4}$$

$$v_i(s_i(t),t) = 0, \quad 0 < t < T_i, \tag{4.5}$$

$$v_{ix}(s_i(t),t) = \mu_i(\dot{s}_i(t), s_i(t)), 0 < t < T_i, \tag{4.6}$$

where $D_{T_i} = \{(x,t) : 0 < x < s_i(t), 0 < t < T_i\}$. Suppose that $b_1 > b_2$,

$$a_1(t) \geq a_2(t), a_1(t) \geq 0, 0 < t < \min\{T_1, T_2\},$$

$$h_1(x) \leq 0, 0 < x < b_1, h_1(x) \leq h_2(x), 0 < x < b_2,$$

where at least one of the above inequalities is not the identity and

$$\mu_1(y_1, x) - \mu_2(y, x) \leq 0 \quad if \quad y_1 \leq y_2, \forall x \geq 0.$$

Then $s_1(t) > s_2(t), \quad 0 \leq t \leq \min\{T_1, T_2\}$.

PROOF: Suppose that there exists $\bar{t} = \min\{t | s_1(t) = s_2(t)\}$. Of course, $\bar{t} > 0$. Put $W((x,t) = v_1(x,t) - v_2(x,t)$ in $D = \{(x,t) : 0 < x < s_2(t), 0 < t < \bar{t}\}$. We have

$$W_{xx} - W_t = 0 \quad \text{in} \quad D,$$

$$W_x(0,t) - W(0,t) = a_1(t) - a_2(t) \geq 0, 0 < t < \bar{t},$$

$$W(s_2(t),t) = v_1(s_2(t),t) < 0, 0 < t < \bar{t},$$

$$W(x,0) = h_1(x) - h_2(x) \leq 0, 0 < x < b_2.$$

Hence W cannot attain its positive maximum on $x = 0$. This implies that $(s_2(\bar{t}), \bar{t})$ is a maximum point of W in $\bar{D}$. Using the strong maximum principle we get $W_x(s_2(\bar{t}), \bar{t}) > 0$. But $W_x(s_2(\bar{t}), \bar{t}) = \mu_1(\dot{s}_1(\bar{t}), s_1(\bar{t})) - \mu_2(\dot{s}_2(\bar{t}), s_2(\bar{t})) \leq 0$ (because $s_1(\bar{t}) = s_2(\bar{t})$ and $\dot{s}_1(\bar{t}) \leq \dot{s}_2(\bar{t})$), a contradiction.

THEOREM 4.1. *Problem I has at most one solution.*

PROOF: Let (T_1, s_1, c_1, C_1) and (T_2, s_2, c_2, C_2) be two maximal solutions of Problem I, $T_2 \leq T_1 \leq +\infty$. We define

$$u_i(x,t) = -\int_x^{s(t)} (c_i(y,t) + q(y))dy, \quad i = 1, 2.$$

Then (T_i, s_i, u_i) solves (4.1)-(4.6) with $b_i = 0, a_i = h(0) + q(0)$, and $\mu_i(y,x) = \Phi(y,x) + q(x)$. For any $\epsilon > 0$, we define

$$s_\epsilon(t) = s_2(t-\epsilon), \quad u_\epsilon(x,t) = u_2(x, t-\epsilon),$$

$$s_{-\epsilon}(t) = s_2(t+\epsilon), \quad u_{-\epsilon}(x,t) = u_2(x, t+\epsilon).$$

Notice that $(T_2 + \epsilon, s_\epsilon, u_\epsilon)$ and $(T_2 - \epsilon, s_{-\epsilon}, u_{-\epsilon})$ are solutions corresponding to the initial data $s_\epsilon(\epsilon) = 0$ and $s_{-\epsilon}(-\epsilon) = 0$. Since Φ is a monotone increasing function, we can apply Lemma 3.1 to $(s_1, u_1), (s_\epsilon, u_\epsilon)$ and $(s_1, u_1), (s_{-\epsilon}, u_{-\epsilon})$ and get

$$s_\epsilon(t) < s_1(t), \quad \epsilon \leq t \leq T_2,$$

$$s_1(t) < s_{-\epsilon}(t), \quad 0 \leq t \leq T_2 - \epsilon.$$

Letting ϵ tend to 0 , from the uniform continuity of s_i (remember that $0 < \dot{s}_i(t) \leq F$) we obtain $s_1(t) \equiv s_2(t), 0 \leq t \leq T_2$. Because of the hypothesis on maximality it follows that $T_1 \equiv T_2$.

REFERENCES

[1]. D. Andreucci and R. Ricci, *A free boundary problem arising from sorption of solvents in glassy polymer*, Quart. Appl. Math. **44** (1987), 649–657.

[2]. G. Astarita and G. C. Sarti, *A class of mathematical models for sorption of swelling solvents in glassy polymers*, Polym. Eng. Sci. **18** (1978), 388–395.

[3]. E. Comparini, *On a class of Non-linear free boundary problems*, Fis. Mat. Suppl., B.U.M.I **2** (1983), 187–202.

[4]. E. Comparini and R. Ricci, *On the swelling of a glassy polymer in contact with a well-stirred solvent*, Math. Meth. Appl. Sci. **7** (1985), 238–250.

[5]. A. Fasano , G. Meyer and M. Primicerio, *On a problem in the polymer industry : theoretical and numerical investigation*, SIAM J. Math. Anal. **17** (1986), 946–960.

[6]. A. Fasano and M. Primicerio, *Su un problema undimensionale di diffusione in un mezzo a contorno mobile con condizioni ai limiti non lineari*, Ann. Mat. Pura Appl. **43** (1972), 333–357..

[7]. A. Friedman, *Partial differential equations of parabolic type, Prentice Hall, 1964.*

[8]. Phan Huu San and Nguyen Dinh Tri, *On a problem of penetration of solvent into polymer*, Internal report, International Centre For Theoretical Physics, Trieste (1990).

DEPARTMENT OF MATHEMATICS

HANOI POLYTECHNICAL UNIVERSITY - VIETNAM.