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THE CONVECTIVE MOTION OF AN INCOMPRESSIBLE VISCOUS FLUID CONTAINED IN AN ELASTIC CYLINDRICAL SHELL

NGO HUY CAN AND TRAN THU HA

I. Introduction.

The problem of convection in viscous fluid contained in a rigid vessel has received much attention in recent years. For the works on this topic we refer to Gershuni G.Z., Zhukhovitskii E.M. [4] and Busse. F.N., N. Riahi [2,3].

Mathematical study of motion of viscous fluid contained in either a rigid or elastic vessel was done by several authors [5, 9, 12, 13, 14, 15].

In this paper we consider the convective motion of incompressible viscous fluid contained in an elastic cylindrical shell.

Let us have a cylindrical shell with the elastic cylindrical part Σ, the lower rigid base S_0 and the rigid upper base S_1. The small motion of the system of elastic shell with heated fluid is described by the following equations (see [5, 12, 13, 15]):

$$-\nu\Delta v + \nabla p - \lambda v - g\beta T k_3 = 0, \quad div v = 0 \qquad \text{in } \Omega, \quad (1.1)$$

$$-\varkappa\Delta T - bv_3 - \lambda T = 0 \qquad \text{in } \Omega, \quad (1.2)$$

$$Eh(1-\mu^2)^{-1}Lu + \rho h\lambda^2 u + \tfrac{1}{2}\tau(v)n + l_1T_1 + l_2T_2 = 0 \qquad \text{on } \Sigma, \quad (1.3)$$

$$h^2\Delta_1 T_1 - \mu_1 T_1 - \mu^* T_2 + \tfrac{h^2}{a}\lambda T_1 = -\tfrac{1}{2}(\mu_1 - \mu_2)\gamma_2 T \qquad \text{on } \Sigma, \quad (1.4)$$

$$h^2\Delta_1 T_2 - 3(\mu_1+1)T_2 - 3\mu^* T_1 + \tfrac{h^2}{a}\lambda T_2 = -3(\mu_1 - \mu_2)\gamma_2 T \qquad \text{on } \Sigma, \quad (1.5)$$

$$T=0,\ v=0 \qquad \text{on } S \cup S_0, \quad (1.6)$$

$$\tfrac{\partial T}{\partial n} = \varkappa_1 T_2/\varkappa h \qquad \text{on } \Sigma, \quad (1.7)$$

$$T_i = 0 (i=1,2) \qquad \text{on } \partial\Sigma, \quad (1.8)$$

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$$\lambda u = v \qquad \text{on } \Sigma, \quad (1.9)$$

$$u = 0, \frac{\partial u}{\partial n} = 0 \qquad \text{on } \Sigma. \quad (1.10)$$

Here the following notations are used:

$\Delta = \frac{\partial^2}{\partial x_1{}^2} + \frac{\partial^2}{\partial x_2{}^2} + \frac{\partial^2}{\partial x_3{}^2}, \Delta_1 = \frac{\partial^2}{\partial \alpha^2} + \frac{1}{R^2}\frac{\partial^2}{\partial \varphi^2}, \tau(v) = (\tau_{ik})^3_{i,k=1},$

$\tau_{ik} = -p\delta_{ik} + \nu(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}),$

$l_1 T_1 = \frac{\beta_1 Eh}{1-\mu}(\frac{\partial}{\partial \alpha}, \frac{1}{R}\frac{\partial}{\partial \varphi}, -\frac{1}{R})T_1, \gamma_2 T = T|_\Sigma,$

$l_2 T_2 = \frac{\beta_1 Eh}{1-\mu}(0, \frac{h}{3R}\frac{\partial}{\partial \varphi}, \frac{h}{3}\Delta_1)T_2,$

$Lu = \sum_{j=1}^3 (\frac{h^2}{12} n_{ij} + l_{ij})u_j, i = \overline{1,3}$

where

$l_{11}u_1 = -\frac{\partial^2}{\partial \alpha^2}u_1 - \frac{1-\mu}{2R^2}\frac{\partial^2}{\partial \beta^2}u_1 - (1-\mu)\frac{u_1}{R_1 R_2},$

$l_{12}u_2 = -\frac{1}{R}\frac{\partial^2}{\partial \alpha \partial \beta}u_2 + \frac{1-\mu}{2R}\frac{\partial^2}{\partial \alpha \partial \beta}u_2,$

$l_{13}u_3 = \frac{1}{R}\frac{\partial u_3}{\partial \alpha} - \frac{1-\mu}{R}\frac{\partial u_3}{\partial \alpha},$

$l_{21}u_1 = -\frac{1}{R}\frac{\partial^2 u_1}{\partial \alpha \partial \beta} + \frac{1-\mu}{2R}\frac{\partial^2 u_1}{\partial \alpha \partial \beta},$

$l_{22}u_2 = -\frac{1}{R^2}\frac{\partial^2 u_2}{\partial \alpha \partial \beta} - \frac{1-\mu}{2}\frac{\partial^2 u_2}{\partial \alpha^2},$

$l_{23}u_3 = \frac{1}{R^2}\frac{\partial u_3}{\partial \beta},$

$l_{31}u_1 = -\frac{1}{R}\frac{\partial u_1}{\partial \alpha} + \frac{1-\mu}{R}\frac{\partial u_1}{\partial \alpha},$

$l_{32}u_2 = -\frac{1}{R^2}\frac{\partial u_2}{\partial \beta},$

$l_{33}u_3 = \frac{1}{R^2}u_3,$

$n_{33}u_3 = (\frac{\partial^2}{\partial \alpha^2} + \frac{1}{R^2}\frac{\partial^2}{\partial \beta^2})(\frac{\partial^2}{\partial \alpha^2} + \frac{1}{R^2}\frac{\partial^2}{\partial \beta^2})u_3,$

$n_{ij} = 0$ if $i + j < 6$.

Here $v = (v_1, v_2, v_3)$ is the fluid velocity, p – the pressure, T – the temperature in the fluid, ν – the coefficient of kinematic viscosity, k_3 – the unit vector of vertical upward axis Ox_3 in the Cartesian system coordinates $X = (x_1, x_2, x_3)$, β – the coefficient of volum expansion, bk_3 – the gradient of equilibrium state temperature in the fluid, $\varkappa$ – the coefficient of heat conductivity of the fluid, E – the modul of elasticity, μ – the Poisson coefficient $(0 < \mu < \frac{1}{2}), 2h$ – the thickness of elastic cylindrical part Σ, (α, φ) orthogonal curvilinear coordinate system on Σ, R – the radius of cylinder, T_1 and T_2 – the characteristic temperatures of the shell, g – the accelerate of gravity, $u = (u_1, u_2, u_3)$ – the displacement of the shell, $\varkappa_1$ – the coefficient of heat conductivity of the shell, β_1 – the coefficient of thermal expansion of the shell, μ^+ – the coefficient of heat transfer between the surface of the elastic shell and the exterior medium,

μ^- – the coefficient of heat transfer between the surface of the elastic shell and the fluid , $\mu_{1,2} = \pm\frac{h}{2}(\mu^+ - \mu^-), \mu^* = \mu_2 - \frac{h}{2R}$, a – the coefficient of thermal conductivity of the shell, ρ – the density of the shell.

We will study the spectrum structure of the problem (1.1)-(1.10).

2. Function spaces and auxiliary problems

The following Hilbert spaces are used throughout

$$L_2(\Sigma) = H_2(\Sigma) \times H_2(\Sigma) \times H_2(\Sigma).$$

The scalar product and the norm are given by

$$(u,v)_{L_2(\Sigma)} = \sum_{j=1}^{3} R \int_\Sigma u_j v_j d\Sigma,$$

$$\|u\|_{L_2(\Sigma)} = [(u,u)_{L_2(\Sigma)}]^{\frac{1}{2}}.$$

The Sobolev spaces $H_2^1(\Sigma), W_2^1(\Sigma)$ are defined as follows:

$$H_2^1(\Sigma) = \{T \in H_2(\Sigma), \nabla T \in L_2(\Sigma)\},$$

$$W_2^1(\Sigma) = H_2^1(\Sigma) \times H_2^1(\Sigma) \times H_2^1(\Sigma),$$

$$(T_1,T_2)_{H_2^1(\Sigma)} = h^2 \int_\Sigma gradT_1 gradT_2 d\Sigma + R\int_{\partial\Sigma} T_1 T_2 d\partial\Sigma,$$

$$\|T_1\|_{H_2^1(\Sigma)} = [(T_1,T_1)_{H_2^1(\Sigma)}]^{\frac{1}{2}}.$$

We define $H_{2,0}^1(\Sigma)$ as the closure of the set of smooth functions vanishing outside some compact subsets of Σ with respect to $\|.\|_{H_2^1(\Sigma)}$ and $W_{2,0}^1(\Sigma)$ as the closure of the set of smooth vector fields vanishing outside some compact subsets of Σ with respect to $\|.\|_{W_2^1(\Sigma)}$.

The Sobolev space $H_2^2(\Sigma)$ is the space of functions in $H_2(\Sigma)$ with derivatives of order less than or equal to 2 in $H_2(\Sigma)$.

The Hilbert space $L_2(\Omega)$ is defined as follows:

$L_2(\Omega) = H_2(\Omega) \times H_2(\Omega) \times H_2(\Omega)$,

$(u,v)_{L_2(\Omega)} = \sum_{j=1}^{3} \int_\Omega u_j v_j d\Omega \|v\|_{L_2(\Omega)} = [(v,v)_{L_2(\Omega)}]^{\frac{1}{2}}$.

It is known that [8, 10, 19]

$$L_2(\Sigma) = \tilde{L}_2(\Omega) \oplus G(\Omega),$$

where $\tilde{L}_2(\Omega) = \{v \in L_2(\Omega), div\ v = 0 \quad \text{in } \Omega, v_n = 0 \quad \text{on } S_0 \cup S_1\}$

$$G(\Omega) = \{v \in L_2(\Omega), v = \nabla p \quad \text{in} \quad \Omega, p = 0 \quad \text{on} \quad \Sigma\}.$$

The Sobolev spaces $H_2^1(\Omega), W_2^1(\Omega)$ are defined as follows

$$(p,q)_{H_2^1(\Omega)} = \varkappa \int_\Omega \nabla p \nabla q d\Omega + \int_{S_0 \cup S_1} pqd(S_0 \cup S_1),$$

$$\|p\|_{H_2^1(\Omega)} = [(p,p)_{H_2^1(\Omega)}]^{\frac{1}{2}},$$

$$(v,w)_{W_2^1(\Omega)} = \sum_{i=1}^{3} \nu \int_\Omega \nabla v_{x_i} \nabla w_{x_i} d\Omega + \int_{S_0 \cup S_1} vwd(S_0 \cup S_1),$$

$$\|v\|_{W_2^1(\Omega)} = [(v,v)_{W_2^1(\Omega)}]^{\frac{1}{2}}.$$

The space $H_{2,0}^1(\Omega)$ is the closure of the set of smooth functions vanishing outside some compact subsets of Ω with respect to $\|.\|_{H_2^1(\Omega)}$ and the space $W_{2,0}^1(\Omega)$ is the closure of the set of smooth solenoidal vector fields vanishing outside some compact subsets of Ω with respect to $\|.\|_{W_2^1(\Omega)}$.

Let $H_2^{-\frac{1}{2}}(\Sigma)$ be the dual space of $H_2^{\frac{1}{2}}(\Sigma)$. We define the spaces H_0, H_+, H_-, K as follows:

$$H_0 = H_2(\Sigma) \ominus \{1\}, H_+ = H_0 \cap H_2^{\frac{1}{2}}(\Sigma), H_- = H_0 \cap H_2^{-\frac{1}{2}}(\Sigma),$$

$$K = L_2(\Sigma) \times \tilde{L}_2(\Omega) \times H_2(\Omega) \times H_2(\Sigma) \times H_2(\Sigma).$$

The operator P_1 is the projection from the space K into the space $L_2(\Sigma)$ and $P_2 = I_5 - P_1$, where

$$I_5 = \begin{pmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{pmatrix}$$

We consider the following auxilliary problems:

PROBLEM 1. *Given a vector-function $f \in L_2(\Sigma)$, is there a vector-function u so that the following equation and conditions are satisfied :*

$$Eh(1-\mu^2)^{-1}Lu + u = f \quad in \quad \Sigma,$$

$$u = 0, \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Sigma \ ?$$

PROBLEM 2. *Given a vector-function $g \in \tilde{L}_2(\Omega)$, are there a vector function $v^{(1)}$ and a function $p^{(1)}$ so that the following equations and conditions are satisfied:*

$$-\nu\Delta v^{(1)} + \nabla p^{(1)} = g \quad divv^{(1)} = 0 \quad in \quad \Omega,$$

$$\tau(v^{(1)})_\Sigma.n = 0 \quad on \quad \Sigma, v^{(1)} = 0 \quad on \quad S_0 \cup S_1 \ ?$$

PROBLEM 3. *Given a function $\psi_1 \in H_-(\Sigma)$, are there a vector function $v^{(2)}$ and a function $p^{(2)}$ so that the following equations and conditions are satisfied:*

$$-\nu\Delta v^{(2)} + \nabla p^{(2)} = 0, div\ v^{(2)} = 0 \quad in \quad \Omega,$$

$$\tau(v^{(2)})_\Sigma.n = \psi_1 \quad on \quad \Sigma, v^{(2)} = 0 \quad on \quad S_0 \cup S_1 \ ?$$

PROBLEM 4. *Given a function $\Phi \in H_2(\Omega)$, is there a function $T^{(1)}$ so that the following equation and conditions are satisfied:*

$$-\varkappa\Delta T^{(1)} = \Phi \quad in \quad \Omega,$$

$$\frac{\partial T^{(1)}}{\partial n} = 0 \quad on \quad \Sigma\ , T^{(1)} = 0 \quad on \quad S_0 \cup S_1 \ ?$$

PROBLEM 5. *Given a function $\Psi_2 \in H_2^{-\frac{1}{2}}(\Sigma)$ is there a function $T^{(2)}$ so that the following equation and conditions are satisfied:*

$$-\varkappa\Delta T^{(2)} = 0 \quad in \quad \Omega$$

$$\frac{\partial T^{(2)}}{\partial n} = \Psi_2 \quad on \quad \Sigma,\ T^{(2)} = 0 \quad on \quad S_0 \cup S_1 \ ?$$

PROBLEM 6. *Given a function $h_i \in H_2(\Sigma)(i = 1,2)$, is there a function T_i so that the following equation and condition are satisfied:*

$$h^2 \Delta_1 T_i = h_i \quad in \quad \Sigma,$$

$$T_i = 0 \quad on \quad \partial\Sigma \ ?$$

In [1, 8, 9, 12, 13] we find the following lemmas.

LEMMA 1. *For a vector-function $f \in L_2(\Sigma)$ there exists a unique generalized solution of Problem 1. This solution is the vector-function u satisfying the equality*

$$(u,s)_{W_2^{1,1,2}(\Sigma)} = (f,s)_{L_2(\Sigma)}, \quad \forall s \in W_2^{1,1,2}(\Sigma),$$

$$W_2^{1,1,2}(\Sigma) = H_{2,0}^1(\Sigma) \times H_{2,0}^1(\Sigma) \times H_{2,0}^1(\Sigma).$$

The solving operator A^{-1} of the problem ($u = A^{-1}f$) is positive and compact, it maps $L_2(\Sigma)$ into $W_2^{1,1,2}(\Omega)$.

LEMMA 2. *For a vector-function $g \in \tilde{L}_2(\Omega)$ there exists a unique generalized solution of Problem 2. This solution is the vector-function $v^{(1)}$ satisfying the equality*

$$(v^{(1)},v)_{W_{2,0}^1(\Omega)} = (g,v)_{\tilde{L}_2(\Omega)}, \quad \forall v \in W_{2,0}^1(\Omega).$$

The solving operator A^{-1} of the problem ($v^{(1)} = A_0^{-1}g$) is positive and compact, it maps $L_2(\Omega)$ into $W_{2,0}^1(\Omega)$.

LEMMA 3. *For a function $\Psi_1 \in H_-$ there exists a unique generalized solution of Problem 3. This is the vector-function $v^{(2)}$ satisfying the equality*

$$(v^{(2)},s)_{W_{2,0}^1} = (\Psi_1, \gamma_1 s)_{H_0(\Sigma)}, \quad \gamma_1 s = s_{3|\Sigma}.$$

The solving operator A^{-1} of the problem $Q_1(v^{(2)} = Q_1\Psi_1)$ is compact and it maps $H_-(\Sigma)$ into $W_{2,0}^1(\Omega)$.

LEMMA 4. *For a function $\Phi \in H_2(\Omega)$ there exists a unique generalized solution of Problem 4. This solution is the function $T^{(1)}$ satisfying the equality*

$$(T^{(1)},T)_{H_{2,0}^1(\Omega)} = (\Phi,T)_{H_2(\Omega)}, \quad \forall T \in H_{2,0}^1(\Omega).$$

The solving operator A^{-1} of the problem $A_1^{-1}(T^{(1)} = A_1^{-1}\Phi)$ is positive and compact, it maps $H_2(\Omega)$ into $H^1_{2,0}(\Omega)$.

LEMMA 5. *For a function $\Psi_2 \in H_2^{-\frac{1}{2}}(\Sigma)$ there exists a unique generalized solution of Problem 5, this solution is the function $T^{(2)}$ satisfying the following equality*

$$(T^{(2)}, T)_{H^1_{2,0}(\Sigma)} = (\Psi_2, \gamma_2 T)_{H_2(\Sigma)}, \quad \forall T \in H^1_{2,0}(\Omega).$$

The solving operator A^{-1} of the problem $Q_2(T^{(2)} = Q_2\Psi_2)$ is compact, it maps $H_2^{-\frac{1}{2}}(\Sigma)$ into $H^1_{2,0}(\Omega)$.

LEMMA 6. *For a function $h_i \in H_2(\Sigma)$ there exists a unique generalized solution of Problem 6, this solution is the function T_i satisfying the following equality :*

$$(T_i, T)_{H^1_{2,0}(\Sigma)} = (h_i, T)_{H_2(\Sigma)}, \quad \forall T \in H^1_{2,0}(\Sigma).$$

The solving operator A^{-1} of the problem $A_2^{-1}(T_i = A_2^{-1}h_i)$ is positive and compact. It maps $H_2(\Sigma)$ into $H^1_{2,0}(\Sigma)$.

3. Spectrum theorems

We seek the solution of the equation (1.1), (1.2) with the conditions (1.6), (1.7) in the form

$$v = v^{(1)} + v^{(2)}, \quad p = p^{(1)} + p^{(2)}, \quad T = T^{(1)} + T^{(2)}.$$

Using lemmas 2-5 we can prove that the equations (1.1), (1.2) with the conditions (1.6). (1.7) are equivalent to the following equations:

$$v^{(1)} = \lambda A_0^{-1}(v^{(1)} + Q_1(\tau(v^{(2)})_{(\Sigma)}.n) + A_0^{-1}N(T^{(1)} + \frac{\varkappa_1}{\varkappa h}Q_2T_2); \tag{3.1}$$

$$T^{(1)} = \lambda A_1^{-1}(T^{(1)} + \frac{\varkappa_1}{\varkappa h}Q_2T_2) + A_1^{-1}M(v^{(1)} + Q_1(\tau(v^{(2)})_{(\Sigma)}.n); \tag{3.2}$$

$$NT \equiv g\beta Tk_3, Mv \equiv b(v.k_3).$$

Using Lemma 6 we can prove that the equations (1.4), (1.5) with the condition (1.8) are equivalent to the following equations:

$$T_1 = \frac{h^2}{a}\lambda A_2^{-1}T_1 - \mu_1 A_2^{-1}T_1 - \mu^* A_2^{-1}T_2 - \frac{\mu_1-\mu_2}{2}A_2^{-1}\gamma_2(T^{(1)} + \frac{\varkappa_1}{\varkappa h}Q_2T_2), \quad (3.3)$$

$$T_2 = \frac{h^2}{a}\lambda A_2^{-1}T_2 - 3\mu^* A_2^{-1}T_1 - 3(\mu_1+1)A_2^{-1}T_2 + \\ + 3(\mu_1-\mu_2)A_2^{-1}\gamma_2(T^{(1)} + \frac{\varkappa_1}{\varkappa h}Q_2T_2). \quad (3.4)$$

It has been proved in [9] that the operator $C_1 = \gamma_1 Q_1$ is self-adjoint, positive and compact in $H_2(\Sigma)$. In the same way we can prove that the operator $C_2 = \gamma_2 Q_2$ is self-adjont, positive and compact in $H_2(\Sigma)$.

Using the following transformation of variables:

$$\xi^1 = A_0^{\frac{1}{2}}v^{(1)}, \tau(v^{(2)})_{(\Sigma)}.n = C^{-\frac{1}{2}}\eta^1, \quad \theta = A_1^{\frac{1}{2}}T^{(1)}, \quad T_i = A_2^{-1}\theta_i,$$

from the equations (3.1)-(3.4) we obtain

$$\xi^1 = \lambda A_0^{-1}(\xi^1 + A_0^{\frac{1}{2}}Q_1C_1^{-\frac{1}{2}}\eta) + A_0^{-\frac{1}{2}}N(A_1^{-\frac{1}{2}}\theta + \frac{\varkappa_1}{\varkappa h}Q_2A_2^{-1}\theta_2), \quad (3.6)$$

$$\theta = \lambda A_1^{-1}(\theta + \frac{\varkappa_1}{\varkappa h}A_1^{\frac{1}{2}}Q_2A_2^{-1}\theta_2) + A_1^{-\frac{1}{2}}M(A_0^{-\frac{1}{2}}\xi^1 + Q_1C_1^{-\frac{1}{2}}\eta^1), \quad (3.7)$$

$$\theta_1 = \frac{h^2}{a}\lambda A_2^{-1}\theta_1 - \mu_1A_2^{-1}\theta_1 - \mu^*A_2^{-1}\theta_2 - \frac{\mu_1-\mu_2}{2}\gamma_2(A_2^{-\frac{1}{2}}\theta + \frac{\varkappa_1}{\varkappa h}Q_2A_2^{-1}\theta_2), \quad (3.8)$$

$$\theta_2 = \frac{h^2}{a}\lambda A_2^{-1}\theta_2 - 3\mu^*A_2^{-1}\theta_1 - 3(\mu_1+1)A_2^{-1}\theta_2 - \\ - 3(\mu_1-\mu_2)\gamma_2(A_2^{-\frac{1}{2}}\theta + \frac{\varkappa_1}{\varkappa h}Q_2A_2^{-1}\theta_2). \quad (3.9)$$

Using the conditions (1.9),(3.5) and Lemmas 2,3 we can write the equation (1.3) in the form:

$$Eh(1-\mu^2)^{-1}L(\gamma_1A_0^{-\frac{1}{2}}\xi^1 + \gamma_1Q_1C_1^{-\frac{1}{2}}\eta^1) + \rho h\lambda^2(\gamma_1A_0^{-\frac{1}{2}}\xi^1 + \gamma_1Q_1C_1^{-\frac{1}{2}}\eta^1) - \\ - \frac{1}{2}\lambda C_1^{-\frac{1}{2}}\eta^1 - \lambda l_1A_2^{-1}\theta_1 - \lambda l_2A_2^{-1}\theta_2 = 0.$$

The operators $l_1A_2^{-1}, l_2A_2^{-1}$ are bounded because of Lemma 6.

From the latter equation it follows that

$$F(K_1\xi^1+\eta^1)+\rho h\lambda^2 C_1(K_1\xi^1+\eta^1)-\frac{1}{2}\lambda(K_1\xi^1+\eta^1)+\frac{1}{2}\lambda K_1\xi^1 \\ -\lambda C_1^{\frac{1}{2}} l_1 A_2^{-1}\theta_1-\lambda C_1^{\frac{1}{2}} l_2 A_2^{-1}\theta_2=0, \quad (3.10)$$

where $K_1=C_1^{-\frac{1}{2}}\gamma_1 A_0^{-\frac{1}{2}}, F=Eh(1-\mu^2)^{-1}C_1^{\frac{1}{2}}LC_1^{\frac{1}{2}}$.

Using Lemma 1 Orazov [13] has proved that the operator V=I+F is unbounded positive, its inverse operator is compact and maps $L_2(\Sigma)$ into $W_2^{1,1,2}(\Sigma)$.

Putting $K_1\xi^1+\eta^1=\tilde{\eta}$ in the equation (3.10) we get

$$\tilde{\eta}-V^{-1}\tilde{\eta}+\rho h\lambda^2 V^{-1}C_1\tilde{\eta}-\frac{1}{2}\lambda V^{-1}\tilde{\eta}+\frac{1}{2}\lambda V^{-1}K_1\xi^1 \\ -\frac{1}{2}\lambda V^{-1}C_1^{\frac{1}{2}} l_1 A_2^{-1}\theta_1-\lambda V^{-1}C_1^{\frac{1}{2}} l_2 A_2^{-1}\theta_2=0,$$

or

$$\eta-V^{-1}\eta+\rho h\lambda^2 V^{-\frac{1}{2}}C_1 V^{-\frac{1}{2}}\eta-\frac{1}{2}\lambda V^{-1}\eta+\frac{1}{2}\lambda V^{-\frac{1}{2}}K_1 A_0^{-\frac{1}{2}}\xi- \\ -\lambda V^{-\frac{1}{2}}C_1^{\frac{1}{2}} l_1 A_2^{-1}\theta_1-\lambda V^{-\frac{1}{2}}C_1^{\frac{1}{2}} l_2 A_2^{-1}\theta_2=0. \quad (3.11)$$

Here $\eta=V^{\frac{1}{2}}\tilde{\eta}, \xi=A_0^{\frac{1}{2}}\xi^1$.

In [8] it has been proved that the operator $K_1^*=A_0^{\frac{1}{2}}Q_1 C_1^{-\frac{1}{2}}$ is an adjoint of the operator K_1. It is easy to see that the operator $K_2^*=A_1^{\frac{1}{2}}Q_2 C_2^{-\frac{1}{2}}$ is an adjoint of the operator $K_2=C_2^{-\frac{1}{2}}\gamma_2 A_1^{-\frac{1}{2}}$.

We rewrite the equations (3.6)-(3.9) in the form:

$$\xi^1=\lambda A_0^{-1}(\xi^1+K_1^*\eta^1)+A_0^{-\frac{1}{2}}NA_1^{-\frac{1}{2}}\theta+\frac{\varkappa_1}{\varkappa h}A_0^{-\frac{1}{2}}NQ_2A_2^{-1}\theta_2, \quad (3.12)$$

$$\theta=\lambda A_1^{-1}\theta+\lambda\frac{\varkappa_1}{\varkappa h}A_1^{-1}K_2^*C_2^{\frac{1}{2}}A_2^{-1}\theta_2+A_1^{-\frac{1}{2}}MA_0^{-\frac{1}{2}}(\xi^1+K_1^*\eta^1), \quad (3.13)$$

$$\theta_1=\frac{h^1}{a}\lambda A_2^{-1}\theta_1-\mu_1A_2^{-1}\theta_1-\mu^*A_2^{-1}\theta_2+\frac{\mu_1-\mu_2}{2}(C_2^{\frac{1}{2}}K_2\theta+\frac{\varkappa_1}{\varkappa h}C_2A_2^{-1}\theta_2), \quad (3.14)$$

$$\theta_2=\frac{h^2}{a}\lambda A_2^{-1}\theta_2-3(\mu_1+1)A_2^{-1}\theta_2-3\mu^*A_2^{-1}\theta_1+ \\ +3(\mu_1-\mu_2)(C_2^{\frac{1}{2}}K_2\theta+\frac{\varkappa_1}{\varkappa h}C_2A_2^{-1}\theta_2). \quad (3.15)$$

Putting $\xi = A_0^{\frac{1}{2}}\xi^1, \eta = V^{\frac{1}{2}}(K_1\xi^1 + \eta^1) = V^{\frac{1}{2}}\tilde{\eta}$ in the equations (3.12), (3.13) we get

$$\xi = \lambda A_0^{-\frac{1}{2}} K_1^* V^{-\frac{1}{2}}\eta + \lambda A_0^{-\frac{1}{2}}(I - K_1^* K_1) A_0^{-\frac{1}{2}}\xi + N A_1^{-\frac{1}{2}}\theta + \frac{\varkappa_1}{\varkappa h} N Q_2 A_2^{-1}\theta_2, \quad (3.16)$$

$$\theta = \lambda A_1^{-1}\theta + \lambda\frac{\varkappa_1}{\varkappa h} A_1^{-1} K_1^* C_2^{\frac{1}{2}} A_2^{-1}\theta_2 + A_1^{-\frac{1}{2}} M A_0^{-\frac{1}{2}} K_1^* V^{-\frac{1}{2}}\eta + $$
$$+ A_1^{-\frac{1}{2}} M A_0^{-\frac{1}{2}}(I - K_1^* K_1) A_0^{-\frac{1}{2}}\xi. \quad (3.17)$$

The system of the equations (3.11), (3.14)-(3.17) can now be written in the form

$$L(\lambda)Z \equiv I_5 Z - D_0 Z - \lambda D_1 Z + \lambda^2 D_2 Z = 0,$$

where $Z = (\eta, \xi, \theta, \theta_1, \theta_2)$

$$D_0 = \begin{pmatrix} V^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & N A_1^{-\frac{1}{2}} & 0 & K_3 \\ A_1^{-\frac{1}{2}} M D^* & A_1^{-\frac{1}{2}} M B^{-1} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\mu_{12} C_2^{\frac{1}{2}} K_2 & -\mu_1 A_2^{-1} & K_4 \\ 0 & 0 & 3\mu_{12} C_2^{\frac{1}{2}} K_2 & -3\mu^* A_2^{-1} & K_5 \end{pmatrix},$$

$$D_1 = \begin{pmatrix} \frac{1}{2}V^{-1} & -\frac{1}{2}D & 0 & V^{-\frac{1}{2}} C_1^{\frac{1}{2}} l_1 A_2^{-1} & V^{-\frac{1}{2}} C_1^{\frac{1}{2}} l_2 A_2^{-1} \\ D^* & B^{-1} & 0 & 0 & 0 \\ 0 & 0 & A_1^{-1} & 0 & \frac{\varkappa_1}{\varkappa h} A_1^{-1} K_1^* C_2^{\frac{1}{2}} A_2^{-1} \\ 0 & 0 & 0 & \frac{h^2}{a} A_2^{-1} & 0 \\ 0 & 0 & 0 & 0 & \frac{h^2}{a} A_2^{-1} \end{pmatrix},$$

$$D_2 = \begin{pmatrix} C & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Here the following notations are used:

$$B^{-1} = A_0^{-\frac{1}{2}}(I - K_1^* K_1)A_0^{-\frac{1}{2}}, C = \rho h V^{-\frac{1}{2}} C_1 V^{-\frac{1}{2}}, D = V^{-\frac{1}{2}} K_1 A_0^{-\frac{1}{2}},$$

$$D^* = A_0^{-\frac{1}{2}} K_1^* V^{-\frac{1}{2}}, \mu_{12} = \mu_1 - \mu_2, \mu_1^+ = \mu_1 + 1,$$

$$K_3 = \frac{\varkappa_1}{\varkappa h} N Q_2 A_2^{-1},$$

$$K_4 = -\mu^* A_2^{-1} + \frac{\varkappa_1 \mu_{12}}{2\varkappa h} C_2 A_2^{-1},$$

$$K_5 = -3\mu_1^+ A_2^{-1} + \frac{3\mu_{12}\varkappa_1}{\varkappa h} C_2 A_2^{-1}$$

In [14] it has been proved that the operator $B^{-1} = A_0^{-\frac{1}{2}}(I - K_1^* K_1)A_0^{-\frac{1}{2}}$ is positive, compact and $D(B^{\frac{1}{4}}) = D(A_0^{\frac{1}{4}})$. So it is clear that the operators D_0, D_1, D_2 are also compact and D_2 is self-adjoint non-negative.

Using Lemma 1 and the results of [14] we can prove that the operator $C_1^{-\frac{1}{2}} V^{-1} C_1^{-\frac{1}{2}}$ is compact. Therefore, the operator $T_{11}^1 = V^{-1} C^{-\frac{1}{2}}$ is compact. By [14] the operator $T_{12}^1 = C^{-\frac{1}{2}} V^{-\frac{1}{2}} K_1 A_0^{-\frac{1}{4}}$ is bounded.

Putting

$$T_{11} = \begin{pmatrix} T_{11}^1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad T_{12} = \begin{pmatrix} 0 & T_{12}^1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$T_{21} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} A_0^{-\frac{1}{4}} T_{12}^{1*} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ A_2^{-1} l_1^* C_1^{\frac{1}{2}} V^{-\frac{1}{2}} C^{-\frac{1}{2}} & 0 & 0 & 0 & 0 \\ A_2^{-1} l_2^* C_1^{\frac{1}{2}} V^{-\frac{1}{2}} C^{-\frac{1}{2}} & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\overline{C_1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & B^{-1} & 0 & 0 & 0 \\ 0 & 0 & A_1^{-1} & 0 & 0 \\ 0 & 0 & 0 & \frac{h^2}{a}A_2^{-1} & 0 \\ 0 & 0 & 0 & 0 & \frac{h^2}{a}A_2^{-1} \end{pmatrix},$$

$$\overline{C_2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\varkappa_1}{\varkappa h}A_1^{-1}K_2^*C_2^{\frac{1}{2}} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \overline{C_3} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\varkappa_1}{\varkappa h}A_2^{-1}C_2^{\frac{1}{2}}K_2 & 0 & 0 \end{pmatrix}$$

we can check that the following equalities are satisfied

$$P_2D_1P_2 = (I_5 + \overline{C_2})\overline{C_1}, P_2D_1^*P_2 = (I_5 + \overline{C_3})\overline{C_1} \equiv D_{22}, \tag{3.19}$$

$$P_1D_1^*P_1 = T_{11}D_2^{\frac{1}{2}} \equiv D_{11}, \tag{3.20}$$

$$P_1D_1^*P_2 = D_2^{\frac{1}{2}}T_{12}\overline{C_1}^{\frac{1}{4}} \equiv D_{12}, \tag{3.21}$$

$$P_2D_1^*P_1 = T_{21}D_2^{\frac{1}{2}} \equiv D_{21}. \tag{3.22}$$

THEOREM 1. *The spectrum of the problem (1.1)-(1.10) consists of discrete characteristic values λ_k which have a finite algebraic multiplicity and unique limit at ∞. Except a finite number of points, the characteristic values λ_k are contained in the small angles, which contain the imaginary axis and positive one.*

PROOF: We denote by $\Lambda_{\epsilon,R}$ the domain:

$$\Lambda_{\epsilon,R} = \{\lambda \in C, \epsilon < |arg\lambda| < \frac{\pi}{2} - \epsilon, |arg\lambda| > \frac{\pi}{2} + \epsilon, |\lambda| > R,$$
$$-\pi \le arg\lambda \le \pi, \epsilon \text{ is sufficiently small and } R = R(\epsilon) \text{ sufficiently big } \}.$$

In the domain $\Lambda_{\epsilon,R}$ we can find an analytic function-operator $X(\lambda)$ which satisfies the following equation:

$$[L(\bar{\lambda})]^*X(\lambda) = I_5. \tag{3.23}$$

Indeed, after operating two sides of the equation (3.23) by P_1 and P_2 we get

$$(I_5 - P_1D_0^* - \lambda P_1D_1^* + \lambda^2 P_1D_2)P_1X(\lambda) = (P_1D_0^* + \lambda P_1D_1^*)P_2X(\lambda) + P_1, \quad (3.24)$$

$$(I_5 - P_2D_0^*P_2 - \lambda P_2D_1^*P_2)P_2X(\lambda) = P_2(D_0^* + \lambda D_1^*)P_1X(\lambda) + P_2. \quad (3.25)$$

From (3.25), (3.19) we obtain

$$R(\lambda)P_2X(\lambda) = P_2(D_0^* + \lambda D_1^*)P_1X(\lambda) + P_2, \quad (3.26)$$

where

$$\begin{aligned} R(\lambda) &= I_5 - P_2D_0^*P_2 - \lambda D_{22} \\ &= I_5 - P_2D_0^*P_2 - \lambda(I_5 + \overline{C_3})\overline{C_1} \\ &= (I_5 + \overline{C_3})(I_5 + R_1(\lambda))(I_5 - \lambda\overline{C_1}), \end{aligned} \quad (3.27)$$

$$R_1(\lambda) = \{S_2 - (I_5 + S_2)P_2D_0^*P_2\}(I_5 - \lambda\overline{C_1})^{-1}.$$

The operator $S_2 = -I_5 + (I_5 + \overline{C_3})^{-1} = -(I_5 + \overline{C_3})^{-1}\overline{C_3}$ is compact. Using results of [16, Lemma 8, pp. 13] we get

$$\|R_1(\lambda)\| = \|[S_2 - (I_5 + S_2)P_2D_0^*P_2](I_5 - \lambda\overline{C_1})^{-1}\| \Rightarrow 0$$

with $|\lambda| \Rightarrow \infty$ and $\lambda \in \Lambda_{\epsilon,R}$. So the operator $I_5 + R_1(\lambda)$ is reversible in $\Lambda_{\epsilon,R}$.

It is clear that the operators $I_5 + \overline{C_3}$ and $I_5 - \lambda\overline{C_1}$ are reversible in $\Lambda_{\epsilon,R}$. So the operator $R(\lambda)$ is reversible in $\Lambda_{\epsilon,R}$. From the equation (3.26) we obtain

$$P_2X(\lambda) = P_2R^{-1}(\lambda)P_2[(P_2D_0^*P_1 + \lambda D_{21})P_1X(\lambda) + P_2]. \quad (3.28)$$

It follows from (3.24) and (3.28) that

$$l(\lambda)P_1X(\lambda) = f(\lambda),$$

where

$$f(\lambda) = P_1 + [P_1D_0^*P_2 + \lambda D_{12}]R^{-1}(\lambda)P_2, \quad (3.29)$$

$$l(\lambda) = I_5 - P_1 D_0^* P_1 - P_1 D_0^* P_2 R^{-1}(\lambda) P_2 D_0^* P_1 - \lambda D_{11} + \lambda^2 D_2 -$$
$$- \lambda P_1 D_0^* P_2 R^{-1}(\lambda) D_{21} - \lambda D_{12} R^{-1}(\lambda) P_2 D_0^* P_1 - \lambda^2 D_{12} R^{-1}(\lambda) D_{21}. \tag{3.30}$$

In the domain $\Lambda_{\epsilon,R}$ the operators $I_5 - i\lambda D_2^{\frac{1}{2}}$ and $I_5 + i\lambda D_2^{\frac{1}{2}}$ are reversible and we can write (3.30) in the form:

$$l(\lambda) = (I_5 - i\lambda D_2^{\frac{1}{2}})(I_5 + T(\lambda))(I_5 + i\lambda D_2^{\frac{1}{2}}), \tag{3.31}$$

where

$$T(\lambda) = (I_5 - i\lambda D_2^{\frac{1}{2}})^{-1}\{P_1 D_0^* P_1 - \lambda D_{11} - P_1 D_0^* P_2 R^{-1}(\lambda) P_2 D_0^* P_1 -$$
$$- \lambda P_1 D_0^* P_2 R^{-1}(\lambda) D_{21} - \lambda D_{12} R^{-1}(\lambda) P_2 D_0^* P_1 -$$
$$- \lambda^2 D_{12} R^{-1}(\lambda) D_{21}\}(I_5 + i\lambda D_2^{\frac{1}{2}})^{-1}. \tag{3.32}$$

Using (3.21),(3.22), (3.27) and [16, lemma 7,pp.13] we obtain an estimation for the last term in (3.32)

$$\|F(\lambda)\| \|(I_5 - i\lambda D_2^{\frac{1}{2}})^{-1} \lambda^2 D_{12} R^{-1}(\lambda) D_{21} (I_5 + i\lambda D_2^{\frac{1}{2}})^{-1}\| \leq$$
$$\leq |\lambda|^2 \|(I_5 - i\lambda D_2^{\frac{1}{2}})^{-1} D_2^{\frac{1}{2}}\| \|T_{12}\| \|\overline{C_1}^{\frac{1}{4}} (I_5 - \lambda \overline{C_1})^{-1}\| \|(I_5 + R_1(\lambda))^{-1}\|.$$
$$.\|(I_5 + \overline{C_3})^{-1}\| \|T_{21}\| \|D_2^{\frac{1}{2}} (I_5 + i\lambda D_2^{\frac{1}{2}})^{-1}\| = o(|\lambda|^{2-2+2\epsilon-\frac{1}{4}})$$
$$= o(|\lambda|^{-\frac{1}{4}+2\epsilon}).$$

In the same way we prove that the other terms in (3.32) are small if ϵ is sufficiently small. Therefore, the operator $I + T(\lambda)$ is reversible in $\Lambda_{\epsilon,R}$. This implies that the operator $l(\lambda)$ is reversible in the domain $\Lambda_{\epsilon,R}$.

From (3.29),(3.31) it follows that

$$l^{-1}(\lambda) f(\lambda) = \{(I_5 + i\lambda D_2^{\frac{1}{2}})^{-1}(I_5 + T(\lambda))^{-1}(I_5 - i\lambda D_2^{\frac{1}{2}})^{-1}\}.$$
$$\{P_1 + [P_1 D_0^* P_2 + \lambda D_{12}] R^{-1}(\lambda) P_2\}. \tag{3.33}$$

Using (3.21), (3.27) and [16, lemmas 7,8,pp.13] we obtain an estimation for the last term in (3.33):

$$\begin{aligned}\|F_1(\lambda)\| &= |\lambda|\|(I_5 + i\lambda D_2^{\frac{1}{2}})^{-1}(I_5 + T(\lambda))^{-1}\|.\\ &.\|(I_5 - i\lambda D_2^{\frac{1}{2}})^{-1} D_{12} P_2 R^{-1}(\lambda) P_2\| \leq\\ &\leq c|\lambda|\|(I_5 - i\lambda D_2^{\frac{1}{2}})^{-1} D_2^{\frac{1}{2}}\|\|T_{12}\|\|\overline{C_1}^{-\frac{1}{4}}(I_5 - \lambda\overline{C_1})^{-1}\|.\\ &.\|(I_5 + R_1^{-1}(\lambda))^{-1}(I_5 + \overline{C_3})^{-1}\| = o(|\lambda|^{-\frac{1}{4}+\epsilon}).\end{aligned}$$

Therefore, if $\lambda \in \Lambda_{\epsilon,R}$ and $\epsilon < \frac{1}{4}$, then $\|F_1(\lambda)\| \Rightarrow 0$. In the same way we prove that the other terms in (3.33) tend to 0 as $|\lambda| \Rightarrow \infty$ if ϵ is sufficiently small and $\lambda \in \Lambda_{\epsilon,R}$. So we obtain

$$P_1 X(\lambda) = l^{-1}(\lambda) f(\lambda) = o(1) \quad \text{when} \quad |\lambda| \Rightarrow \infty, \lambda \in \Lambda_{\epsilon,R}. \tag{3.34}$$

This implies that $P_1 X(\lambda)$ is an analytic function in the domain $\Lambda_{\epsilon,R}$.

In the same way we prove that $P_2 X(\lambda)$ is an analytic function in the domain $\Lambda_{\epsilon,R}$ and

$$\|P_2 X(\lambda)\| \Rightarrow o(|\lambda|^{\epsilon}), |\lambda| \Rightarrow \infty, \lambda \in \Lambda_{\epsilon,R}. \tag{3.35}$$

Therefore, the operator function $X(\lambda)$ is analytic in the domain $\Lambda_{\epsilon,R}$ and

$$\|X(\lambda)\| \Rightarrow o(|\lambda|^{\epsilon}) \quad \text{as} \quad |\lambda| \Rightarrow \infty, \lambda \in \Lambda_{\epsilon,R}. \tag{3.36}$$

This implies that the operator $(L(\bar{\lambda}))^*$ has a right hand reversible operator in the domain $\Lambda_{\epsilon,R}$.

Similarly, the operator $L(\lambda)$ has a right hand reversible operator. This implies that there exists an analytic operator $X_1(\bar{\lambda})$ in $\Lambda_{\epsilon,R}$ so that

$$L(\lambda) X_1(\bar{\lambda}) = I_5. \tag{3.37}$$

Because of the symmetry of the domain $\Lambda_{\epsilon,R}$ from (3.23) and (3.37) it follows that the operator $L(\lambda)$ is reversible in the domain $\Lambda_{\epsilon,R}$. Using results of [6,pp. 325] we obtain that the spectrum of the operator $L(\lambda)$ consists of discrete characteristic values λ_k with finite algebraic multiplicity.

Now we will prove that in the domain

$$\Lambda_{\epsilon,R} = \{\lambda \in C, |arg\lambda| < \frac{\pi}{2} - \epsilon, |arg\lambda| > \frac{\pi}{2} + \epsilon, |\lambda| > R,$$
$$-\pi < arg\lambda \leq \pi, \epsilon \text{ is sufficiently small and } R = R(\epsilon) \text{ sufficiently big } \}$$

the characteristic values λ_k of the operator $L(\lambda)$ have a limit at infinity.

We rewrite (3.24) in the form

$$R_2(\lambda)P_1X(\lambda) = [P_1D_0^* + \lambda P_1D_1^*]P_2X(\lambda) + P_1, \tag{3.38}$$

where

$$R_2(\lambda) = I_5 - P_1D_0^*P_1 - \lambda P_1D_1^*P_1 + \lambda^2 P_1D_2P_1 = [I_5 + T_2(\lambda)](I_5 + \lambda^2 D_2).$$

Because of (3.20),

$$T_2(\lambda) = -(P_1D_0^*P_1 + \lambda T_{11}D_2^{\frac{1}{2}})(I_5 + \lambda^2 D_2)^{-1}.$$

Using [16, lemmas 7,8] we get

$$\|T_2(\lambda)\| \Rightarrow 0 \quad \text{as} \quad |\lambda| \Rightarrow \infty, \quad \lambda \in \Lambda_{\epsilon,R}.$$

Therefore $R_2(\lambda)$ is reversible in the domain $\Lambda^1_{\epsilon,R}$ and we obtain from (3.38)

$$P_1X(\lambda) = R_2^{-1}(\lambda)[P_1D_0^* + \lambda P_1D_1^*]P_2X(\lambda) + R_2^{-1}(\lambda)P_1. \tag{3.39}$$

We rewrite the operator $R_2(\lambda)$ in the form

$$R_2(\lambda) = (I_5 - i\lambda D_2^{\frac{1}{2}})(I_5 + T_3(\lambda))(I_5 + i\lambda D_2^{\frac{1}{2}}). \tag{3.40}$$

Here the operator

$$T_3(\lambda) = (I_5 - i\lambda D_2^{\frac{1}{2}})^{-1}T_2(\lambda)(I_5 - i\lambda D_2^{\frac{1}{2}})$$

is bounded in $\Lambda_{\epsilon,R}$.

From (3.39), (3.25) it follows that

$$l_2(\lambda)P_2X(\lambda) = f_2(\lambda),$$

$$f_2(\lambda) = P_2 + P_2(D_0^* + \lambda D_1^*)R_2^{-1}(\lambda)P_1,$$

where

$$\begin{aligned} l_2(\lambda) &= I_5 - P_2 D_0^* P_2 - \lambda P_2 D_1^* P_2 - \\ &- P_2(D_0^* + \lambda D_1^*) P_1 R_2^{-1}(\lambda)(P_1 D_0^* + \lambda P_1 D_1^*) P_2 = \\ &= (I_5 + \overline{C_3})[(I_5 + \overline{C_3})^{-1} - (I_5 + \overline{C_3})^{-1} P_2 D_0^* P_2 - \\ &- B_1(\lambda) \lambda^{\frac{1}{4}} \overline{C_1}^{\frac{1}{4}} - S_0(\lambda) - \lambda \overline{C_1}], \end{aligned}$$

and

$$\begin{aligned} S_0(\lambda) &= (I_5 + \overline{C_3})^{-1}[P_2 D_0^* P_1 R_2^{-1}(\lambda) P_1 D_0^* P_2 + \\ &+ \lambda P_2 D_0^* P_1 R_2^{-1}(\lambda) P_1 D_1^* P_2 + \lambda P_2 D_1^* P_1 R_2^{-1}(\lambda) P_1 D_0^* P_2], \\ B_1(\lambda) &= (I_5 + \overline{C_3})^{-1} \lambda^{\frac{7}{4}} P_2 D_1^* P_1 R_2^{-1}(\lambda) D_2^{\frac{1}{2}} T_{12}. \end{aligned}$$

By [16, lemma 7,8] from (3.38) and (3.40) we see $S_0(\lambda)$ and $B_1(\lambda)$ are analytic operator functions and $\|S_0(\lambda)\| \Rightarrow 0, \|B_1(\lambda)\| \Rightarrow 0$ as $|\lambda| \Rightarrow \infty, \lambda \in \Lambda_{\varepsilon,R}$.

Using [17, Theorem 1, pp.399] we get

$$N(r, l_2(\lambda)) \approx N(r, \overline{C_1}),$$

where $N(r, l_2(\lambda)), N(r, C_1)$ are the distribution functions of the operators $l_2(\lambda)$ and $\overline{C_1}$ with characteristic values $\lambda_k(|\lambda_k| < r)$.

Since the characteristic values of the operator $\overline{C_1}$ have a limit at infinity, the characteristic values of either $l_2(\lambda)$ or $L(\lambda)$ have a limit at infinity.

The proof of Theorem 1 is now complete.

Let λ_k be an eigenvalue, Z_k an eigenfunction of the bundle $L(\lambda)$ and $Z_{k,s}(s = \overline{1, m_k})$ the associated eigenfunctions of Z_k.

We denote by M^2 the closure of the variety which consists of all the vector functions:

$$f_{k,s} = (Z_{k,s}, \lambda_k Z_{k,s} + Z_{k,s-1}).$$

It is easy to see that

$$\begin{aligned} &((Z_{k,s}, \lambda_k Z_{k,s} + Z_{k,s-1}), [(I_5 - D_0^*) P_2 \xi, -D_1^* P_2 \xi]) = \\ &(P_2 L(\lambda_k) Z_{k,s} + P_2 L_\lambda(\lambda_k) Z_{k,s}, \xi) = 0 \quad \forall \xi \in K. \end{aligned}$$

So the orthogonal subspace to M^2 in $K^2 = K \times K$ is the subspace N^2 consisting of all vector-functions:

$$\{(I_5 - D_0^*)P_2\xi, -D_1^*P_2\xi\}, \forall\xi \in K.$$

THEOREM 2. $K^2 = N^2 \oplus M^2$.

PROOF: In the domain $\Lambda_{\epsilon,R}$ we construct the function

$$Z(\lambda) = [L^*(\bar{\lambda})]^{-1}(\lambda g_1 + g_0), \tag{3.41}$$

where $f = (g_0, g_1) \in K^2$.

After operating two sides of the equation (3.41) by $P_1L^*(\bar{\lambda})$ and $P_2L^*(\bar{\lambda})$ we get

$$(I_5 - P_1D_0^*P_1 - \lambda D_{11} + \lambda^2 D_2)P_1Z(\lambda) = (P_1D_0^*P_2 + \lambda D_{12})P_2Z(\lambda) + \lambda P_1g_1 + P_1g_0, \tag{3.42}$$

$$(I_5 - P_2D_0^*P_2 - \lambda D_{22})P_2Z(\lambda) = (P_2D_0^* + \lambda D_{21})P_1Z(\lambda) + \lambda P_2g_1 + P_2g_0. \tag{3.43}$$

As in the proof of Theorem 1 we can prove that if $|\lambda| \Rightarrow \infty, \lambda \in \Lambda_{\epsilon R}$, then

$$\|P_1Z(\lambda)\| = o(|\lambda|), \|P_2Z(\lambda)\| = o(|\lambda|^{1+\epsilon}). \tag{3.44}$$

Let us assume that there exists a vector-function $f = (g_0, g_1) \in K^2$ and $f \perp N^2, f \perp M^2$. Using [16,Lemma 2, pp.9] we obtain that the vector-function (3.41) is entire. From (3.42), (3.43) it follows that

$$P_1Z(\lambda) = l^{-1}(\lambda)f_1(\lambda), \lambda \in \Lambda_{\epsilon,R},$$

where

$$f_1(\lambda) = P_1g_0 + \lambda P_1g_1 + [P_1D_0^*P_2 + \lambda D_{12}]R^{-1}(\lambda)P_2(g_0 + \lambda g_1).$$

Suppose that the operator $(I_5 - P_iD_0^*P_i)$ is reversible. We rewrite the operator $l(\lambda)$ in the form:

$$l(\lambda) = S_1(\lambda)l_1(\lambda), \tag{3.45}$$

where

$$S_1(\lambda) = (I_5 - P_1 D_0^* P_1)(I_5 - (I_5 - P_1 D_0^* P_1)^{-1} P_1 D_0^* P_2 R^{-1}(\lambda) P_2 D_0^* P_1), \quad (3.46)$$

$$\begin{aligned} l_1(\lambda) = I_5 - \lambda S_1^{-1}(\lambda) T_{11} D_2^{\frac{1}{2}} - \lambda S_1^{-1}(\lambda) P_1 D_0^* P_2 R^{-1}(\lambda) T_{21} D_2^{\frac{1}{2}} + \\ + \lambda^2 S_1^{-1}(\lambda) D_2 - \lambda S_1^{-1}(\lambda) D_2^{\frac{1}{2}} T_{12} \overline{C}^{-\frac{1}{4}} R^{-1}(\lambda) P_2 D_0^* P_1 - \\ - \lambda^2 S_1^{-1}(\lambda) D_2^{\frac{1}{2}} T_{12} \overline{C}^{-\frac{1}{4}} R^{-1}(\lambda) T_{21} D_2^{\frac{1}{2}}. \end{aligned} \quad (3.47)$$

Because $D_2 \in \delta_{p_1}(p_1 < \infty)$ [14] (here δ_q is the class of compact operators with order less than or equal to q), using results of [11,18] we can show that

$$\lim_{r \Rightarrow \infty} \frac{\ln \ln M(r)}{\ln r} = q < \infty, \quad (3.48)$$

where

$$M(r) \doteq \max(e, |E_1(\lambda)|),$$
$$E_1(\lambda) = (P_1 Z(\lambda), x^1) = (l^{-1}(\lambda) f_1(\lambda), x^1), x^1 \in L_2(\Sigma).$$

According to the Fragmen-Lindelof Theorem this implies that (3.44) is satisfied in the whole plane [7].

Using the Liouville theorem we get

$$P_1 Z(\lambda) = \eta^0 = \quad \text{const.}$$

Similarly, from the condition $\overline{C_1} \in \delta_{p_2}(p_2 < \infty)$ we obtain

$$P_2 Z(\lambda) = \lambda \xi^1 + \xi^0. \quad (3.50)$$

Putting (3.49), (3.50) into (3.42), (3.43) we get

$$\eta^0 = 0, \xi^1 = 0,$$

$$g_0 = (I_5 - D_0^*) P_2 \xi^0, g_1 = -D_1^* P_2 \xi^0.$$

This implies that $f = (g_0, g_1) \in N^2$ and $f \perp N^2$. But this is possible only when $f = 0$.

According to the method of M. G. Krien [6,pp.318], in the case when the operator $(I_5 - P_i D_0^* P_i)$ is not reversible we investigate instead of $L^*(\bar{\lambda})$ the bundle:

$$L^*(\bar{\lambda} + \bar{a}) = I_5 - D_{01}^* - \lambda D_{10}^* + \lambda^2 D_2.$$

Note that for this bundle the operator $(I_5 - P_i D_{01}^* P_i)$ is reversible.

REMARK: If we construct the operator

$$Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix},$$

where

$$Q_1 = (I_5 + P_2[(I_5 - P_2 D_0^* P_2)^{-1}]^* P_2 D_0) P_1 (I_5 +$$
$$+ D_0^* P_2 (I_5 - P_2 D_0^* P_2)^{-1} P_2),$$

$$Q_2 = (I_5 - P_2[(P_1 + D_{22})^{-1}]^* P_2 D_{12}^*) P_1 (I_5 - D_{12} P_2 (P_1 + D_{22})^{-1} P_2),$$

then the system $Q_1 Z_{k,s}, \lambda_k Q_2 Z_{k,s} + Q_2 Z_{k,s-1}\}_{k=1}^{\infty}$ is complete in $Q_1 K \times Q_2 K$.

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