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ON THE CONTROLLABILITY OF SINGULAR SYSTEMS IN BANACH SPACES

VU NGOC PHAT AND K. BALACHANDRAN

Abstract. The paper presents necessary and suficient conditions for global controllability to a subset of singular linear discrete-time systems with constrained controls in Banach spaces. The main tool of the proofs is based on extension of the surjectivity theorem to set-valued functions.

1. Introduction

Singular systems (or descriptor systems or generate state systems) have found wide applications in systems and control theory, signal processing communication and other areas [1,3]. With the development of digital computers, electrical network, and analysis of difference systems, controllability of singular discrete-time systems has attracted the attention of many authors [4,6,7]. Constrained controllability of normal discrete-time systems has been studied in [2,9-11,14]. The aim of this paper is to give the corresponding development of the controllability of abstract singular system

$$Ex(k+1) = Ax(k) + Bu(k) \quad , \quad k = 0, 1, \dots \tag{1}$$

where E is a singular operator.

Let M, Ω be nonempty subsets. Descriptor system (1) is said to be globally controllable to M (GC_M) if for every state $x_0 \in X$ there exist a number $K > 0$ and controls $u(k) \in \Omega$, $k = 0, 1, \dots, K-1$ such that the solution $x(k)$ of system (1) satisfies $x(0) = x_0, x(K) \in M$. We may notice that in several papers devoted the controllability to a subset, an essential assumption

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is that the target set M is A-invariant, i.e. $AM \subset M$, and a common tool of the proof is based on the Banach open mapping theorem. In addition, in the case $M \neq \{0\}$ we should concern with set-valued operators constructing the controllable set of the given control system. In this paper, not making the above assumption on M, we shall use some extensions of the Banach open mapping theorem to set-valued cases obtained in [8,12] to derive controllability conditions for singular system (1) with constrained controls in Banach spaces.

2. Preliminaries

Let X, U be infinite-dimensional Banach spaces, X^* be the dual topological space of X. By $< x^*, x >$ we denote the value of x^* at $x \in X$. The interior, the closure, the linear hull of a set M are denoted by intM, clM, spM, respectively. By B_x, B_u we denote the open unit balls in X, U. Let M^+ denote the dual positive cone of a set M at $0 \in M$ defined by

$$M^+ = \{x^* \in X^* :< x^*, x >\geq 0 \quad \forall x \in M\}.$$

For each $k > 0$, M^k denotes the set of all elements $x^k = (x_1, x_2, \ldots, x_k)$ with $x_i \in M$ and X^k stands for a Banach space with the norm

$$||x^k|| = ||x_1|| + ||x_2|| + \cdots + ||x_k||.$$

Moreover, we denote by riM the relative interior of a set M, i.e. the interior relative to the subspace generated by M, $W = \text{sp } M$.

To any set-valued function $T : U \to X$ we associate the image, the graph of T by

$$\text{Im} \quad T = \{Tu, u \in domT\},$$

$$\text{gr} \quad T = \{(u, x) \quad x \in Tu\}.$$

DEFINITION 2.1. A set valued function T is called convex, closed and odd if grT is convex, closed and $grT = -grT$.

For example, $Tx = Ax + M$, where A is a linear bounded operator, M is a convex, closed and odd set or closed subspace, is a convex, closed and odd set-valued function.

The following lemma is recalled from the author's report [8] which can be considered as a set-valued version of the Banach open mapping theorem [13]. Detailed explanations at each stage of the proof given below may be often omitted, for which we refer the reader to [8] .

LEMMA 2.1. *Let $T : U \to X$ be a convex, closed and odd set-valued function. If ImT is a set of the second category, then $0 \in intImT$.*

PROOF: : Let V be an arbitary neighborhood of zero in U. Let

$$V_n = r/2^n B_u \quad , n = 0, 1, \dots$$

where r is chosen so that $V_0 \subset V$.

Since T (.) is convex, odd, for every $n \geq 1$ we have

$$clT(V_n \supseteq clT(1/2(V_{n+1} - V_{n+1}) \supseteq 1/2clT(V_{n+1}) - 1/2clT(V_{n+1}). \quad (2)$$

Let $n \geq 1$ be some fixed number. Since

$$U = \bigcup_{k=1}^{\infty} kV_{n+1},$$

then

$$T(U) = \bigcup_{k=1}^{\infty} T(kV_{n+1}).$$

By the Baire theorem of categories, there is a number $K \geq 1$ such that $intclT(KV_{n+1}) \neq \emptyset$. By the convexity of T and $0 \in T(0)$, it follows that for every $k \geq 1$

$$T(kV_{n+1}) \subseteq kT(V_{n+1})$$

which implies

$$\text{int} \quad clT(V_{n+1}) \neq \emptyset.$$

In virtue of (2) we conclude

$$0 \in \quad \text{int} \quad clT(V_n).$$

This means that there is a sequence of positive numbers $\beta_n \to 0$ such that

$$\beta_n B_x \subseteq cl(1/2^n T(V_n)). \tag{3}$$

To prove the lemma it suffices to show that

$$1/2clT(V_1) \subset T(V).$$

Indeed, let $y_1 \in 1/2clT(V_1)$. From (3) it follows that there exist sequences y_n, u_n, z_n such that

$$y_{n+1} = y_n - z_n, y_{n+1} \in 1/2^{n+1}clT(u_{n+1}).$$

and $||y_n|| < \beta_n$.

Therefore, $y_n \to 0$ and the sequence $\sum_{k=1}^{n} 1/2^k x_k$ is a Cauchy sequence which converges to some $u_0 \in V$. On the other hand, clearly

$$\sum_{k=1}^{n} z_k \in \sum_{k=1}^{n} 1/2^k T(x_k).$$

By the convexity of T, we have

$$\sum_{k=1}^{n} 1/2^k T(x_k) \subseteq T(\sum_{k=1}^{n} 1/2^k x_k).$$

Since $y_1 = \lim_{n\to\infty} \sum_{k=1}^{n} z_k$ and by the closedness of T we obtain

$$y_1 \in T(u_0) \subset T(V),$$

which completes the proof.

We conclude this section with the Robinson surjectivity theorem given in [12] and the Krein-Rutman theorem, whose proof was improved in [5].

LEMMA2.2. (Robinson's surjectivity theorem). *Let $T : U \to X$ be a convex, closed set-valued function. Let $x_0 \in intT(U)$. Then for every $(x_0, u_0) \in grT$,* *there is a number* $\beta > 0$ *such that for every* $\epsilon \in [0,1]$

$$x_0 + \epsilon\beta B_x \subset T(u_0 + \epsilon B_u).$$

LEMMA 2.3. (Krein-Rutman theorem). *Let* K *be a convex cone,* $intK \neq \emptyset$ *and* $K \neq X$. *Let* A *be a linear bounded operator in* X. *If* K *is* A*-invariant, then* A^* *has an eigenvector in* K^+ *with a nonnegative eigenvalue.*

3. Controllability results

Throughout this section we always assume that Ω is a convex closed cone and $ri\Omega \neq \emptyset$. We first consider the singular system(1) with $A = I_x$, the identity operator in X :

$$\begin{cases} Ex(k+1) & = x(k) + Bu(k), \quad k = 0,1,... \\ u(k) & \in \Omega \subset U. \end{cases} \tag{4}$$

By direct computation, for every initial state $x_0 \in X$ and controls $u(k) \in \Omega$, the state $x(k)$ of system (1) is given by

$$E^k x(k) = x_0 + \sum_{i=0}^{k-1} E^i Bu(i). \tag{5}$$

THEOREM 3.1. *Assume that* $E^k(M)$ *is closed,* $k = 1, 2, \ldots$ *Let* M *be a closed subspace in* X. *Singular system (4) is* GC_M *if and only if*

i) $sp\ \{ BW, EBW, \ldots, E^{k-1}BW, E^k(M) \} = X\ ,\ W = sp\ \Omega$ (6)

ii) E^* *has no eigenvector in* $E(M)^+ \cap (-B\Omega)^+$ *with a nonnegative eigenvalue.*

PROOF: *Necessity.* We define the following convex, closed, odd set-valued function

$$T_k u^k : W^k \to X,$$

by

$$T_k u^k = -F_k u^k + E^k(M),$$

where

$$F_k u^k = \sum_{i=0}^{k-1} E^i B u(i).$$

It is easily verified that T_k is convex, closed and odd and

$$C_4 = \bigcup_{k=1}^{\infty} T_k(W^K) = X,$$

where C_4 denotes the controllable to M set of system (4) with unconstrained controls $u(k) \in W$. By the Baire theorem of categories, there is a number $k \geq 1$ such that $T_k(W^k)$ is of the second category. In virtue of Lemma 2.1. , we have

$$0 \in \quad \text{int} \quad T_k(W^k).$$

Since $T_k(W^k)$ is a linear subspace, it follows that the first condition i) is satisfied. To prove ii) we assume the contrary. Let $0 \neq x^* \in (EM)^+ \cap (-B\Omega)^+$ be an eigenvector of E^* with an eigenvalue $\beta \geq 0$.

For every $x_0 \in X$, there exist a number $K > 0$ and controls $u(0), u(1), \ldots, u(K-1) \in \Omega$ such that $x(K) \in M$. From (5) it follows that

$$< x^*, E^K x(K) > = < x^*, x_0 > + \sum_{i=0}^{K-1} < x^*, E^i B u(i) >,$$

or

$$< x^*, x_0 > = \beta^{K-1} < x^*, Ex(K) > - \sum_{i=0}^{K-1} \beta^i < x^*, Bu(i) > .$$

Then, $< x^*, x_0 > \geq 0$ for any $x_0 \in X$ which contradicts the condition $x^* \neq 0$.

Sufficiency . From i) it follows that system (4) with $u(k) \in W$ is GC_M after K steps. Then

$$T_K(W^K) = X.$$

Since T_K is convex, closed and $ri\Omega^K \neq \emptyset$, by Lemma 2.2 we claim

$$intT_K(\Omega^K) \neq \emptyset.$$

Indeed, let $u_0^K \in int\Omega^K$ (interior relative to W^K). For some $\beta_1 > 0$

$$u_0^K + \beta_1 B_{W^K} \subset \Omega^K.$$

Let $x_0 \in T_K(u_0^K)$. Since $x_0 \in T_K(W^K)$, by Lemma 2.2, there is a number $\delta > 0$ such that for every $\epsilon \in [0,1]$

$$x_0 + \epsilon\delta B_x \subset T_K(u_0^K + \epsilon B_{W^K}).$$

Taking $\beta_2 \in [0,1], \beta_2 < \beta_1$, we have

$$x_0 + \beta_2\delta B_x \subset T_K(u_0^K + \beta_2 B_{W^K}) \subset T_K(\Omega^K).$$

This implies that

$$\text{int} \quad T_K(\Omega^K) = \quad \text{int} \quad C_{4,K} \neq \emptyset,$$

or

$$\text{int} \quad C_4 \neq \emptyset.$$

On the other hand, by simple computation we can see that C_4 is E-invariant, i.e.

$$EC_4 \subset C_4.$$

Since C_4 is convex, by Lemma 2.3, if $C_4 \neq X$, then E^* has an eigenvector in C_4^+ with a nonnegative eigenvalue. It is easily seen from the definition that

$$E(M) \subset C_4 \quad , \quad -B\Omega \subset C_4.$$

Therefore, E^* has an eigenvector in $(EM)^+ \cap (-B\Omega)^+$ which contradicts the second condition ii). Thus, $C_4 = X$, i.e. the system (4) is GC_M.

REMARK 3.1: In the case M is an arbitrary convex closed odd subset, not necessarily subspace, Theorem 3.1 still holds where the condition (6) is replaced by

$$0 \in \quad \text{int} \quad sp\{BW, ..., E^{K-1}BW, E^K(M)\}. \tag{7}$$

In this case, as we have already seen in the preceding proof, we arrived at the fact that

$$0 \in \quad \text{int} \quad T_K(W^K).$$

Since Ω is a convex cone and T_K is convex, we complete the proof by the same way given above.

REMARK 3.2: It is easy to see that if $M = \{0\}$, then the global controllability of system (4) follows immediately from the global reachability of the following normal system

$$\begin{cases} x(k+1) = Ex(k) - Bu(k), \\ u(k) \in \Omega. \end{cases}$$

However, in the case $M \neq \{0\}$ this fact, in general, is not true.

We are now considering singular system (1), where $A \neq I_x$. We shall say that operator A is semi-invertible if

$$\exists P \quad : \quad AP = I_x \text{ or } PA = I_x.$$

By left (or right) multiplying both sides of system (1) with P we shall lead system (1) to system (4) and then applying Theorem 3.1 we can prove the following

THEOREM 3.2. *Assume that A is semi-invertible and $E^k(M)$ is closed for every $k \geq 1$. System (1) is GC_M if and only if conditions i) and ii) of Theorem 3.1 hold (or conditions (7) and ii) hold for the case M is a convex closed odd set), where E is replaced by PE (or EP), B is replaced by PB (or BP).*

4. Example

Consider the following singular system in Hilbert space l_2 of the form

$$Ex(k+1) = Ax(k) + u(k) \tag{8}$$

where

$$E : (\beta_1; \beta_2, \ldots) \to (\beta_2, \beta_3, \ldots),$$

$$A : (\beta_1, \beta_2, ...) \to (\beta_1, 0, \beta_2, 0, ...),$$

$$\Omega = \{(\beta_1, 0, \beta_3, \beta_4, ...) \in l_2 : \beta_i \leq 0\},$$

$$M = \{\beta = (\beta_1, \beta_2, ...) \in l_2 : ||\beta|| \leq 1, \beta_{2i} = 0, \forall i\}.$$

Note that A is right invertible, i.e. $PA = I_x$, where

$$P : (\beta_1, \beta_2, ...) \to (\beta_1, \beta_3, \beta_5, ...).$$

We have

$$E^* : (\beta_1, \beta_2, ...) \to (0, \beta_1, \beta_2, ...).$$

Then E^* has no eigenvectors. A simple computation shows that

$$(PE)^k : (\beta_1, \beta_2, ...) \to (\beta_{2^k}, \beta_{2.2^k}, \beta_{3.2^k}, ...).$$

Hence, for every $m \in M$, we have

$$(PE)^k m = (0, 0, ...) \in l_2.$$

Therefore, we can easily verify that the conditions of Theorem 3.2 are satisfied, and system (8) is GC_M.

5. Concluding remarks

In this paper we have established necessary and sufficient conditions for global controllability to a target set of linear singular discrete- time systems in Banach spaces. We have assumed that the operator A is semi-invertible. The key to global controllability for singular systems is reduced to the global controllability of normal "identity" system (4). The investigation is based on multivalued convex techniques. The concept developed in this paper thus gives us a wealth of open problems of global controllability of singular system (1), where the operator A is an arbitrary linear operator.

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Institute of Mathematics
P.O.Box 631, Bo Ho
Hanoi, Vietnam.

Department of Mathematics
Bharathiar University, Coimbatore,
641046, Tamil Nadu, India.