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ON ANALYSIS AND DISCRETIZATION OF NONLINEAR ABEL INTEGRAL EQUATIONS OF FIRST KIND

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Dedicated to the memory of Professor Dr. Lothar Collatz

Abstract. For $0 \le x \le B, 0 < \beta < 1$, we consider the integral equation

$$\int_0^x (x-t)^{-\beta} K(x,t,y(t))\,dt = f(x)$$

under appropriate Lipschitz-like conditions on the function K and some of its derivatives, the most essential condition being

$$K_u(x,t,u) \ge c > 0 \quad \text{for} \quad 0 \le t \le x \le B, u \in \mathbb{R}.$$

After a survey on theorems of existence, uniqueness and stability of the solution we generalize a numerical method, proposed and investigated 1976 by H.W. Branca for the particular case $\beta = 1/2$, to all $\beta \in (0,1)$ and show it to be $O(h^2)$ convergent for all $\beta \in [0.2118, 1)$ if the solution y is sufficiently smooth. The method is based on piecewise linear interpolation, one-point weighted Gauss quadrature on parition intervals of equal lenght h, and collocation.

Introduction

Consider the nonlinear first kind Abel integral equation

$$\int_0^x (x-t)^{-\beta} K(x,t,y(t))\,dt = f(x), \quad 0 \le x \le B,\ B > 0, \quad \beta \in (0,1)$$

for determining a function $y : [0,B] \to \mathbb{R}$. We generalize a discretization method presented in 1976 by H. W. Branca for the particular case $\beta = 1/2$ (see[1] and [2]) to all $\beta \in (0,1)$.

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This method presupposes that there exists a unique solution $y \in C^3[0,B]$ and that certain partial derivatives of the kernel K exist and are continuous or Lipschitz-continuous or bounded. In particalur the partial derivative of K with respect to the third argument is supposed to be bounded away from zero, so that unique solvability of the nonlinear system of equations resulting from the discretization is guaranteed. The interval $[0,B]$ is equidistantly partitioned by a step-length $h = B/N$ where $N \in \mathbb{N}$, and the integral equation is collocated in the gridpoints $x_i = ih$ for $i = 1,2,\ldots,N$. For the exact solution y a continuous piecewise linear ansatz is made, and the resulting integrals on the intervals $[jh,(j+1)h]; j = 0,1,\ldots,i-1; \quad i = 1,\ldots,N$, are evaluated by aid of a weighted one-point Gauss formula. By ignoring error terms one obtains a triangular nonlinear system of equations for the values $\tilde{y}(x_i)$; $i = 0,1,\ldots,N$ of the approximate solution $\tilde{y}$, for which the initial value $\tilde{y_0} = \tilde{y}(0)$ may be calculated from the limit relation (see Lemma 1.2)

$$K(0,0,y(0)) = \lim_{x\to 0}(1-\beta)x^{\beta-1}f(x)$$

or from its approximation

$$K(0,0,\tilde{y_0}) = (1-\beta)h^{2\beta-2}f(h^2).$$

Branca, under appropriate smoothness assumptions, proved $0(h^2)$ convergence of $\tilde{y}$ towards y as $h \to 0$, for the particular case $\beta = 1/2$. We are going to prove $0(h^2)$ convergence first for all $\beta \in [2-\frac{\ln 3}{\ln 2}, 1)$ and then, by a modification of the method of proof, also for $\beta \in [0.2118, 2-\frac{\ln 3}{\ln 2})$. Notice that $2-\frac{\ln 3}{\ln 2} \approx 0.4150375$. These restrictions are caused by non-negativity conditions for the coefficients of the discretization method, conditions required for applicability of Branca's central lemma on a nonlinear system of difference inequalities.

Let us remark that the bounds just mentioned also occurred in a paper of Weiss [11] of 1972 on a product integration method for the generalized linear first kind Abel integral equation and that Eggermont [4] succeeded in 1981 to show that Weiss' method is $0(h^2)$ convergent for all $\alpha \in (0,1)$, by a method of proof differing from that of Weiss.

Our work consists of three chapters. In Chapter I we give statements on existence, uniqueness and differentiability of the solution. In Chapter II we first describe the method and then prove a series of lemmas required for the proof of convergence. Our calculations are much more complicated than those of Branca who could profit from the easy algebraic manipulability of differences of square roots. Finally the statement on $0(h^2)$ convergence is proved, first for $2-\frac{\ln 3}{\ln 2}\le\beta<1$, then for $0.2118\le\beta<2-\frac{\ln 3}{\ln 2}$. In Chapter III the actual application of the method to concrete problems is described, and the results of numerical case studies are displayed, also for the value $\beta=0.1\notin[0.2118,1)$. Some problems of practical computation are discussed.

Chapter I. Theoretical foundations

In order to have a firm analytical basis for analysis of the discretization method we give in Theorem 1.5 conditions on existence and uniqueness of a continuous solution of the nonlinear first kind Abel integral equation and in Theorem 1.6 a statement on differentiability of the solution. For logical completeness we do not hesitate to formulate a few theorems and lemmas of elementary character. Variants Theorem 1.5, with somehow relaxed conditions, can be found in the book [3] of Brunner and van der Houwen and in Gorenflo and Vessella [6]. Let us also stress the importance of Lemma 1.2, the limit relation for the solution's initial value $y(0)$. Concerning notation: If P is a fuction of n variables, we denote the operator of partial differentiation with respect to the $j-th$ argument by D_j for $j=1,2,\ldots,n$. By $\mathbb{R}^+$ we mean the set of positive real numbers, by $\mathbb{R}_0^+$ the set of non-negative real numbers.

From Branca ([1],[2]) we take

THEOREM 1.1. *Let* $B\in\mathbb{R}^+, I=[0,B], T=\{(x,t)|(x,t)\in\mathbb{R}^2, 0\le t\le x\le B\}$.

Assume (i), (ii), (iii) and (vi):

(i) $k\in C^1(T\times\mathbb{R})$.

(ii) *There exists* $L\in\mathbb{R}_0^+$ *such that* $|D_1k(x,t,z)-D_1k(x,t,\tilde z)|\le L|z-\tilde z|$ *for all* $(x,t)\in T$, $z,\tilde z\in\mathbb{R}$.

(iii) *There exists* $k_0 \in \mathbf{R}^+$ *such that* $D_3 k(x,x,z) \geq k_0 > 0$ *for all* $(x,z) \in I \times \mathbf{R}$.

(iv) $F \in C^1(I), F(0) = 0$.

Then the Volterra integral equation

$$\int_0^x k(x,t,y(t))\,dt = F(x)$$

has exactly one continuous solution.

From Hurwitz [8] we take

THEOREM 1.2. (extended Dirichlet formula). *Let* $a, b \in \mathbf{R}, a < b, T$ *as in Theorem 1.1,* $g \in C^0(T), \lambda, \mu, \nu \in [0,1], w(x,y) = (x-y)^{-\nu}(b-x)^{-\lambda}(y-a)^{-\mu}$. *Then*

$$\int_a^b \int_a^x g(x,y)w(x,y)\,dy\,dx = \int_a^b \int_y^b g(x,y)w(x,y)dxdy.$$

LEMMA 1.1. *Let* $s, x \in \mathbf{R}, s < x, \alpha, \beta \in \mathbf{R}^+$ *Then*

$$\int_s^x (x-t)^{\alpha-1}(\tau-s)^{\beta-1}\,d\tau = (x-s)^{\alpha+\beta-1}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

PROOF: With $\tau = \eta(x-s)+s$ the left hand side is equal to

$$(x-s)^{\alpha+\beta-1}\int_0^1 (1-\eta)^{\alpha-1}\eta^{\beta-1}\,d\eta,$$

and the last integral is equal to $B(\alpha,\beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$, where B is Euler's Beta function. See [7].

THEOREM 1.3. *Let* $\alpha \in \mathbf{R}^+, \beta \in (-\infty, 1), a, b \in \mathbf{R}, a < b$, *and* $M = (a,b)$ *or* $[a,b)$*or* $(a,b]$ *or* $[a,b]$. *Let, furthermore,* $f \in C^0(M \times [0,1])$, $D_1 f = C^0(M \times [0,1]), F(x) = \int_0^1 (1-t)^{\alpha-1}t^{-\beta}f(x,t)\,dt$. *Then* $F \in C^1(M)$ *and*

$$F'(x) = \int_0^1 (1-t)^{\alpha-1}t^{-\beta}D_1 f(x,t)\,dt.$$

PROOF: Exercise in classical analysis.

THEOREM 1.4. *Let $a, b \in \mathbb{R}, a < b, \beta \in (-\infty, 1), g \in C^1[a,b], g(a) = 0$, and*

$$G(x) = \int_a^x (x-t)^{-\beta} g'(t)\,dt \quad \text{for} \quad a \le x \le b.$$

Then $G \in C^1[a,b], G(a) = G'(a) = 0$, and

$$G'(x) = \int_a^x (x-t)^{-\beta} g'(t)\,dt \quad \text{for} \quad a \le x \le b.$$

PROOF: Consider first the case $a < x \le b$. With $t = \eta(x-a) + a$ we obtain

$$G(x) = (x-a)^{1-\beta} \int_0^1 (1-\eta)^{-\beta} g(\eta(x-a)+a)\,d\eta.$$

Define $f(\xi, \eta) = g(\eta(\xi - a) + a)$ for $(\xi, \eta) \in (a, b] \times [0,1]$. Then f and $D_1 f$ are in $C^0((a,b] \times [0,1])$, and, by Theorem 1.3 and back-substitution,

$$G'(x) = (1-\beta)(x-a)^{-\beta} \int_1^0 \frac{f(\xi,\eta)d\eta}{(1-\eta)^\beta} + (x-a)^{1-\beta} \int_0^1 \frac{g'(\eta(x-a)+a)\eta d\eta}{(1-\eta)^\beta}$$

$$= (1-\beta)(x-a)^{-1} \int_a^x \frac{g(t)dt}{(x-t)^\beta} + (x-a)^{-1} \int_a^x \frac{g'(t)(t-a)dt}{(x-t)^\beta}$$

Using $g(a) = 0$ and hence

$$\int_a^x (x-t)^{-\beta} g(t)\,dt = \frac{1}{1-\beta} \int_a^x (x-t)^{1-\beta} g'(t)\,dt$$

we obtain

$$G'(x) = \int_a^x (x-t)^{-\beta} g'(t)\,dt.$$

Continuity of G' at $x \in (a,b]$ now follows from the uniform continuity of g'.

The remaining case $x = a$ is handled by considering $G(x)$, $(G(x) - G(a))/(x-a)$ and $G'(x)$ for $x \to a+$

THEOREM 1.5. *Let $\beta \in (0,1), B \in \mathbb{R}^+, T = \{(x,t) | (x,t) \in \mathbb{R}^2, 0 \le t \le x \le B\}$. Assume furthermore* (i'), (ii'), (iii'), (iv'), (v'):

(i') $K \in C^1(T \times \mathbb{R})$.

(ii') *There exists a constant $L \in \mathbb{R}_0^+$ such that* $|D_1 K(x,t,z) - D_1 K(x,t,\tilde{z})| \le L|z - \tilde{z}|$ *for all* $(x,t) \in T, z, \tilde{z} \in \mathbb{R}$.

(iii') *There exists a constant* $K_0 \in \mathbf{R}^+$ *such that* $D_3K(x,x,z) \geq K_0 > 0$ *for all* $x \in [0,B], z \in \mathbf{R}$.

(iv') $f \in C^0[0,B], f(0) = 0$.

(v') *The function* F , *defined on* [0,B] *by* $F(x) = \int_0^x (x-t)^{\beta-1} f(t)\,dt$, *is in* $C^1[0,B]$.

Then the integral equation

$$\int_0^x (x-t)^{-\beta} K(x,t,y(t))\,dt = f(x), \quad 0 \leq x \leq B, \tag{1.1}$$

has exactly one solution $y \in C^0[0,B]$.

REMARK. *If* (iv') *is satisfied and* $f \in C^1[0,B]$ *then, by Theorem 1.4,* (v') *is fulfilled.*

PROOF: Define

$$k(x,t,z) = \int_t^x (x-s)^{\beta-1}(s-t)^{-\beta} K(s,t,z)\,ds \quad \text{for} \quad 0 \leq t < x \leq B, z \in \mathbf{R},$$

$$k(x,x,y) = K(x,x,z)\Gamma(\beta)\Gamma(1-\beta) \quad \text{for} \quad 0 \leq x \leq B, z \in \mathbf{R},$$

and consider the integral equation

$$\int_0^x k(x,t,y(t))\,dt = F(x), \quad 0 \leq x \leq B, \tag{1.2}$$

with F defined in (v'). We shall show $(\alpha), (\beta), (\gamma)$.

(α) k and F satisfy the condition of Theorem 1.1, so that (1.2) has exactly one continuous solution.

(β) Every continuous solution of (1.1) solves (1.2).

(γ) The continuous solution of (1.2) solves (1.1). $(\alpha), (\beta)$ and (γ) imply that (1.1) has exactly one continuous solution.

Proof of (α). We have to show that conditions (i), (ii), (iii), (iv) of Theorem 1.1 are satisfied.

(i) : With the representation $s = \tau(x-t)+t$ we find, for $t < x$,
$k(x,t,z) = \int_0^1 (1-\tau)^{\beta-1)}\tau^{-\beta} K(\tau(x-t)+t,t,z)\,d\tau$
which, because of Lemma 1.1, is also valid for $t = x$.

We have

$$D_1 k(x,t,z) = \int_0^1 \tau^{1-\beta}(1-\tau)^{\beta-1} D_1 K(\tau(x-t)+t,t,z)\,d\tau,$$

$$D_3 k(x,t,z) = \int_0^1 \tau^{-\beta}(1-\tau)^{\beta-1} D_3 K(\tau(x-t)+t,t,z)\,d\tau,$$

$$D_2 k(x,t,z) = \int_0^1 \tau^{-\beta}(1-\tau)^{\beta-1}(D_1 + D_2 K(\tau(x-t)+t,t,z))\,d\tau.$$

The continuity of these partial derivatives is a consequence of the uniform continuity of K on $T \times J$ for any compact subset $J \subset \mathbb{R}$.

(ii): a consequence of (ii').

(iii): $D_3 k(x,x,z) = \Gamma(1-\beta)\Gamma(\beta) D_3 K(x,x,z)$, hence (iii') implies (iii).

(iv): $F \in C^1[0,B]$ is a restatement of (v'), and $|F(x)| \leq \beta^{-1} x^{\beta} \|f\|_\infty$ implies $F(0) = 0$.

The proof of (α) is completed.

Proof of (β). Use the extended Dirichlet formula (Theorem 1.2).

Proof of (γ). $F \in C^1[0,B]$ and Theorem 1.4 imply, for $0 \leq x \leq B$,

$$\int_0^x (x-z)^{-\beta} F'(z)\,dz = \frac{d}{dx}\int_0^x (x-z)^{-\beta} F(z)\,dz,$$

and use of the definition of F, Theorem 1.2 and Lemma 1.1 yield

$$= \frac{d}{dx}(\int_0^x f(s)\,ds)\Gamma(\beta)\Gamma(1-\beta) = f(x)\Gamma(\beta)\Gamma(1-\beta).$$

On the other hand, using (1.2) and again Theorem 1.2 and Lemma 1.1, one obtains

$$\frac{d}{dx}\int_0^x (x-z)^{-\beta} F(z)\,dz = \Gamma(\beta)\Gamma(1-\beta)\frac{d}{dx}\int_0^x \int_0^s (s-t)^{-\beta} K(s,t,y(t))\,dtds$$

for the continuous solution y of the integral equation (1.2). Consider the function ϕ, defined on [0,B] by $\phi(s) = \int_0^s (s-t)^{-\beta} K(s,t,y(t))\,dt$, and abbreviate $\tilde{K}(x,t) = K(x,t,y(t))$. With $M = \max\{\tilde{K}(x,t) | (x,t) \in T\}$ we then have for $0 \leq t < s \leq B$ the estimates

$$|\phi(s) - \phi(t)| \leq \int_0^t \left|\frac{\tilde{K}(s,t)}{(s-\tau)^{\beta}} - \frac{\tilde{K}(t,\tau)}{(t-\tau)^{\beta}}\right| d\tau + \int_t^s \frac{|\tilde{K}(s,\tau)|}{(s-\tau)^{\beta}}\,d\tau$$

$$\leq M \int_0^t \left(\frac{1}{(t-\tau)^\beta} - \frac{1}{(s-\tau)^\beta}\right) d\tau + \int_0^t \frac{|\tilde{K}(s,t) - \tilde{K}(t,\tau)|}{(t-\tau)^\beta} d\tau + \frac{M}{1-\beta}(s-t)^{1-\beta}$$

from which we deduce continuity of ϕ , and hence

$$\frac{d}{dx} \int_0^x (x-z)^{-\beta} F(z)\, dz = \Gamma(\beta)\Gamma(1-\beta) \int_0^x (x-t)^{-\beta} K(x,t,y(t))\, dt.$$

Collecting partial results we see that (1.1) is fulfilled.

THEOREM 1.6. *Let $m \in N$ and assume, that in addition to the assumptions of Theorem 1.5 the assumptions* (vi'), (vii') *and* (viii') *are satisfied:*

(vi') $K \in C^m(T \times \mathbf{R})$.

(vii') $D_1^{m+1} K \in C^0(T \times \mathbf{R})$.

(viii') $F \in C^{m+1}[0,B]$, *where F is defined in* (v').

Then the solution y of the integral equation (1.1) is in $C^m[0,B]$.

PROOF: The unique continuous solution y of (1.1) fulfils, as has been shown as part (β) of the proof of Theorem 1.5, the integral equation

$$\int_0^x k(x,t,y(t))\, dt = F(x),$$

and (part (α)) $k \in C^1(T \times \mathbf{R}), F \in C^1[a,b]$. By differentiation:

$$F'(x) = k(x,x,y(x)) + \int_0^x D_1 k(x,t,y(t))\, dt.$$

In analogy to calculations in the proof of Theorem 1.5 one finds
(!) $k \in C^m(T \times \mathbf{R})$, (!!) $D_1^{m+1} k \in C^0(T \times \mathbf{R})$.
Now, if $m = 1$, define, for $(x,z) \in [0,B] \times \mathbf{R}$,

$$G(x,z) = k(x,x,z) - F'(x) + \int_0^x D_1 k(x,t,y(t))\, dt.$$

Then (!) and (!!) imply $G \in C^1([0,B] \times \mathbf{R})$, and because of Theorem 1.5 (iii') (see proof of part (α)) we have $D_2 G(x,z) = D_3 k(x,x,z) > 0$ for all $(x,z) \in [0,B] \times \mathbf{R}$. From the continuity of y and from $G(x,y(x)) = 0$ for $0 \leq x \leq B$ we conclude by the theorem on differentiability of implicitly defined functions that $y \in C^1[0,B]$.

If $m > 1$ we find

$$F''(x) = (2D_1 + D_2 + y'(x)D_3)k(x,x,y(x)) + \int_0^x D_1^2 k(x,t,y(t))\,dt,$$

$$y'(x) = \frac{1}{D_3 k(x,x,y(x))}\{F''(x) - (2D_1 + D_2)k(x,x,y(x)) - \int_0^x D_1^2 k(x,t,y(t))dt\},$$

and by successive differentiation we obtain continuous differentiability of $y^{(j)}$ for $j = 1, 2, \ldots, m-1$.

REMARK: The condition $f \in C^{m+1}[0,B], f^{(j)}(0) = 0$ for $j = 0, 1, \ldots, m$, is sufficient for validity of (viii'). To see this use Theorem 1.4.

LEMMA 1.2. *Under the conditions of Theorem 1.5 we have the limit relation*

$$K(0,0,y(0)) = (1-\beta) \lim_{x \to 0} x^{\beta-1} f(x). \tag{1.3}$$

PROOF: For $0 \le x \le B$ there exists $\xi(x) \in [0,x]$ such that

$$K(x,\xi(x),y(\xi(x))) \int_0^x (x-t)^{-\beta}\,dt = f(x),$$

$$K(x,\xi(x),y(\xi(x))) = (1-\beta)x^{\beta-1} f(x).$$

By $x \to 0$ we obtain (1.3).

Chapter II. The discretization method and its convergence

II.1 Derivation of the method. In order to discretize the integral equation

$$\int_0^x (x-t)^{-\beta} K(x,t,y(t))\,dt = f(x), 0 \le x \le B, \tag{2.1}$$

where $0 < \alpha < 1, B \in \mathbb{R}^+$, we partition the interval $[0,B]$ by grid-points $x_j = jh$, with $h = B/N, N \in \mathbb{N}, j = 0, 1, \ldots, N$. It is our intention to approximate the solution y by a piecewise linear function with nodes x_j, and we therefore state the well known theorem on *piecewise linear interpolation*.

THEOREM 2.1. *Let* $B \in \mathbb{R}^+, N \in \mathbb{N}, y \in C^2[0,B], x_j = jh,$
$y_j = y(x,y)$, *and define*

$$\bar{y}(t) = \frac{1}{h}(x_{j+1}-t)y_j + \frac{1}{h}(t-x_j)y_{j+1} \quad \text{for} \quad x_j \le t \le x_{j+1}, \quad j = 0,1,\ldots,N-1.$$

Then, with suitable $\xi_j \in (x_j, x_{j+1})$,

$$y(t) - \bar{y}(t) = \frac{1}{2}(x_{j+1}-t)(x_j-t)y''(\xi_j) \quad \text{for} \quad x_j \le t \le x_{j+1}.$$

Collocation of (2.1) at the grid-points x_i with the piecewise linear interpolant yields, for $i = 1,2,\ldots,N$,

$$f(x_i) = \int_0^{x_i} (x_i - t)^{-\beta} K(x_i, t, \bar{y}(t))\, dt + E_{1i},$$

$$E_{1i} = \int_0^{x_i} (x_i - t)^{-\beta} \{K(x_i,t,y(t)) - K(x_i,t,\bar{y}(t))\}\, dt,$$

$$f(x_i) = \sum_{j=0}^{i-1} \int_{x_j}^{x_{j+1}} (x_i - t)^{-\beta} K(x_i, t, \bar{y}(t))\, dt + E_{1i}$$

$$= h^{1-\beta} \sum_{j=0}^{i-1} \int_0^1 (i-j-\eta)^{-\beta} K(x_i, (j+\eta)h, \bar{y}((j+\eta)h))\, d\eta + E_{1i}.$$

For evaluation of the latter integrals we use a weighted *one-point Gauss formula.*

THEOREM 2.2. *Let* $k \in \mathbb{N}, G \in C^2[0,1], 0 < \beta < 1,$

$$a_k = \int_0^1 (k-t)^{-\beta}\, dt, \quad w_k = \frac{1}{a_k}\int_0^1 t(k-t)^{-\beta}\, dt,$$

$$r_k = \frac{1}{2}\int_0^1 (k-t)^{-\beta}(t-w_k)^2\, dt,$$

Then, with a suitable number $\varphi_k \in (0,1)$ *depending on* G, *we have*

$$\int_0^1 (k-t)^{-\beta} G(t)\, dt = a_k G(w_k) + r_k G''(\varphi_k).$$

NOTATION: Instead of $r_k G''(\varphi_k)$ we sall also write $R_k[G(.)]$.

REMARK: (see Lemmas 2.2 and 2.3).

$$a_k = \frac{1}{1-\beta}\{k^{1-\beta} - (k-1)^{1-\beta}\},$$

$$w_k = \frac{1}{(1-\beta)(2-\beta)a_k}\{k^{2-\beta} - (k-1)^{2-\beta} - (2-\beta)(k-1)^{1-\beta}\} \in (\;\,,1),$$

$$(2-\beta)^2 2r_k = \frac{k^{3-\beta}}{3-\beta} - \frac{(k-1)^{3-\beta}}{3-\beta} - (1-\beta)\frac{(k-1)^{1-\beta}k^{1-\beta}}{k^{1-\beta}-(k-1)^{1-\beta}}.$$

PROOF OF THEOREM 2.2: Observe, that the function H_k, defined on [0,1] by

$$H_k(t) = 2(t-w_k)^2(G(t) - G(w_k)) - 2(t-w_k)^1 G'(w_k) \text{ for } t \neq w_k,$$

$H_k(w_k) = G''(w_k)$, is continuous. This is trivial if $t \neq w_k$. If $t = w_k$ we have for $\tau \neq w_k$, by Taylor's formula,

$$G(\tau) = G(w_k) + G'(w_k)(\tau - w_k) + \frac{1}{2}G''(\eta_k)(\tau - w_k)^2,$$

η_k suitably taken genuinely between τ and w_k. We have $H_k(\tau) = G''(\eta_k)$, and the continuity of G'' at w_k implies that of H_k.

Now

$$G(t) = G(w_k) + G'(w_k)(t - w_k) + \frac{1}{2}H_k(t)(t-w_k)^2,$$

$$\int_0^1 (k-t)^{-\beta}G(t)\,dt = a_k G(w_k) + \frac{1}{2}\int_0^1 (k-t)^{-\beta}H_k(t)(t-w_k)^2\,dt$$
$$= a_k G(w_k) + r_k H_k(\tilde{\varphi}_k)$$

with suitable $\tilde{\varphi}_k \in (0,1)$. Well, if $\tilde{\varphi}_k = w_k$. put $\varphi_k = \tilde{\varphi}_k$. Otherwise, according to the above proof of continuity of H_k, there exists η_k gennuiely between $\tilde{\varphi}_k$ and w_k with $H_k(\tilde{\varphi}_k) = G_k''(\eta_k)$. Put $\varphi_k = \eta_k$.

The proof of the theorem is completed.

The state of the matter now is as follows. For $i = 1, 2, \ldots, N$ we have

$$f(x_i) = h^{1-\beta}\sum_{j=0}^{i-1} a_{i-j}K(x_i, (j + w_{i-j})h, \bar{y}((j+w_{i-j})h)) + E_{2i} + E_{1i}$$

with

$$E_{2i} = h^{1-\beta} \sum_{i=0}^{i-1} R_{i-j}[K(x_i, (j+.)h, \bar{y}((j+.)h))]$$

Replace $\bar{y}$ by its defining formula (see Theorem 2.1), write $\tilde{y}_j$ instead of y_j and omit the error terms E_{1i} and E_{2i}. You get the *nonlinear system of equations*

$$h^{1-\beta} \sum_{j=0}^{i-1} a_{i-j} K(x_i, (j+w_{i-j})h, (1-w_{i-j})\tilde{y}_j + w_{i-j}\tilde{y}_{j+1}) = f(x_i)$$

for $i = 1, 2, ..., N$
from which, if the value $\tilde{y}_0$ *is given, successively* $\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_N$ *can be uniquely calculated (because* $D_3 K > 0$*). If known or calculable by Lemma 1.2 one may put* $\tilde{y}_0 = y(0)$*. In case of sufficient smoothness* $\tilde{y}_0$ *from*

$$K(0, 0, \tilde{y}_0) = (1-\beta)h^{2\beta-2} f(h^2)$$

is a reasonable approximation to y(0) if h is small.

II.2.Preparatory lemmas. We shall present a series of lemmas required for the proof of the convergence of the discretization method. The important Lemma 2.1 on stability of a nonlinear system of difference inequalities is taken from Branca ([1] and [2]), whereas the other lemmas generalize lemmas stated by Branca for $\beta = 1/2$ only. For the sake of completeness and readability we reproduce in essence the proof Branca has provided in [1] for the complicated Lemma 2.1. *In these lemmas we shall define quantities* $a_k, b_k, c_k, d_k, e_k, r_k, q_k$ *that will alway have the some meaning.*

LEMMA 2.1. *Let* $B, K_1 \in \mathbb{R}^+, 2 \le p \in \mathbb{N}, q \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}$, *and let for* $i, j \in \mathbb{N}_0, j \le i, h \in \mathbb{R}^+$ *the real numbers* $\epsilon_i, A(i,j,h), B(i,j,h), C(i,j,h)$, $D(i,h) \ge 0$. *Assume, for all* $h = B/N, N \in \mathbb{N}$, *the conditions (i), (ii), (iii), (iv), (v), (vi) to be satisfied.*

(i) $\epsilon_i \le K_1 h^p$ *for* $i = 0, 1, \ldots, q$.

(ii) (α) *or* (β) *for* $i = q, q+1, \ldots, N-1$.

$(\alpha) \epsilon_{i+1} \le \epsilon_i$.

$(\beta)\epsilon_{i+1}\{1-\sum_{j=q}^{i} C(i,j,h)\epsilon_j\} \leq \sum_{j=q}^{i}\{A(i,j,h)+\epsilon_i B(i,j,h)\}\epsilon_j +$
$+D(i,h)$.

(iii) *There exists a number* $\Delta B \in (0,B]$ *and a constant* $K_2 \in \mathbb{R}^+$ *such that for all* $h \leq \Delta B$ *and* $i = q, q+1, \ldots, N-1$ *there is a quantity* $\delta(i,h) \in \mathbb{R}_0^+$ *with* (γ) *and* (δ).

$(\gamma) \sum_{j \in M_1(i,h)} A(i,j,h) \leq 1 - \delta(i,h)$,

$(\delta) \sum_{j \in M_2(i,h)} A(i,j,h) \leq K_2 \delta(i,h)$,

where

$M_1(i,h) = \{j | j \in \mathbb{N}, \max(q, i+1-\frac{\Delta B}{h}) \leq j \leq i\}$,

$M_2(i,h) = \{j | j \in \mathbb{N}, q \leq j < i+1-\frac{\Delta B}{h}\}$.

(iv) *There exists a constant* $K_3 \in \mathbb{R}^+$ *such that*
$h^{p-1}\sum_{j=q}^{i} B(i,j,h) \leq K_3\delta(i,h)$ *for* $i = q, q+1, \ldots, N-1$.

(v) *There exists a constant* $K_4 \in \mathbb{R}^+$ *such that*
$h^{p-1}\sum_{j=q}^{i} C(i,j,h) \leq K_4\delta(i,h)$ *for* $i = q, q+1, \ldots, N-1$.

(vi) *There exists a constant* $K_5 \in \mathbb{R}^+$ *such that*
$D(i,h) \leq K_5 h^p \delta(i,h)$ *for* $i = q, q+1, \ldots, N-1$.

Then there exist $h_0 \in (0, \Delta B], K \in \mathbb{R}^+, \lambda \in \mathbb{R}_0^+$ *such that*
$\epsilon \leq K e^{i\lambda h} h^p$ *for all* $h = B/N \leq h_0, i = 0, 1, \ldots, N$.

PROOF: The proof is by induction on $i \in \{q, \ldots, N\}$. Let $h = B/N \leq h_0$ with

$$\lambda = \max\{0, \ln(4K_2)/\Delta B\}, K = \max(K_1, 4K_5),$$

$$h_0 = \min\{\Delta B, \min(\frac{1}{K_3}\frac{1}{K_4})/(4Ke^{\lambda B})\}.$$

Assume that for some $i \in \{q, \ldots, N-1\}$ we have proved

$$\epsilon_j \leq K e^{j\lambda h} h^p \quad \text{for} \quad j = 0, 1, \ldots, i$$

(this is true by (i) for $i = q$). By (ii) we then have (α) or (β). In case of (α) we have

$$\epsilon_{i+1} \leq \epsilon_i \leq K e^{\lambda i h} h^p \leq K e^{\lambda(i+1)h} h^p.$$

In case (β) we have, using (iii) (δ) and the definition of λ,
$\sum_{j\in M_2(i,h)} A(i,j,h)e^{-\lambda(i+1-j)h} \le e^{-\lambda\Delta B}\sum_{j\in M_2(i,h)} A(i,j,h),$
$\le e^{-\lambda\Delta B}K_2\delta(i,h) \le \frac{1}{4}\delta(i,h),$
furthermore, using (iii)(γ),

$$\sum_{j=q}^{i} A(i,j,h)e^{-\lambda(i+1-j)h} \le \sum_{j\in M_1(i,h)} A(i,j,h) + \sum_{j\in M_2(i,h)} A(i,j,h)e^{-\lambda(i+1-j)h}$$

$$\le 1-\delta(i,h)+\frac{1}{4}\delta(i,h) = 1-\frac{3}{4}\delta(i,h).$$

By (vi), (v), $h \le h_0$ and the definition of h_0 we have

$$Kh^p e^{\lambda B} \le \sum_{j=q}^{i} B(i,j,h) \le Khe^{\lambda B}K_3\delta(i,h) \le \frac{1}{4}\delta(i,h),$$

$$Kh^p e^{\lambda B} \le \sum_{j=q}^{i} C(i,j,h) \le Khe^{\lambda B}K_4\delta(i,h) \le \frac{1}{4}\delta(i,h).$$

By (vi) we see that $D(i,h) \le \frac{1}{4}Kh^p\delta(i,h)$.

Hence, using (*) we get

$$\epsilon_{i+1}(1-\sum_{j=q}^{i} C(i,j,h)\epsilon_j) \ge \epsilon_{i+1}(1-\sum_{j=q}^{i} C(i,j,h)Ke^{\lambda jh}h^p)$$

$$\ge \epsilon_{i+1}(1-Khe^{\lambda B}\sum_{j=q}^{i} C(i,j,h) \ge \epsilon_{i+1}(1-\frac{1}{4}\delta(i,h))$$

and

$$\begin{aligned}
&\textstyle\sum_{j=q}^{i}(A(i,j,h)+\epsilon_i B(i,j,h))\epsilon_j + D(i,h)\\
&\le \textstyle\sum_{j=q}^{i} Ke^{\lambda jh}h^p\{A(i,j,h)+Ke^{\lambda jh}h^p)B(i,j,h)h^p\} + \frac{1}{4}Kh^p\delta(i,h)\\
&\le Kh^p e^{(i+1)\lambda h}h^p \textstyle\sum_{j=q}^{i}\{A(i,j,h)+Ke^{\lambda B}h^pB(i,j,h))e^{-\lambda(i+1-j)h}+\\
&\qquad\qquad +\tfrac{1}{4}e^{-\lambda(i+1)h}\delta(i,h)\}\\
&\le Kh^p e^{(i+1)\lambda h}\{1+(-\tfrac{3}{4}+\tfrac{1}{4}+\tfrac{1}{4})\delta(i,h)\} = Kh^p e^{(i+1)\lambda h}(1-\tfrac{1}{4}\delta(i,h)).
\end{aligned}$$

Now, $1-\frac{1}{4}\delta(i,h) > 0$ (this follows from the fact that because of $h \le \Delta B$ the index $i \in M_1(i,h)$ and therefore by (iii) (γ)$1-\delta(i,h) \ge A(i,i,h) \ge 0$).

Collecting our estimates and using (β) we find

$$\epsilon_{i+1}(1-\frac{1}{4}\delta(i,h)) \le \epsilon_{i+1}(1-\sum_{j=q}^{i} C(i,j,h)\epsilon_j) \le Kh^p e^{(i+1)\lambda h}(1-\frac{1}{4}\delta(i,h)),$$

hence $\epsilon_{i+1} \le Kh^p e^{(i+1)\lambda h}$.

The proof is completed

LEMMA 2.2. *For $k \in N, \beta \in (0,1)$ let* (as in Theorem2.2),

$$a_k = \int_0^1 (k-t)^{-\beta}\,dt, \quad w_k = \frac{1}{a_k}\int_0^1 (k-t)^{-\beta} t\,dt.$$

Then

(i) $a_k = \frac{1}{1-\beta}\{k^{1-\beta} - (k-1)^{1-\beta}\} > 0$ *for* $k \in \mathbf{N}$.

(ii) $w_k = \frac{1}{(1-\beta)(2-\beta)a_k}\{k^{2-\beta} - (k-1)^{2-\beta} - (2-\beta)(k-1)^{1-\beta}\} \in (0,1)$ *for* $k \in \mathbf{N}$.

(iii) $a_1 = \frac{1}{(1-\beta)}, \quad w_1 = \frac{1}{(2-\beta)}$.

PROOF: Trivial calculations.

LEMMA 2.3. *For $k \in \mathbf{N}, \beta \in (0,1)$ let (as in Theorem2.2),*

$$r_k = \frac{1}{2}\int_0^1 (k-t)^{-\beta}(t-w_k)^2\,dt$$

with w_k as in lemma 2.2 . Then

(i) $0 < r_k < \frac{1}{2}(k-1)^{-\beta}$ *for* $k \ge 2, 0 < r_1 < \frac{1}{2}(1-\beta)^{-1}$.

(ii) $0 < \sum_{k=2}^{i+1} r_k < \frac{1}{2(1-\beta)} i^{1-\beta}$ *for all* $i \in \mathbf{N}$.

(iii) $0 < r_k - r_{k+1} < \beta(k-1)^{-(1+\beta)}$ *for all* $k \ge 2, r_1 - r_2 > 0$.

(iv) $0 \le \sum_{k=2}^{i}(r_k - r_{k+1})(k-1) < \frac{\beta}{1-\beta} i^{1-\beta}$ *for all* $i \in \mathbf{N}$.

PROOF:

(i): For $k \ge 2$ we have $0 < r_k < \frac{1}{2}\int_0^1 (k-t)^{-\beta}\,dt < \frac{1}{2}(k-1)^{-\beta}$. Furthermore $0 < r_1 < \frac{1}{2}\int_0^1 (1-t)^{-\beta}\,dt = \frac{1}{2}.\frac{1}{1-\beta}$.

(ii): By (i) we have

$$0 < \sum_{k=2}^{i+1} r_k < \frac{1}{2}\sum_{k=2}^{i+1}(k-1)^{-\beta} < \frac{1}{2}\int_0^i s^{-\beta}\,ds = \frac{i^{1-\beta}}{2(1-\beta)}.$$

(iii): This part of the proof requires some effort. Verify by elementary calculation that for all $k \in \mathbb{N}$ (2.2), (2.2') and (2.2") are valid.

$$(2.2)\quad (2-\beta)^2 2r_k = \frac{k^{3-\beta}}{3-\beta} - \frac{(k-1)^{3-\beta}}{3-\beta} - (1-\beta)\frac{(k-1)^{1-\beta}k^{1-\beta}}{k^{1-\beta}-(k-1)^{1-\beta}},$$

$$(2.2')\qquad 2r_k = \int_0^1 \frac{t^2}{(k-t)^\beta}\,dt - \{\int_0^1 \frac{dt}{(k-t)^\beta}\}^{-1}\{\int_0^1 \frac{t\,dt}{(k-t)^\beta}\}^2,$$

$$(2.2")\qquad (2-\beta)^2 2(r_k - r_{k+1}) = U_k + V_k,$$

$$U_k = \frac{1}{3-\beta}\{-(k+1)^{3-\beta} + 2k^{3-\beta}(k-1)^{3-\beta}\}$$

$$V_k = -(1-\beta)\{\frac{(k-1)^{1-\beta}k^{1-\beta}}{k^{1-\beta}-(k-1)^{1-\beta}} - \frac{k^{1-\beta}(k+1)^{1-\beta}}{(k+1)^{1-\beta}-k^{1-\beta}}\}.$$

We now show (a), (b), (c). Part (c) will be very complicated.

(a) $r_k - r_{k+1} < \beta(k-1)^{-(1+\beta)}$ for $k \geq 2$.

(b) $r_1 - r_2 > 0$.

(c) $r_k - r_{k+1} > 0$ for $k \geq 2$.

(a): (2.2') implies

$$2r_k < \frac{1}{3}(k-1)^{-\beta} - (k-1)^\beta(\frac{1}{2}k^{-\beta})^2,$$

$$2r_{k+1} > \frac{1}{3}(k+1)^{-\beta} - (k+1)^\beta(\frac{1}{2}k^{-\beta})^2,$$

$$2(r_k - r_{k+1}) < \frac{1}{3}\{(k-1)^{-\beta} - (k+1)^{-\beta}\} + \frac{1}{4}k^{-2\beta}\{(k+1)^\beta - (k-1)^\beta\},$$

and from the strict concavity of the functions $x \longrightarrow -x^{-\beta}$ and $x \longrightarrow x^\beta$, $x > 0$, we deduce that the right hand side is

$$< \frac{2}{3}\beta(k-1)^{-(1+\beta)} + \frac{2}{4}k^{-2\beta}(k-1)^{\beta-1},$$

hence $r_k - r_{k+1} < \beta(k-1)^{-(1+\beta)}$.

(b): (2.2") implies

$$(2-\beta)^2 2(r_1 - r_2) = \frac{1}{3-\beta}\cdot\frac{2^{1-\beta}}{2^{1-\beta}-1}g(\beta)$$

with $g(x) = 9 - 2^{3-x} - 2^x - 4x + x^2$. Consider this function for $0 \le x \le 1$. From $g''(x) = -(\ln 2)^2 2^{3-x} - (\ln 2)^2 2^x + 2 \le -(\ln 2)^2.(4+1) + 2 < 0$ we decuce that g is strictly concave. This fact and $g(0) = g(1) = 0$ imply $g(x) > 0$ for $0 < x < 1$, hence $g(\beta) > 0$.

(c): Let $k \ge 2$ and consider (2.2"). By Taylor's theorem

$$(k+\eta)^{3-\beta} - k^{3-\beta} < \eta(3-\beta)k^{2-\beta} + \binom{3-\beta}{2}k^{1-\beta} + \eta\binom{3-\beta}{3}k^{-\beta}$$

for $\eta = 1$ and for $\eta = -1$. Hence $U_k > -(2-\beta)k^{1-\beta}$, $(2-\beta)^2 2(r_k - r_{k+1}) > k^{1-\beta}W_k$, where $W_k = \beta - 2 + k^{\beta-1}U_k =$

$$\frac{-k^{1-\beta}(k+1)^{1-\beta} + \beta(k-1)^{1-\beta}(k+1)^{1-\beta} + (2-\beta)k^{2-2\beta} - k^{1-\beta}(k-1)^{1-\beta}}{(k^{1-\beta} - (k-1)^{1-\beta})((k+1)^{1-\beta} - k^{1-\beta})}$$

The denominator of the last expression is positive. With

$$X_k = k^{\beta-1}(k^{1-\beta} - (k-1)^{1-\beta})((k+1)^{1-\beta} - k^{1-\beta})(2-\beta)^2 2(r_k - r_{k+1})$$

the above estimate can be rewriten as

$$X_k > k^{1-\beta}\{-(k+1)^{1-\beta} + 2k^{1-\beta} - (k-1)^{1-\beta}\} - \beta(k^2)^{1-\beta} - (k^2-1)^{1-\beta} = Y_k,$$

and it remains to show that $Y_k > 0$. Now $k^{-1} \le \frac{1}{2} < 1$, and we can, by Taylor series development, write

$$Y_k = k^{2-2\beta}\sum_{n=1}^{\infty}\gamma_n k^{-2n} \quad \text{with} \quad \gamma_n = \binom{1-\beta}{n}\beta(-1)^n - 2\binom{1-\beta}{2n}.$$

By induction we show that all $\gamma_n \ge 0$ (and hence $Y_k \ge 0$ as wanted). Obviously $\gamma_1 - 0$. Assume that $\gamma_n \ge 0$ has been proved for an index $n \in \mathbb{N}$. Then

$(n+1)(2n+1)\gamma_{n+1} = \gamma_n(n-1+\beta)(2n+1) - \binom{1-\beta}{2n}(2-\beta)(1-\beta) \ge -\binom{1-\beta}{2n}(2-\beta)(1-\beta)$ which is $> o$ because the binomial coefficient's product expansion consists of one positive factor and $2n-1$ negative factors.

The proof of (c), and thus of (iii), is completed.

(iv): Trivial for $i = 1$ (then the sum is empty). For $i \ge 2$ we use (iii) and find

$$0 < \sum_{k=2}^{i}(r_k - r_{k+1})(k-1) < \sum_{k=2}^{i}\beta(k-1)^{-\beta}$$

$$< \beta\int_0^{i-1} s^{-\beta}\,ds < \frac{\beta}{1-\beta}i^{1-\beta}.$$

LEMMA 2.4. *For $k \in \mathbb{N}, \beta \in (0,1)$ let*

$$b_k = a_k w_k, \quad c_k = a_k - b_k = (1-w_k)a_k.$$

Then we have

(i) $0 < b_k < \frac{1}{2}(k-1)^{-\beta}$ *for* $k \geq 2, b_1 = \frac{1}{(1-\beta)(2-\beta)} > 0$.
(ii) $0 < b_k - b_{k+1} < \frac{5}{2}\beta k^{-(1+\beta)}$ *for* $k \geq 2, b_1 - b_2 > 0$.
(iii) $0 \leq \sum_{k=2}^{i} b_k \leq \frac{1}{2(1-\beta)}(i-1)^{1-\beta}$ *for* $i \in \mathbb{N}$
(iv) $0 < c_k < k^{-\beta}$ *for* $k \in \mathbb{N}, c_1 = 1/(2-\beta)$.
(v) $0 < c_k - c_{k+1} < \frac{5}{2}k^{-(1+\beta)}$ *for* $k \in \mathbb{N}$
(vi) $0 \leq \sum_{k=1}^{i-1} c_k \leq \frac{1}{1-\beta}(i-1)^{1-\beta}$ *for* $i \in \mathbb{N}$
(vii) $b_1 - b_2 - c_1 \geq 0$ *for* $\beta \in [2 - \frac{\ln 3}{\ln 2}, 1)$, $\quad b_1 - b_2 - c_1 < 0$ *for* $\beta \in (0, 2 - \frac{\ln 3}{\ln 2})$.

REMARK: (vii) is the place where we must restrict β.

PROOF: (i): b_1 by direct calculation. For $k \geq 2$ we have

$$0 < b_k = \int_0^1 (k-t)^{-\beta} t\,dt < \frac{1}{2}(k-1)^{-\beta}.$$

(ii): The positivity of the difference is trivial. For $k \geq 2$ we have

$$b_k - b_{k+1} = \int_0^1 t\{(k-t)^{-\beta} - (k+1-t)^{-\beta}\}\,dt$$

$$< \frac{1}{1-\beta}\{k^{1-\beta} - (k-1)^{1-\beta} - (k+1)^{1-\beta} + k^{1-\beta}\}.$$

By appropriately estimating the second order error term in the Taylor formula for $(k \pm 1)^{1-\beta} - k^{1-\beta}$ we obtain

$$\leq \frac{\beta}{2}((k-1)^{-(1+\beta)} + k^{-(1+\beta)}) \leq \frac{\beta}{2}k^{-(1+\beta)}(2^{(1+\beta)} + 1) < \frac{5\beta}{2}k^{-(1+\beta)}.$$

(iii): Trivial for $i = 1$. If $i \geq 2$ use (i) and estimate the sum by an integral.

(iv): For $k = 1$ by direct calculation:

$$c_1 = (1 - w_1)a_1 = \frac{1}{(2-\beta)} = w_1 \in (1/2, 1).$$

For $k \geq 2$ we have

$$0 < c_k = \int_0^1 (1-t)(k-t)^{-\beta}\, dt < \int_0^1 (1-t)(k-1)^{-\beta}\, dt = \frac{1}{2}(k-1)^{-\beta} \leq k^{-\beta}.$$

(v): We have

$$0 < c_k - c_{k+1} = \int_0^1 (1-t)\{(k-t)^{-\beta} - (k+1-t)^{-\beta}\}\, dt$$
$$< \frac{1}{1-\beta}\{k^{1-\beta} - (k-1)^{1-\beta} - (k+1)^{1-\beta} + k^{1-\beta}\},$$

and, compare proof of (ii), for $k \geq 2$ this is $< \frac{5}{2}\beta k^{-(1+\beta)} < \frac{5}{2}k^{-(1+\beta)}$. For $k = 1$ we have $c_1 - c_2 < c_1 < 1 < 5/2$.

(vi): Evident for $i = 1$. If $i \geq 2$ use (iv) and estimate the sum by an integral.

(vii): Consequence of $(1-\beta)(2-\beta)(b_1 - b_2 - c_1) = 3 - 2^{2-\beta}$.

LEMMA2.5. *Let $d_k = c_k + b_{k+1}, e_k = d_k - d_{k+1}$ for $k \in \mathsf{N}, \beta \in (0,1)$. Then*

(i) $\frac{3}{4}k^{-\beta} < d_k < \frac{3}{2}k^{-\beta}$ *for* $k \in \mathsf{N}$.

(ii) $0 < e_k < 5k^{-(1+\beta)}$ *for* $k \in \mathsf{N}$.

(iii) $0 \leq \sum_{k=1}^{i-1} e_k < d_1$ *for* $i \in \mathsf{N}$

(iv) $0 \leq \sum_{k=1}^{i-1} e_k + b_1 - b_2 - c_1 < b_1 - \frac{3}{4}i^{-\beta}$ *for* $i \in \mathsf{N}, \beta \in [2 - \frac{\ln 3}{\ln 2}, 1)$.

(v) $0 \leq \sum_{k=1}^{i-1} ke_k \leq 5.\frac{1}{1-\beta}(i-1)^{1-\beta}$ *for* $i \in \mathsf{N}$.

PROOF:

(i): By Lemma 2.4 (i) and (iv) we get $d_k < \frac{3}{2}k^{-\beta}$. On the other hand

$$d_k = \int_0^1 (1-t)(k-t)^{-\beta}\, dt + \int_0^1 t(k+1-t)^{-\beta}\, dt > \frac{1}{2}(k^{-\beta} + (k+1)^{-\beta})$$

$$> \frac{1}{2}k^{-\beta}(1+2^{-\beta}) > \frac{3}{4}k^{-\beta}.$$

(ii): Using Lemma 2.4 (ii) and (v) we find

$$0 < e_k = c_k - c_{k+1} + b_{k+1} - b_{k+2}$$
$$< \frac{5}{2}\{k^{-(1+\beta)} + \beta(k+1)^{-(1+\beta)}\} < 5k^{-(1+\beta)}.$$

(iii): Trivial for $i = 1$. For $i \geq 2$, using (i) and (ii), we see that

$$0 < \sum_{k=1}^{i-1} e_k = d_1 - d_i < d_1.$$

(iv): From (ii) and Lemma 2.4 (vii) follows nonnegativity. Using (i) we find

$$\sum_{k=1}^{i-1} e_k + b_1 - b_2 - c_1 = b_1 - d_i < b_1 - \frac{3}{4}i^{-\beta}.$$

(v): Trivial for $i = 1$. Incase $i \geq 2$ use (ii) and estimate the sum by an integral.

LEMMA 2.6. *Let $q_k = \int_0^1 (1-t)t(k-t)^{-\beta}\,dt$ for $k \in \mathbb{N}, \beta \in (0,1)$. Then*

(i) $0 < q_k$ *for* $k \in \mathbb{N}, q_k < \frac{1}{4}(k-1)-\beta$ *for* $k \geq 2$.

(ii) $0 < q_k - q_{k+1} < \frac{1}{4}\beta(k-1)^{-(1+\beta)}$ *for* $k \geq 2$

(iii) $0 < \sum_{k=1}^{i} q_k < \frac{1}{4(1-\beta)}i^{1-\beta}$ *for* $i \in \mathbb{N}$

(iv) $0 \leq \sum_{k=2}^{i}(q_k - q_{k+1})(k-1) \leq \frac{\beta}{4(1-\beta)}(i-1)^{1-\beta}$ *for* $i \in \mathbb{N}$.

(v) $q_1 = \frac{1}{(2-\beta)(3-\beta)}$.

PROOF: We shall use the fact that the function $t \longrightarrow (1-t)t, 0 \leq t \leq 1$, is stricly concave, nonnegative and attains its maximum $1/4$ at $t = 1/2$.

(i) Positivity is evident. For $k \geq 2$ we have $q_k < \frac{1}{4}(k-1)^{-\beta}$.

(ii) Positivity is evident. By the above mentioned fact we obtain

$$q_k - q_{k+1} < \frac{1}{4(1-\beta)}\{k^{1-\beta} - (k-1)^{1-\beta} - (k+1)^{1-\beta} + k^{1-\beta}\},$$

and this is (compare the proof of Lemma 2.4 (i))

$$\leq \frac{1}{4}\cdot\frac{\beta}{2}((k-1)^{-(1+\beta)} + k^{(1+\beta)}) \leq \frac{\beta}{4}(k-1)^{-(1+\beta)}.$$

(iii) Positivity is trivial because of (i). Furthermore,

$$\sum_{k=1}^{i} q_k < \frac{1}{4}\sum_{k=1}^{i}\int_0^1 (k-t)^{-\beta}\,dt = \frac{1}{4(1-\beta)} i^{1-\beta}.$$

(iv) Nonnegativity is trivial by (ii). For $i = 1$ the sum is empty, and the right side inequality is trivial. For $i \geq 2$ use (ii) and estimate the sum by an integral.

(v) by calculation.

Just for reference purposes we state a well-known lemma on finite sums.

LEMMA 2.7. (partial summation): *Let $i \in \mathbb{N}, u_j, v_j \in \mathbb{R}$ for $j = 0, 1, \ldots, i-1$. Then*

$$\sum_{j-0}^{i-1} u_j v_j = v_{i-1}\sum_{j=0}^{i-1} u_j + \sum_{j=0}^{i-2}(v_j - v_{j+1})\sum_{k=0}^{j} u_k.$$

LEMMA 2.8. *Let $\beta \in [0.2118, 2 - \frac{\ln 3}{\ln 2})$ and $\gamma(\beta) = 2^{2-\beta} - 3$. Then (with the quantities b_j, c_j, e_j defined in Lemmas 2.4 and 2.5)*
(i) $e_k - \gamma(\beta)e_{k-1} > 0$ for $k = 2, 3, 4, \ldots,$
(ii) $(1 + \gamma(\beta))b_1 - b_2 - c_1 = 0$.

REMARK: (ii) implies $b_2 + c_1 - b_1 >$ for $\beta \in [0.2118, 2 - \frac{\ln 3}{\ln 2})$. The bound 0.2118 is rather sharp : $e_2 - \gamma(0.2117)e_1 < 0$.

PROOF: (ii) by calculation.
(ii) for $k = 2$: We have

$$(e_2 - \gamma(\beta)e_1)(1-\beta)(2-\beta) = 2^{2-\beta}h(\beta),$$

$$h(x) = 9.3^{-x} - 16.2^{-x} + 9 - 2.2^{x}.$$

We shall show that $h(x) > 0$, hence $e_2 - \gamma(\beta)e_1 > 0$, for $x \in [0.2118, 2 - \frac{\ln 3}{\ln 2})$. We have $h'(x) = (\tilde{h}(x) - 2\ln 2)2^x$, with
$\tilde{h}(x) = -9\ln 3.6^{-x} + 16\ln 2.4^{-x}, \tilde{h}'(x) = 9\ln 3.\ln 6.6^{-x} - 16\ln 2.\ln 4.4^{-x}$.
The continuous function $\tilde{h}'$ possesses exactly one real zero, namely

$$x^* = \ln \frac{16\ln 2.\ln 4}{9\ln 3.\ln 6}.\frac{1}{\ln 4 - \ln 6} \in (0.3496, 0.3497),$$

and from (check, e.g., by aid of a pocket computer)

$$\tilde{h}'(0) > 2.3415 > 0, \tilde{h}'(1) < -0.8909 < 0$$

we conclude that $\tilde{h}$ is strictly increasing on $(-\infty, x^*)$, strictly decreasing on (x^*, ∞). In particular

$$\tilde{h}(x) \geq \tilde{h}(0.2) > 1.49 > 2\ln 2 \text{for} \quad x \in [0.2, x^*],$$

$$\tilde{h}(x) \geq \tilde{h}(0.5) > 1.50 > 2\ln 2 \quad \text{for} \quad x \in [x^*, 0.5].$$

Hence the continuous function h' has no zero in [0.2,0.5], and because of $h'(0.3) > 0.19 > 0$ is positive in this interval.

Using $h(0.2118) > 8.10^{-6} > 0$ we see that $h(x) > 0$ for all $x \in [0.2118, 0.5]$, the more so for $x \in [0.2118, 2 - \frac{\ln 3}{\ln 2})$.
Notice, please, that $h(0.2117) < -5.10^{-6} < 0$.

(i) for $k \geq 3$: Develop into a Taylor series in powers of $1/k$ and separate odd and even powers. You obtain

$$(e_k - \gamma(\beta)e_{k-1})(1-\beta)(2-\beta)k^{\beta-2}$$

$$= \sum_{\lambda=1}^{\infty} \binom{2-\beta}{2\lambda+1}(\frac{1}{k})^{2\lambda+1}(-2^{2\lambda+1}+2)(4-2^{2-\beta})$$

$$+\sum_{\lambda=1}^{\infty} \binom{2-\beta}{2\lambda+2})(\frac{1}{k})^{2\lambda+2}(-2^{2\lambda+2}+4)(2^{2-\beta}-2)$$

$$= \sum_{\lambda=1}^{\infty} \binom{2-\beta}{2\lambda+1}(\frac{1}{k})^{2\lambda+1}(-1)(2^{2\lambda+1}-2)\gamma_\lambda(k),$$

$$\gamma_\lambda(k) = 4 - 2^{2-\beta} + \frac{1-\beta-2\lambda}{2\lambda+2}.\frac{1}{k}.2(2^{2-\beta}-2).$$

Always remembering the restriction of β and $k \geq 3$ we now consider the indices $\lambda \geq 2$. By caculation we find $4-2^{2-\beta}+\gamma_\lambda(k) \geq 8-2.2^{2-\beta}+\frac{4}{3} > 0.1 > 0$, on the other hand, $\gamma_\lambda(k) < 4 - 2^{2-\beta}$. Hence

$$|\gamma_\lambda(k)| < 4 - 2^{2-\beta} \quad \text{for} \quad \lambda \geq 2, k \geq 3$$

Furthermore,

$$|\binom{2-\beta}{2\lambda+1}| \leq |\binom{2-\beta}{5}| = \frac{1}{120}(2-\beta)(1-\beta)\beta(1+\beta)(2+\beta).$$

Because in any case $\beta < 1/2$ we obtain

$$|\sum_{\lambda=2}^{\infty}\binom{2-\beta}{2\lambda+1}(-1)(\frac{1}{k})^{2\lambda+1}(2^{2\lambda+1}-2)\gamma_\lambda(k)| \leq |\binom{2-\beta}{5}|\sum_{\lambda=2}^{\infty}(\frac{2}{k})^{2\lambda+1}(4-2^{2-\beta})$$

$$\leq \frac{1}{6}(2-\beta)(1-\beta)\beta.\frac{3}{16}(4-2^{2-\beta})(\frac{2}{k})^5.(1-\frac{4}{k^2})^{-1}$$

Taking into account also the index $\lambda = 1$ in the series and the fact that (remember the resstriction of β) $3 \leq 212 - \beta < 3.454$ we find

$$(\epsilon_k - \gamma(\beta)\epsilon_{k-1})(1-\beta)(2-\beta)k^{\beta-2} \geq$$

$$\frac{(2-\beta)(1-\beta)\beta}{6}\{6.(\frac{1}{k})^3(4-2^{2-\beta}-\frac{1+\beta}{2k}(2^{2-\beta}-2))-\frac{3}{16}(4-2^{2-\beta})(\frac{2}{k})^5\frac{k^2}{k^2-4}\}.$$

Here (pay attention to $k \geq 3$)

$$k^3\{...\} \geq 6\{(4-2^{2-\beta}) - \frac{3/2}{6}(2^{2-\beta}-2)\} - \frac{3}{2}(4-2^{2-\beta})\frac{1}{k^2-4}$$

This completes the proof of Lemma 2.8.

LEMMA 2.9. *If $\beta \in [0.2118, 2 - \frac{\ln 3}{\ln 2})$ then $d_k - \gamma(\beta)d_{k-1} > \frac{1}{6}k^{-\beta}$ for $2 \leq k \in \mathbb{N}$. Take here $\gamma(\beta)$ as in Lemma 2.8.*

PROOF: We begin with the case $k = 2$. We calculate $(d_2 - \gamma(\beta)d_1)(1-\beta)(2-\beta) = 2^{-\beta}h(\beta)$ with $h(x) = 9.(3/2)^{-x} - 16.2^{-x} - 5.2^{-x} + 12$. We have $h'(x) = (\tilde{h}(x) - 5\ln 2).2^x$ with $\tilde{h}(x) = -9\ln\frac{3}{2}.3^{-x} + 16\ln 2.2^{-2x}$, $\tilde{h}'(x) = 9\ln\frac{3}{2}.\ln 3.3^{-x} - 32(\ln 2)^2.2^{-2x}$.

The continuous function $\tilde{h}'$ has exactly one real zero, namely

$$x^* = (\ln\frac{4}{3})^{-1}.\ln\{\frac{32}{9}.\frac{(\ln 2)^2}{\ln(3/2).\ln 3}\} > 4,$$

and $\tilde{h}'(0) < -11$. Hence $\tilde{h}$ is strictly decreasing on $[0.2118, 2-\frac{\ln 3}{\ln 2})$, and in this interval $\tilde{h}(x) > \tilde{h}(2-\frac{\ln 3}{\ln 2}) > 3.9 > 5\ln 2$, hence h' positive, h strictly increasing, $h(x) \geq h(0.2118) > 0.6 > 1/6$.

Now, let $k \geq 3$. Then

$$(d_k - \gamma(\beta)d_{k-1})(1-\beta)(2-\beta)k^{\beta-2}$$
$$= (1+\tfrac{1}{k})^{2-\beta} - (2+\gamma(\beta)) + (1+2\gamma(\beta))(1-\tfrac{1}{k})^{2-\beta} - \gamma(\beta)(1-\tfrac{2}{k})^{2-\beta}.$$
$$= (1-\beta)(2-\beta)(\tfrac{1}{k})^2(1-\gamma(\beta)) + \textstyle\sum_{j=3}^{\infty}\binom{2-\beta}{j}(\tfrac{1}{k})^j\{1+(-1)^j-(-)^j\gamma(\beta)(2^j-2)\}.$$

From $\gamma(\beta) < 0.454 < 1/2$ (see proof of Lemma 2.8), $\{..\} = 2-\gamma(\beta)(2^j-2)$ for $j \in 2\mathbb{N}+2, = \gamma(\beta)(2^j-2)$ for $j \in 2\mathbb{N}+1$ we infer that $|\{...\}| < 2^{j-1}$ for all $j \geq 3$. It follows (observe $2-(\ln 3)/(\ln 2) < 1/2$) that

$$|\sum_{j=3}^{\infty}\ldots| < \frac{(2-\beta)(1-\beta)\beta}{6}.\frac{1}{2}\sum_{j=3}^{\infty}(\frac{2}{k})^j == \frac{2}{3}.\frac{(2-\beta)(1-\beta)\beta}{k^3}.\frac{1}{1-\frac{2}{k}},$$

$$d_k - \gamma(\beta)d_{k-1} > k^{-\beta}\{1-\gamma(\beta) - \frac{2\beta}{3(k-2)}\} > k^{-\beta}(1-\frac{1}{2}-\frac{1}{3}) = \frac{1}{6}k^{-\beta}.$$

The proof is completed.

II.3 The Convergence Theorem. For approximate solution of the integral equation (2.1),

$$\int_0^1 (x-t)^{-\beta}K(x,t,y(t))\,dt = f(x), 0 \leq x \leq B,$$

we use *the discretization method* (DM).

(DM) *Take* $N \in \mathbb{N}, x_j = jh$ *for* $j = 0,1,\ldots,N$.
$a_k - \int_0^1 (k-t)^{-\beta}\,dt$ *and* $w_k = \frac{1}{a_k}\int_0^l t(k-t)^{-\beta}\,dt$ *for* $k = 1,2,\ldots,N$.
If $y(0)$ *is known or can be computed from*
$K(0,0,\tilde{y}_0) = (1-\beta)\lim_{x\to 0} x^{\beta-1}f(x)$,
put $\tilde{y}_0 = y(0)$ *else compute* $\tilde{y}_0$ *from* $K(0.0,\tilde{y}_0) = (1-\beta)h^{2(\beta-1)}f(h^2)$.
Then compute $\tilde{y}_i$ *successively for* $i = 1,2,\ldots,N$ *from*

(rec) $h^{1-\beta}\sum_{j=0}^{i-1} a_{i-j}K(x_i,(j+w_{i-j})h,(1-w_{i-j})\tilde{y}_j + w_{i-j}\tilde{y}_{j+1}) = f(x_i)$.

REMARK: For motivation of DM see II.1 . For integral-free expressions for a_k and w_k see Lemma 2.2 . The label (rec) means "recursion".

THEOREM 2.3. (Convergence of (DM)): *Let*

$$B \in \mathbb{R}^+, \beta \in [0.2118, 1), T = \{(x,t) | (x,t) \in \mathbb{R}^2, 0 \le t \le x \le B\},$$

and let y be a solution of the integral equation (2.1). Assume that the condition (i), (ii), (iii) *are satisfied .*

(i) $y \in C^3[0, B]$.

(ii) $K \in C^1(T \times \mathbb{R}), D_3K$ *is on* $T \times \mathbb{R}$ *bounded and Lipschitz continuous with respect to its first argument,* D_3D_3K, D_3D_2K *and* D_2D_2K *exist and are on* $T \times R$ *bounded and Lipschitz-continuous with respect to all three arguments.*

(iii) *There exists a constant* $K_0 \in \mathbb{R}^+$ *such that*

$$D_3K(x,t,z) \ge K_0 > 0 \quad \textit{for all} \quad (x,t,z) \in T \times \mathbb{R}.$$

Then there exists a constant $M_0 \in \mathbb{R}^+$ *and an index* $N_0 \in \mathbb{N}$ *such that, if* $N_0 \le N$, *we have*

$$|y(x_j) - \tilde{y}_j| \le M_0 h^2 \quad \textit{for} \quad j = 0, 1, \ldots, N.$$

REMARK (A): The global conditions (ii) and (iii) can be replaced by local ones.

(ii') There exists $\delta \in \mathbb{R}^+$ such that $K \in C^3(W_\delta)$ with $W_\delta = \{(x,t,z) | (x,t,z) \in T \times \mathbb{R}, |z - y(x)| \le \delta\}$.

(iii') $D_3K(x, x, y(x)) \ne 0$ for $0 \le x \le B$.

Branca's proof of his corresponding Theorem 2.3 in [1] can be carried over to our more general case.

REMARK (B): The conditions of Theorem 2.3 are differet from (but not contradictory to) those of Theorem 1.5 . You may unite the two sets of conditions if you want.

REMARK (C): The method (DM) is well defined. Because of (iii) the values $\tilde{y}_i$ can be computed uniquely.

REMARK (D): For the following proof observe and keep in mind: (ii) implies Lipschitz-continuity of D_3K with respect to all three arguments; (i), (ii) and the compactness of T imply $\tilde{K} \in C^1(T)$ and Lipschitz continuity of $\tilde{K}$, where $\tilde{K}(x,t) = K(x,t,y(t)),(x,t) \in T$.

The Proof of the Theorem 2.3 is based on Lemma 2.1 applied to

$$\epsilon_j = \max\{|y(x_k) - \tilde{y}_k||k = 0,1,\dots,j\}.$$

The whole proof is very very long and complicated. We divide it into two main parts:

(A) *Proof for* $2 - \frac{\ln 3}{\ln 2} \le \beta < 1$. (B) *Proof for* $0.2118 \le \beta < 2 - \frac{\ln 3}{\ln 2}$.

Each of these parts is subdivided into three subparts:

(α) Derivation of the difference inequality.

(β) Estimation of coefficients.

(γ) Check of the conditions of Lemm 2.1.

Subpart (Aα): For abbreviation let

$f_j = f(x_j), y_j = y(x_j)$ for $j = 0,1,\dots,N$,

$\bar{y}(t) = \frac{x_{j+1}-t}{h} y_j + \frac{t-x_j}{h} y_{j+1}$ for $x_j \le t \le x_{j+1}, j = 0,1,\dots,N-1$,

$\tilde{y}(t) = \frac{x_{j+1}-t}{h} \tilde{y}_j + \frac{t-x_j}{h} \tilde{y}_{j+1}$ for $x_j \le t \le x_{j+1}, j = 0,1,\dots,N-1$,

For $i = 1,2,\dots,N$ we then have, with

$$(2.3) \quad \tilde{Q}_i = h^{\beta-1} \int_0^{x_k} \frac{K(x_i,t,y(t)) - K(x_i,t,\bar{y}(t)}{(x_i-t)^\beta} dt \quad \text{for} \quad i = 1,2,\dots,N,$$

by collocation, appropriate decomposition and substitution,

$$f_i = \sum_{j=0}^{i-1} h. \frac{K(x_i,(j+\eta)h,\bar{y}((j+\eta)h)}{((i-j-\eta)h)^\beta} d\eta + h^{1-\beta}\tilde{Q}_i.$$

Hence, applying Theorem 2.2 (one-point Gauss quadrature), we get

$$f_i = h^{1-\beta} \sum_{j=0}^{i-1} \{a_{i-j} K(x_i,(j+w_{i-j})h,\bar{y}((j+w_{i-j})h)$$
$$+R_{i-j}[K(x_i,(j+.)h,\bar{y}((j+.)h)]\} + h^{1-\beta}\tilde{Q}_i.$$

Considering recursion (rec) of (DM) we see that

$$0 = \sum_{j=0}^{i-1} \{a_{i-j} K(x_i, (j+w_{i-j})h, \tilde{y}((j+w_{i-j})h) - K(x_i, (j+w_{i-j})h, \bar{y}((j+w_{i-j})h)\}$$

$$- \sum_{j=0}^{i-1} R_{i-j}[K(x_i, (j+.)h, \bar{y}((j+.)h)] + \tilde{Q}_i$$

Now, define

$$\tilde{\epsilon}_k = \tilde{y}_k - y_k \quad \text{for} \quad k = 0, 1, 2, \ldots, N. \tag{2.4}$$

Then, with $\xi_{k,j}$ suitably chosen between

$$\bar{y}((j+w_{k-j})h) \quad \text{and} \quad \tilde{y}((j+w_{k-j})h),$$

$$\xi_{k,j} = (1-w_{k-j})y_j + w_{k-j}y_{i+1} + \Theta_{k,j}\{(1-w_{k-j})\tilde{\epsilon}_j + w_{k-j}\tilde{\epsilon}_{j+1}\}, \Theta_{k,j} \in (0,1),$$

$$K_y^{i,j} = D_3 K(x_i, (j+w_{i-j})h, \xi_{i,j}, \tag{2.5}$$

$$\tilde{S}_i = \sum_{j=0}^{i=1} R_{i-j}[K(x_i, (j+.)h, \bar{y}((j+.)h)))] \tag{2.6}$$

we have

$$0 = \sum_{j=0}^{i-1} a_{i-j} K_y^{i,j} \{(1-w_{i-j})\tilde{\epsilon}_j + w_{i-j}\tilde{\epsilon}_{j+1}\} - \tilde{S}_i - \tilde{Q}_i,$$

and using

$$b_k = a_k w_k, \quad c_k = a_k(1-w_k) = a_k - b_k \quad \text{for} \quad k = 1, \ldots, N, \tag{2.7}$$

we obtain, for $i = 1, 2, \ldots, N$,

$$0 = c_i K_y^{i,0}\tilde{\epsilon}_0 + \sum_{j=1}^{i-1}\{c_{i-j}K_y^{i,j} + b_{i-j+1}K_y^{i,j-1}\}\tilde{\epsilon}_j + b_1 K_y^{i,i-1}\tilde{\epsilon}_i - \tilde{S}_i - \tilde{Q}_i. \tag{2.8}$$

Now, let $i \in \{1, 2, \ldots, N-1\}$, subtract equation (2.8) from the corresponding equation with index i replaced by index $i+1$, and solve for $b_1 K_y^{i+1,i}$. Defining quantities

$$Q_k = \tilde{Q}_{k+1} - \tilde{Q}_k, S_k = \tilde{S}_{k+1} - \tilde{S}_k, \tag{2.9}$$
$$P_k = (c_{k+1}K_y^{k+1,0} - c_k K_y^{k,0})\tilde{\epsilon}_0$$

for $k = 1, \ldots, N-1$, you obtain (after rearrangement)

(2.10)
$$\begin{aligned} b_1 K_y^{i+1,i}\tilde{\epsilon}_{i+1} = \sum_{j=1}^{i-1}\{ & e_{i-j}K_y^{i+1,j} + d_{i-j}(K_y^{i+1,j} - K_y^{i,j}) \\ & + b_{i+1-j}(K_y^{i,j-1} - K_y^{i,j}) + b_{i+2-j}(K_y^{i+1,j} - K_y^{i+1,j-1})\}\tilde{\epsilon}_y \\ & + \{(b_1 - b_2 - c_1)K_y^{i+1,i} + (b_1 - b_2)(K_y^{i+1,i-1} - K_y^{i+1,i}) \\ & + b_1(K_y^{i,i-1} - K_y^{i+1,i-1})\}\tilde{\epsilon}_i - (P_i - Q_i - S_i). \end{aligned}$$

Here we have used the quantities

$$\begin{aligned} d_k &= c_k + b_{k+1} \quad \text{for} \quad k = 1, \ldots, N-1 \\ e_k &= d_k - d_{k+1} \quad \text{for} \quad k = 1, \ldots, N-2. \end{aligned} \tag{2.11}$$

By aid of Lemmas 2.4 and 2.5 (in particular Lemma 2.4 (vii): $b_1 - b_2 - c_1 \geq 0$ where we need $2 - \frac{\ln 3}{\ln 2} \leq \beta < 1$) we estimate

$$\begin{aligned} b_1 K_y^{i+1,i}|\tilde{\epsilon}_{i+1}| \leq \sum_{j=1}^{i-1}\{ & e_{i-j}|K_y^{i+1,i}| + e_{i-j}|K_y^{i+1,i} - K_y^{i+1,j}| \\ & + d_{i-j}|K_y^{i+1,j} - K_y^{i,j}| + b_{i+1-j}|K_y^{i,j} - K_y^{i,j-1}| \\ & + b_{i+2-j}|K_y^{i+1,j} - K_y^{i+1,j-1}|\}|\tilde{\epsilon}_j| \\ & + \{(b_1 - b_2 - c_1)|K_y^{i+1,i}| + (b_1 - b_2)|K_y^{i+1,i} - K_y^{i+1,i-1}| \\ & + b_1|K_y^{i+1,i-1} - K_y^{i,i-1}|\}|\tilde{\epsilon}_i| + |P_i - Q_i - S_i| \end{aligned}$$

Here all occurring coefficients $b_k, d_k, e_k > 0$, and $b_1 - b_2 - c_1 \geq 0$.

Now divide by $b_1 K_y^{i+1,i}$, use condition (iii) of the Theorem, estimate the mean value theorem the differences of the $K_y^{k,\lambda}$, and set

$$\epsilon_j = \max\{|\tilde{\epsilon}_k| \quad |k = 0, \ldots, j\} \quad \text{for} \quad j = 0, 1, \ldots, N. \tag{2.12}$$

you find that there exists a constant $L_1 \in \mathbb{R}^+$ such that

(2.13)
$$
\begin{aligned}
|K_y^{k+1,\lambda} - K_y^{k+1,j}| &\le L_1\{(\lambda - j)h + |\tilde{\epsilon}_\lambda| + |\tilde{\epsilon}_{\lambda+1}| \\
&\qquad + |\tilde{\epsilon}_j| + |\tilde{\epsilon}_{j+1}|\} \quad \text{for} \quad 0 \le j < \lambda \le k, \\
|K_y^{k+1,j} - K_y^{k+1,j-1}| &\le L_1\{h + |\tilde{\epsilon}_{j+1}| + |\tilde{\epsilon}_j| + |\tilde{\epsilon}_{j-1}|\} \quad \text{for} \quad 1 \le j \le k, \\
|K_y^{k+1,j} - K_y^{k-j,j}| &\le L_1\{h + |\tilde{\epsilon}_j| + |\tilde{\epsilon}_{j+1}|\} \quad \text{for} \quad 0 \le j \le k-1.
\end{aligned}
$$

After rearrangement one obtains

$$
\begin{aligned}
|\tilde{\epsilon}_{i+1}| \le {} & \sum_{j=1}^{i-1} \frac{e_{i-j}}{b_1} |\tilde{\epsilon}_j| + \frac{b_1 - b_2 - c_1}{b_1} |\tilde{\epsilon}_i| \\
& + \frac{b_1 h}{b_1 K_0} \{ \sum_{j=1}^{i-1} (e_{i-j}(i-j) + d_{i-j} + b_{i-j+1} + b_{i-j+2}) |\tilde{\epsilon}_j| \\
& + (2b_1 - b_2)|\tilde{\epsilon}_i| \} \\
& + \frac{3L_1}{b_1 K_0} \epsilon_i \sum_{j=0}^{i-1} (e_{i-j}(i-j) + d_{i-j} + b_{i-j+1} + b_{i-j+2}) |\tilde{\epsilon}_j| \\
& + \frac{2L_1}{b_1 K_0} \epsilon_i . (2b_1 - b_2) . |\tilde{\epsilon}_i| \\
& + \frac{L_1}{b_1 K_0} |\tilde{\epsilon}_{i+1}| \{ \sum_{j=1}^{i-1} e_{i-j} |\tilde{\epsilon}_j| + (b_1 - b_2)|\tilde{\epsilon}_i| \} + \frac{|P_i - Q_i - S_i|}{b_1 K_0}.
\end{aligned}
$$

With the coefficients

(2.14)
$$
\begin{aligned}
c_0 &= \frac{L_1}{b_1 K_0}(b_1 - b_2), \\
C_k &= \frac{L_1}{b_1 K_0} e_k \quad \text{for} \quad k = 1, 2, ..., N-2, \\
B_0 &= \frac{L_1}{b_1 K_0} . 2(2b_1 - b_2), \\
B_k &= \frac{L_1}{b_1 K_0} . 3(e_k k + d_k + b_{k+1} + b_{k+2}) \quad \text{for} \quad k = 1, 2, .., N-2,
\end{aligned}
$$

$$A_0(h) = \frac{1}{b_1}(b_1 - b_2 - c_1) + hB_0,$$
$$A_k(h) = \frac{1}{b_1}e_k + hB_k \quad \text{for} \quad k = 1, 2, .., N-2,$$
$$D_k = \frac{1}{b_1 K_0}|P_k - Q_k - S_k| \quad \text{for} \quad k = 1, 2, .., N-1,$$

we obtain

$$|\tilde{\epsilon}_{i+1}| \leq |\tilde{\epsilon}_{i+1}|| \sum_{j=1}^{i} C_{i-j}\epsilon_j + \sum_{j=1}^{i} A_{i-j}(h)\epsilon_j + \epsilon_i \sum_{j=1}^{i} B_{i-j}\epsilon_j + D_i,$$

hence, for $i = 1, 2, ..., N-1$, the system of difference inequalities

(2.15)
$$\begin{cases} \epsilon_{i+1} = \epsilon_i \\ \quad \text{or} \\ \epsilon_{i+1}(1 - \sum_{j=1}^{i} C_{i-j}\epsilon_j) \leq \sum_{j=1}^{i} A_{i-j}(h)\epsilon_j + \epsilon_i \sum_{j=1}^{i} B_{i-j}\epsilon_j + D_i, \end{cases}$$

Subpart(Aβ):

PRELIMINARY REMARK: The following estimates are valid for all $\beta \in (0, 1)$. In their derivation properties of the function K and Lemma 2.3, 2.4(i), 2.4(iv), 2.4(v) and 2.6 are used.

(a) Estimation of the quantities S_i.

Define, for $\lambda = 1, 2, ..N$, functions

$G_\lambda(t) = K(x_\lambda, t, \bar{y}(t)) \quad \text{for} \quad x_j \leq t \leq x_{j+1}, j = 0, 1, .., \lambda - 1.$

Then

$G''_\lambda(t) = \{1 + \bar{y}'(t) + (\bar{y}'(t))^2\}(D_1D_2 + 2D_3D_2 + D_3D_3)K(x_\lambda, t, \bar{y}(t)),$

and because of the boundedness of $\bar{y}'(t) = (y_{j+1} - y_j)/h$ and the assumptions on K there exists a constant $L_2 \in \mathbf{R}^+$ such that all $|G''_\lambda(t)| \leq L_2$. Hence, by aid of Theorem 2.2, $R_{\lambda-j}[K(x_\lambda, (j+.)h, \bar{y}((j+.)h)] = r_{\lambda-j}G''_\lambda((\varphi_{\lambda-j}+j)h)h^2$ with suitable $\varphi_{\lambda-j} \in (0, 1)$, and

$$(2.16) \qquad |R_{\lambda-j}[K(x_\lambda, (j+.)h, \bar{y}((j+.)h)]| \leq r_{\lambda-j}L_2h^2$$

for $\lambda = 1, 2, ..., N; j = 0, 1, ..., \lambda - 1.$

Now, let $i \in \{1, 2, ..., N-1\}$. Remember (2.6) and (2.9). Then, with $\phi_{\lambda,j} = j + \varphi_{\lambda-j}$, we obtain

$$|S_i| = h^2 |\sum_{j=0}^{i} r_{i+1-j} G''_{i+1}(\phi_{i+1,j}) - \sum_{j=0}^{i-1} r_{i-j} G''_i(\phi_{i,j})|$$

$$\leq h^2 \{ |r_1 G''_{i+1}(\phi_{i+1,j}) - \sum_{j=0}^{i-1} G''_i(\phi_{i,j})(r_{i-j} - r_{i+1-j})|$$

$$+ \sum_{j=0}^{i-1} r_{i+1-j} |G''_{i+1}(\phi_{i+1,j}) - G''_i(\phi_{i,j})| \}.$$

Hence, by use of Lemma 2.7 and the abbreviations $H_{p,q} = G''_p(\phi_{p,q})$,

$$|S_i| \leq h^2 \{ |r_1 H_{i+1,j} - H_{i,i-1} \sum_{j=0}^{i-1} (r_{i-j} - r_{i+1-j}) -$$

$$- \sum_{j=0}^{i-2} (H_{i,j} - H_{i,j+1}) \sum_{k=0}^{j} (r_{i-k} - r_{i+1-k})| + \sum_{j=0}^{i-1} r_{i+1-j} |H_{i+1,j} - H_{i,j}| \}.$$

Using $\sum_{k=0}^{i-1} (r_{i-k} - r_{i+1-k}) = r_1 - r_{i+1} \leq r_1$, adding and subtracting $H_{i+1,i}(r_1 - r_{i+1})$ and estimating we get

$$|S_i| \leq h^2 \{ |H_{i+1,i}| r_{i+1} + |H_{i+1,i} - H_{i,i-1}| r_1 +$$
$$+ \sum_{j=0}^{i-2} |H_{i,j+1} - H_{i,j}| \sum_{k=0}^{j} (r_{i-k} - r_{i+1-k}) + \sum_{j=0}^{i-1} r_{i+1-j} |H_{i+1,j} - H_{i,j}| \}.$$

By aid of our assuptions on y and K and their derivatives there exists a constant $L_3 \in \mathbb{R}^+$ such that all the occurring differences of the $H_{p,q}$ are in absolute value $\leq L_3 h$, hence

$$|S_i| \leq h^2 \{ L_2 r_{i+1} + L_3 h r_1 + L_3 h (\sum_{j=0}^{i-2} \sum_{k=0}^{j} (r_{i-k} - r_{i+1-k}) + \sum_{j=0}^{i-1} r_{i+1-j}) \}.$$

Here, by Lemma 2.7 and Lemma 2.3 (iv),
$\sum_{j=0}^{i-2} \sum_{k=0}^{j} (r_{i-k} - r_{i+1-k}) = \sum_{k=0}^{i-2} (r_{i-k} - r_{i+1-k}) \sum_{j=k}^{i-2} 1$

$= \sum_{k=2}^{i}(r_k - r_{k+1})(k-1) < \frac{\beta}{1-\beta} i^{1-\beta}$,
and with $h = B/N < B/i \le Bi^{-\beta}$ and Lemma 2.3 (i), (ii) we see that

$$|S_i| \le L_4 h^2 i^{-\beta} \quad \text{for} \quad i = 1, 2, ..., N-1 \tag{2.17}$$

with a constant $L_4 \in \mathbf{R}^+$.

(b) Estimation of the quantities Q_i

Let $i \in \{1, 2, ..., N-1\}$. Put $V_k(t) = K(x_k, t, y(t)) - K(x_k, t, \bar{y}(t)), y''_k = y''(x_k)$, and observe that for $k = 1, 2, ..., N, \quad j = 0, 1, ...k-1$ we have (see Lemma 2.6)

$$\int_{x_j}^{x_{j+1}} \frac{(x_{j+1} - t)(t - x_j)}{(x_k - t)^\beta} \, dt = h^{3-\beta} \int_0^1 \frac{s(1-s)}{(k-j-s)^\beta} \, ds = h^{3-\beta} q_{k-j}. \tag{2.18}$$

By (2.9) and (2.3)

$$h^{\beta-1} Q_i = \int_0^{x_{i+1}} \frac{V_{k+1}(t)}{(x_{i+1} - t)^\beta} \, dt - \int_0^{x_i} \frac{V_k(t)}{(x_i - t)^\beta} \, dt.$$

Now, using Theorem 2.1 on piecewise linear interpolation and the properties of the functions y and K we find, uniformly for $x_j \le t \le x_{j+1}$,

$$V_k(t) = \frac{1}{2} y''_k D_3 K(x_k, t, y(t))(x_{j+1} - t)(x_j - t) + O(h^3),$$

hence

$$\begin{aligned} 2h^{1-\beta} Q_i = y''_{i+1} &\int_{x_i}^{x_{i+1}} \frac{(x_{i+1} - t)(x_i - t)}{(x_{i+1} - t)^\beta} D_3 K(x_{i+1}, t, y(t)) \, dt \\ &+ \sum_{j=0}^{i-1} y''_{j+1} \cdot \int_{x_j}^{x_{j+1}} (x_{j+1} - t)(x_j - t) \{ \frac{D_3 K(x_{i+1}, t, y(t))}{(x_{i+1} - t)^\beta} \\ &- \frac{D_3 K(x_i, t, y(t))}{(x_i - t)^\beta} \} dt + O(h^3) = \rho_i + O(h^3). \end{aligned}$$

We split

$$\rho_i = y''_{i+1} \int_{x_i}^{x_{i+1}} \frac{(x_{i+1} - t)(x_i - t)}{(x_{i+1} - t)^\beta} D_3 K(x_{i+1}, t, y(t)) \, dt$$

$$+\sum_{j=0}^{i-1} y''_{j+1} \int_{x_j}^{x_{j+1}} (x_{j+1}-t)(x_j-t) D_3K(x_{i+1},t,y(t))\{\frac{1}{(x_{i+1}-t)^\beta} - \frac{1}{(x_i-t)^\beta}\}dt$$

$$+\sum_{j=0}^{i-1} y'_{j+1} \int_{x_j}^{x_{j+1}} \frac{(x_{j+1}-t)(x_j-t)}{(x_i-t)^\beta}\{D_3K(x_{i+1},t,y(t)) - D_3K(x_i,t,y(t))\}dt.$$

Using here (and without explicit mention later) Lemma 2.6 we see that the last sum is equal to $= O(h)h^{3-\beta}\sum_{j=0}^{i-1} q_{i-j} = O(h^{4-\beta})i^{1-\beta} = O(h^{4-\beta})O(N).i^{-\beta} = O(h^{3-\beta})i^{-\beta}$, because $i \leq N = \frac{B}{h} = O(h^{-1})$. Hence $\rho_i = \rho_i^* + O(h^{3-\beta})i^{-\beta}$.

With suitable numbers $t_{i,j} \in (x_j, x_{j+1})$ we have now to find an appropriate estimate of

$$-\rho_i^* = y''_{i+1} M_{i,i} \int_{x_i}^{x_{i+1}} \frac{(x_{i+1}-t)(t-x_i)}{(x_{i+1}-t)^\beta}\,dt$$

$$+\sum_{j=0}^{i-1} y''_{j+1} M_{i,j} \int_{x_j}^{x_{j+1}} (\frac{1}{(x_{i+1}-t)^\beta} - \frac{1}{(x_i-t)^\beta)})(x_{j+1}-t)(t-x_j)\,dt$$

where

$$M_{i,j} = D_3K(x_{i+1}, t_{i,j}, y(t_{i,j})).$$

Here $D_3K > 0$ and the mean value theorem of integral calculus have been used. Thus

$$-\rho_i^* = h^{3-\beta}\{y''_{i+1}M_{i,i}q_1 + \sum_{j=0}^{i-1} y''_{j+1} M_{i,j}(q_{i+1-j} - q_{i-j}),$$

and by aid of Lemma 2.7 (or direct verification)

$$-\rho_i^* = h^{3-\beta}\{y''_1 M_{i,0} q_{i+1} + \sum_{j=1}^{i} (y''_{j+1}M_{i,j} - y''_j M_{i,j-1})q_{i+1-j}\}.$$

Considering the position of the intermediate points $t_{k,\lambda}$ we observe that

$$(\dots) = y''_{j+1}(M_{i,j} - M_{i,j-1}) + (y''_{j+1} - y''_j)M_{i,j-1} = O(h),$$

hence (take account of Lemma 2.6)

$$\rho_i^* = h^{3-\beta}\{i^{-\beta} + O(h)\sum_{j=1}^{i} q_{i+1-j}\} = h^{3-\beta}\{O(i^{-\beta}) + O(h).i^{1-\beta}\}$$
$$= O(h^{3-\beta})\{i^{-\beta} + O(hN)i^{-\beta}\} = O(h^{3-\beta})i^{-\beta}.$$

Putting everything together yields

$$\begin{aligned} h^{1-\beta}Q_i &= O(h^3) + \rho_i = O(h^3) + O(h^{3-\beta})i^{-\beta} + \rho_i^* \\ &= O(h^3) + O(h^{3-\beta})i^{-\beta} + O(h^{3-\beta})i^{-\beta}, \\ Q_i &= O(h^{2+\beta}) + O(h^2)i^{-\beta} \\ &= O(h^2)N^{-\beta} + O(h^2)i^{-\beta} = O(h^2)i^{-\beta}. \end{aligned}$$

We have, with a constant $L_5 \in \mathbb{R}^+$, found the estimate

(2.19) $$|Q_i| \le L_5 h^2 i^{-\beta} \quad \text{for} \quad i = 1, 2, ..., N-1.$$

(c) Estimation of ϵ_0 and ϵ_1.

Remember the formulas (2.4) and (2.12), defining the error quantities $\tilde{\epsilon}_k = \tilde{y}_k - y_k$ and $\epsilon_k = \max\{|\tilde{\epsilon}_k| | k = 0, 1, ..., j\}$. We want to prove that all $\epsilon_k = O(h^2)$, and we begin here with ϵ_0 and ϵ_1.

If the starting value $\tilde{y}_0$ is taken exactly as $y_0 = y(0)$ (either by a priori knowledge or from the limit relation in Lemma 1.2) then trivially $\epsilon_0 = \tilde{\epsilon}_0 = 0 = O(h^2)$.

Otherwise we calculate $\tilde{y}_0$ from

$$K(0, 0, \tilde{y}_0) = h^{2\beta-2} f(h^2)(1-\beta).$$

Using condition (iii) of our Theorem 2.3 we get

$$|K(0, 0, \tilde{y}_0) - K(0, 0, y_0)| \ge K_0 |\tilde{y}_0 - y_0| = K_0 \epsilon_0,$$

$$\epsilon_0 = |\tilde{\epsilon}_0| \le \frac{1}{K_0} |h^{2\beta-2} f(h^2)(1-\beta) - K(0, 0, y_0)|$$

$$= \frac{1}{K_0} |(1-\beta)h^{2\beta-2} \int_0^{h^2} \frac{K(h^2, t, y(t))}{(h^2-t)^\beta} dt - K(0, 0, y_0)|.$$

With $\tau \in (0, h^2)$ suitably chosen the integral here is

$$= K(h^2, \tau, y(\tau)) \int_0^{h^2} (h^2 - t)^{-\beta} \, dt = \frac{h^{2-2\beta}}{1-\beta} K(h^2, \tau, y(\tau)),$$

and hence $\epsilon_0 \leq \frac{1}{K_0} |K(h^2, \tau, y(\tau)) - K(0, 0, y(0))| = O(h^2)$.

To estimate ϵ_1 we consider (2.8) for the index $i = 1$:

$$0 = c_1 K_y^{1,0} \tilde{\epsilon}_0 + b_1 K_y^{1,0} \tilde{\epsilon}_1 - \tilde{S}_1 - \tilde{Q}_1.$$

By (2.6) and (2.16) we find

$$|\tilde{S}_1| = |R_1[K(x_1, (.)h, \bar{y}((.)h)]| \leq r_1 L_2 h^2,$$

and by (2.3) $\tilde{Q}_1 = h^{\beta-1} \int_0^h (h-t)^{-\beta} \{K(h, t, y(t)) - K(h, t, \bar{y}(t)\} dt$. With a suitable constant $M \in \mathbf{R}^+$ we get

$$\begin{aligned} |\tilde{Q}_1| &\leq h^{\beta-1} M \int_0^h (h-t)^{-\beta} |y(t) - \bar{y}(t)| dt \\ &\leq M h^{\beta-1} . \frac{1}{8} \max_{0 \leq t \leq B} |y''(t)|.h^2. \frac{h^{1-\beta}}{1-\beta} \\ |\tilde{Q}_1| &= O(h^2). \end{aligned}$$

Using $K_y^{1,0} \geq K_0 > 0$, $b_1 > 0$, $c_1 > 0$ (Lemma 2.4), we obtain

$$|\tilde{\epsilon}_1| \leq \frac{c_1}{b_1} |\tilde{\epsilon}_0| + \frac{1}{b_1 K_0} \{|\tilde{S}_1| + |\tilde{Q}_1|\} = O(h^2).$$

We have shown that, in any case, with a constant $K_1 \in \mathbf{R}^+$,

$$\epsilon_0 = |\tilde{\epsilon}_0| \leq K_1 h^2, \quad \epsilon_1 = \max\{|\tilde{\epsilon}_0|, |\tilde{\epsilon}_1|\} \leq K_1 h^2. \tag{2.20}$$

(d) Estimation of the quantities P_i

Let $i \in \{1, 2, ..., N-1\}$. Then, by aid of (2.9), Lemma (2.4), and formula (2.13)

$$\begin{aligned} |P_i| &= |c_{i+1} K_y^{i+1,0} - c_i K_y^{i,0}| |\tilde{\epsilon}_0| \\ &\leq \{|c_{i+1} - c_i| K_y^{i+1,0} + c_i |K_y^{i+1,0} - K_y^{i,0}|\} |\tilde{\epsilon}_0| \\ &= \{O(1) i^{-(1+\beta)} + i^{-\beta} O(1)(h + |\tilde{\epsilon}_0| + |\tilde{\epsilon}_1|)\} K_1 h^2. \end{aligned}$$

(2.20) and $h = B/N \le Bi^{-1}$ now imply, with a constant $L_6 \in \mathbf{R}^+$,

$$|P_i| \le L_6 i^{-(1+\beta)} h^2 \quad \text{for} \quad i = 1, 2, ..., N-1 \tag{2.21}$$

Subpart (Aγ)

We shall now check the conditions of Lemma 2.1 for our nonlinear system (2.15) of difference inequalities. Looking at (2.15) we see that we have to take $q = 1$ and, for the relevant indices,

$$A(i,j,h) = A_{i-j}(h), B(i,j,h) = B_{i-j}, C(i,j,h) = C_{i-j}, D(i,h) = D_i.$$

Put $p = 2$ and conclude from (2.20) that condition (i) is fulfilled. The system of difference inequalities (2.15) is equivalent to condition (ii).

To check the remaining conditions we must specify the constans K_2, K_3, K_4, K_5, the bound ΔB for h, the sets $M_1(i,h), M_2(i,h)$, and the quantities $\delta(i,h)$. Remember the formulas (2.14), defining the coefficients $A_k(h), B_k, C_k, D_k$, and use Lemmas (2.4) and (2.5) to obtain the required estimates.

Let $i \in \{1, 2, ..., N-1\}$ and put

$$\begin{aligned} K_3 &= 98 L_1 b_1 B / K_0, \quad \Delta B = \min\{1, (2K_3)^{\frac{1}{(\beta-1)}}\}.B, \\ K_2 &= 6(B/\Delta B)^\beta + K_3, K_4 = 4 L_1 b_1 B / K_0, \\ K_5 &= 2(L_4 + L_5 + L_6)/K_0, \\ \delta(i,h) &= \frac{1}{2b_1}.i^{-\beta}. \end{aligned}$$

Then

$$\begin{aligned} h\sum_{j=1}^{i} B_{i-j} &= h\sum_{k=0}^{i-1} B_k \\ &= \frac{2L_1}{K_0}(2 - \frac{b_2}{b_1})h + \sum_{k=1}^{i-1} \frac{3L_1}{b_1 K_0}(e_k k + d_k + b_{k+1} + b_{k+2})h \\ &\le \frac{4L_1}{K_0}h + \frac{3L_1}{b_1 K_0} h i^{1-\beta}\left(\frac{5}{1-\beta} + \frac{1}{1-\beta} + \frac{1}{1-\beta} + \frac{1}{2(1-\beta)}\right) \\ &\le \frac{L_1}{K_0} h i^{1-\beta}\{4 + \frac{45}{2b_1(1-\beta)}\} < \frac{L_1}{K_0} B i^{-\beta}.49 = K_3 \delta(i,h). \end{aligned}$$

The last " $<$ " is valid because of $1/(b_1(1-\beta)) = 2-\beta < 2$ and $h = B/N < B/i$. We have shown that condition (iv) is satisfied.

Furthermore

$$h\sum_{j=1}^{i} C_{i-j} = h\frac{L_1}{K_0 b_1}(b_1 - b_2 + \sum_{k=1}^{i-1} e_k) < h\frac{L_1}{K_0 b_1}(b_1 - b_2 + b_2 + c_1)$$
$$= \frac{hL_1}{K_0}(2-\beta) < \frac{2hL_1}{K_0} < \frac{L_1}{K_0}B4b_1 i^{-\beta}\frac{1}{2b_1} = K_4\delta(i,h),$$

and we see that condition (v) is fulfilled.

Using (2.17), (2.19), (2.21) we find

$$D_i \le \frac{1}{b_1 K_0}\{|P_i| + |S_i| + |Q_i|\}$$
$$\le \frac{1}{b_1 K_0}\{L_4 i^{-\beta} + L_5 i^{-\beta} + L_6 i^{-(1+\beta)}\}h^2$$
$$\le \frac{1}{b_1 K_0}h^2 i^{-\beta}\{L_4 + L_5 + L_6\} = h^2\delta(i,h)K_5,$$

and condition (vi) is satisfied.

Now, let $h \le \Delta B$, and define the sets $M_1(i,h)$ and $M_2(i,h)$ as in condition (iii), with $q = 1$. With $k^* = \min\{i, [\frac{\Delta B}{h}]\}$ we get

$$\sum_{j\in M_1(i,h)} A_{i-j}(h) \le \frac{1}{b_1}\{b_1 - b_2 - c_1 + \sum_{k=1}^{i-1} e_k + h\sum_{i\in M_1(i,h)} B_{i-j}$$
$$\le 1 - \frac{3}{4b_1}i^{-\beta} + h\sum_{k=0}^{k^*-1} B_k$$
$$\le 1 - \frac{3}{4b_1}i^{-\beta} + 49\frac{L_1}{K_0}(k^*)^{1-\beta}h$$
$$\le 1 - \frac{3}{4b_1}i^{-\beta} + \frac{1}{4b_1}i^{-\beta} = 1 - \delta(i,h).$$

The last " $\le$ " follows from

$$(k^*)^{1-\beta}h \le h^\beta(\Delta B)^{1-\beta} \le B^\beta i^{-\beta}(\frac{1}{2K_3})B^{1-\beta} = \frac{K_0}{196L_1 b_1}i^{-\beta}.$$

Thus (iiiα) is satisfied.

If the set $M_2(i,h)$ is not empty (we assume this in the sequel) we have $i+1-\frac{\Delta B}{h}>1$, $\{j|j\in\mathbb{N}, j<i+1-\frac{\Delta B}{h}\}=\{j|j\in\mathbb{N}, j\leq i-[\frac{\Delta B}{h}]\}$, and with $\hat{k}=[\frac{\Delta B}{h}]$ we get

$$\sum_{j\in M_2(i,h)} A_{i-j}(h)=\sum_{k=\hat{k}}^{i-1}A_k(h)\leq\frac{1}{b_1}\sum_{k=\hat{k}}^{i-1}e_k+\sum_{k=0}^{i-1}B_kh\leq\frac{1}{b_1}d_{\hat{K}}+K_3\delta(i,h)$$
$$\leq\frac{3}{2b_1}\hat{k}^{-\beta}+K_3\delta(i,h)\leq\frac{3}{2b_1}(\frac{\Delta B}{2h})^{-\beta}+K_3\delta(i,h)$$
$$\leq\frac{6}{2b_1}(\frac{B}{\Delta B})^{\beta}i^{-\beta}+K_3\delta(i,h)=K_2\delta(i,h)$$

and we recognize also (iiiβ) as satisfied.

Part (A) of the proof of Theorem 2.3 is completed.

We now shall treat **Main Part B** , the case

$$0.2118\leq\beta\leq 2-\frac{\ln 3}{\ln 2},$$

by deriving again our system of difference inequalities, but with modified coefficients. We shall use some identities and estimates of Main Part A that do not require the restriction $\beta\in[2-\frac{\ln 3}{\ln 2},1)$.

Let

$$\gamma(\beta)=2^{2-\beta}-3$$

REMARK: $\gamma(\beta)>0$ for $\beta<2-\frac{\ln 3}{\ln 2}$.

Subpart (Bα)

For $i=1,2,..,N-1$ we have by (2.10) (before rearrangement)

$$b_1K_y^{i+1,i}\tilde{\epsilon}_{i+1}=\sum_{j=1}^{i-1}\{c_{i-j}K_y^{i,j}+b_{i+1-j}K_y^{i,j-1}-c_{i+1-j}K_y^{i+1,j}$$
$$-b_{i+2-j}K_y^{i+1,j-1}\}\tilde{\epsilon}_j+\{b_1K_y^{i,i-1}-c_1K_y^{i+1,i}-b_2K_y^{i+1,i-1}\}\tilde{\epsilon}_i$$
$$+P_i+Q_i+S_i,$$

Subtract from this equation the equation obtained by replacing the index i by the index $i-1$. After introduction of the quantity $\gamma(\beta)$ you obtain, for $i=2,3,...N-1$,

(2.22)
$$b_1 K_y^{i+1,i}\tilde{\epsilon}_{i+1} = \sum_{j=1}^{i-2} u_{i,j}\tilde{\epsilon}_j + v_i\tilde{\epsilon}_{i-1} + z_i\tilde{\epsilon}_i - (P_i - S_i - Q_i - \gamma(\beta)(P_{i-1} - S_{i-1} - Q_{i-1}))$$

with the abbreviations

$$\begin{aligned} u_{i,j} &= c_{i-j}K_y^{i,j} + b_{i+1-j}K_y^{i,j-1} - c_{i+1-j}K_y^{i+1,j} - b_{i+2-j}K_y^{i+1,j-1} \\ &\quad - \gamma(\beta)(c_{i-1-j}K_y^{i-1,j} + b_{i-j}K_y^{i-1,j-1} - c_{i-j}K_y^{i,j} - b_{i+1-j}K_y^{i,j-1}, \\ v_i &= c_1 K_y^{i,i-1} + b_2 K_y^{i,i-2} - c_2 K_y^{i+1,i-1} - b_3 K_y^{i+1,i-2} \\ &\quad - \gamma(\beta)(b_1 K_y^{i-1,i-2} - c_1 K_y^{i,i-1} - b_2 K_y^{i,i-2}), \\ z_i &= (1+\gamma(\beta))b_1 K_y^{i,i-1} - c_1 K_y^{i+1,i} - b_2 K_y^{i+1,i-1}. \end{aligned}$$

Define, for $i = 3, 4, .., N-1$,

(2.23)
$$\hat{P}_i = \{(1+\gamma(\beta))(c_{i-1}K_y^{i,1} + b_i K_y^{i,0}) - c_i K_y^{i+1,1} - b_{i+1}K_y^{i+1,0} - \gamma(\beta)(c_{i-2}K_y^{i-1,1} + b_{i-1}K_y^{i-1,0})\}\tilde{\epsilon}_1.$$

Then we have, for $i = 3, 4, ..., N-1$,

$$b_1 K_y^{i+1,i}\tilde{\epsilon}_{i+1} = \sum_{j=2}^{i-2} u_{i,j}\tilde{\epsilon}_j + v_i\tilde{\epsilon}_{i-1} + z_i\tilde{\epsilon}_i - (P_i - S_i - Q_i - \gamma(\beta)(P_{i-1} - S_{i-1} - Q_{i-1})) + \hat{P}_i.$$

Obsreve now that ('), ("), ("') hold for $i = 3, 4, ..., N-1$.

(') for $j = 2, 3, ..., j-2$:

$$u_{i,j} = \{c_{i-j} + b_{i+1-j} - c_{i+1-j} - b_{i+2-j} - \gamma(\beta)(c_{i-1-j} + b_{i-j} - c_{i-j} - b_{i+1-j})\}K_y^{i+1,j}$$

$$+\{c_{i-j}+b_{i+1-j}-\gamma(\beta)(c_{i-1-j}+b_{i-j}c_{i-j}-b_{i+1-j})\}(K_y^{i,j}-K_y^{i+1,j})$$
$$+b_{i+2-j}(K_y^{i+1,j}-K_y^{i+1,j-1})+(1+\gamma(\beta))b_{i+1-j}(K_y^{i,j-1}-K_y^{i,j})$$
$$+\gamma(\beta)(c_{i-1-j}+b_{i-j})(K_y^{i,j}-K_y^{i-1,j})+\gamma(\beta)b_{i-j}(K_y^{i-1,j}-K_y^{i-1,j-1}).$$

Note that all coefficients of the $K_y^{\mu,\lambda}$ and of their occurring differences are *positive* (by Lemma 2.4, 2.5, 2.8) and as a consequence of

$$d_{i-j}-\gamma(\beta)e_{i-j-1}>e_{i-j}-\gamma(\beta)e_{i-j-1}.$$

(") $$v_i=\{(c_1+b_2)(1+\gamma(\beta))-c_2-b_3-\gamma(\beta)b_1\}K_y^{i+1,i-1}$$
$$+b_3(K_y^{i+1,i-1}-K_y^{i+1,i-2})$$
$$+\{(c_1+b_2)(1+\gamma(\beta))-\gamma(\beta)b_1\}(K_y^{i,i-1}-K_y^{i+1,i-1})$$
$$+\gamma(\beta)b_1(K_y^{i,i-1}-K_y^{i,i-2})+\gamma(\beta)b_1(K_y^{i,i-2}-K_y^{i-1,i-2})$$
$$+(1+\gamma(\beta))b_2(K_y^{i,i-2}-K_y^{i,i-1}).$$

Again by our lemmas all coefficients here are nonnegative, in particular the first one is equal to $e_1+\gamma(\beta)(c_1+b_2-b_1)>0$.

("') $$z_i=\{b_1(1+\gamma(\beta))-b_2-c_1\}K_y^{i+1,i}$$
$$+\{b_1(1+\gamma(\beta))-b_2\}(K_y^{i+1,i-1}-K_y^{i+1,i})$$
$$+b_1(1+\gamma(\beta))(K_y^{i,i-1}-K_y^{i+1,i-1}).$$

By Lemmas 2.4 and (2.8) all coeficients here are nonnegative.

Using $K_y^{i+1,i}\geq K_0>0,\quad b_1>0$, the inequalities (2.13), the quantities

(2.24) $$\hat{e}_k=e_k-\gamma(\beta)e_{k-1}\quad\text{for}\quad k=2,3,..N-2$$

and the definition (2.12) we obtain

$$|\tilde{\epsilon}_{i+1}|\leq$$
$$\sum_{j=2}^{i-2}\tfrac{1}{b_1}\hat{e}_{i-j}|\tilde{\epsilon}_j|+\tfrac{1}{b_1}\{\hat{e}_1+\gamma(\beta)c_1+b_2-b_1\}|\tilde{\epsilon}_{i-1}|$$
$$+\tfrac{1}{b_1}\{b_1(1+\gamma(\beta))-b_2-c_1\}|\tilde{\epsilon}_i|+\tfrac{L_1h}{b_1K_0}\{\sum_{j=2}^{i-2}(\hat{e}_{i-j}(i-j)$$
$$+d_{i-j}(1+\gamma(\beta))+b_{i-j+2}+(1+\gamma(\beta))b_{i-j+1}+\gamma(\beta)b_{i-j})|\tilde{\epsilon}_j|$$
$$+(e_1+2\gamma(\beta)(c_1+b_2-b_1)+d_1+b_3+2\gamma(\beta)b_1+(1+\gamma(\beta))b_2)|\tilde{\epsilon}_{i-1}|$$

$+ (2b_1(1+\gamma(\beta)) - b_2)|\tilde{\epsilon}_i|\} + \frac{3L_1\epsilon_i}{b_1K_0}\{\sum_{j=2}^{i-2}(\hat{e}_{i-j}(i-j)$
$+ d_{i-j}(1+\gamma(\beta)) + b_{i-j+2} + (1+\gamma(\beta))b_{i-j+1} + \gamma(\beta)b_{i-j}|\tilde{\epsilon}_j|$
$+ (e_1 + 2\gamma(\beta))(c_1 + b_2 - b_1) + d_1 + b_3 + 2\gamma(\beta)b_1 + (1+\gamma(\beta))b_2)|\tilde{\epsilon}_{i-1}|$
$+ \frac{2L_1\epsilon_i}{b_1K_0}\{2b_1(1+\gamma(\beta)) - b_2\}|\tilde{\epsilon}_i|$
$+ \frac{L_1}{b_1K_0}\{\sum_{j=2}^{i-2}\hat{e}_{i-j}|\tilde{\epsilon}_j| + (e_1 + \gamma(\beta)(c_1+b_2-b_1))|\tilde{\epsilon}_{i-1}| + (b_1(1+\gamma(\beta)) - b_2)|\tilde{\epsilon}_i|\}|\tilde{\epsilon}_{i+1}|$
$+ \frac{1}{b_1K_0}\{|P_i - Q_i - S_i| + \gamma(\beta)|P_{i-1} - Q_{i-1} - S_{i-1}| + |\hat{P}_i|.$

Now define

(2.25)
$$\hat{C}_0 = \frac{L_1}{b_1K_0}\{b_1(1+\gamma(\beta)) - b_2\},$$
$$\hat{C}_1 = \frac{L_1}{b_1K_0}\{e_1 + \gamma(\beta)(c_1 + b_2 - b_1)\},$$
$$\hat{C}_k = \frac{L_1}{b_1K_0}\hat{e}_k \quad \text{for} \quad k = 2, 3, \ldots, N-2,$$
$$\hat{B}_0 = \frac{L_1}{b_1K_0}.2\{2b_1(1+\gamma(\beta)) - b_2\},$$
$$\hat{B}_1 = \frac{L_1}{b_1K_0}.3\{e_1 + (1+2\gamma(\beta))d_1 + b_3 + (1+\gamma(\beta))b_2\},$$
$$\hat{B}_k = \frac{L_1}{b_1K_0}.3\{k\hat{e}_k + d_k(1+\gamma(\beta)) + b_{k+2} + b_{k+1}(1+\gamma(\beta)) + \gamma(\beta)b_k\}$$
$$\text{for} \quad k = 2, 3, ..., N-2,$$

$$\hat{A}_0(h) = \frac{1}{b_1}\{b_1(1+\gamma(\beta)) - b_2 - c_1\} + h\hat{B}_0,$$
$$\hat{A}_1(h) = \frac{1}{b_1}\{e_1 + \gamma(\beta)(b_2 + c_1 - b_1)\} + h\hat{B}_1,$$
$$\hat{A}_k(h) = \frac{1}{b_1}\hat{e}_k + h\hat{B}_k \quad \text{for} \quad k = 2, 3, ..., N-2,$$
$$\hat{D}_k(h) = \frac{1}{b_1K_0}\{|P_k - Q_k - S_k| + |P_{k-1} - Q_{k-1} - S_{k-1}| + |\hat{P}_k|$$
$$\text{for} \quad k = 3, ..., N-1.$$

Then, observing (2.12) again, we obtain for $i = 3, 4, ..., N-1$ that

$\epsilon_{i+1} = \epsilon_i$ or
$\epsilon_{i+1}(1 - \sum_{j=2}^{i}\hat{C}_{i-j}\epsilon_j) \leq \sum_{j=2}^{i}\hat{A}_{i-j}(h)\epsilon_j + \epsilon_i\sum_{j=2}^{i}\hat{B}_{i-j}\epsilon_j + \hat{D}_i.$

We want this difference inequality also to be valid for the index $i = 2$. From (2.22) for $i = 2$ we deduce, using the definition

$$(2.26)\quad \hat{P}_2 = \{(c_1 K_y^{2,1} + b_2 K_y^{2,0})(1+\gamma(\beta)) - c_2 K_y^{3,1} - b_3 K_y^{3,0} - \gamma(\beta) b_1 K_y^{1,0}\}\tilde{\epsilon}_1,$$

that

$$\begin{aligned} b_1 K_y^{3,2}\tilde{\epsilon}_3 = &\{1+\gamma(\beta))b_1 - c_1 - b_2\} K_y^{3,2}\tilde{\epsilon}_2 \\ &+ \{(1+\gamma(\beta))b_1 - b_2\}(K_y^{3,1} - K_y^{3,2})\tilde{\epsilon}_2 \\ &+ (1+\gamma(\beta))b_1(K_y^{2,1} - K_y^{3,1})\tilde{\epsilon}_2 \\ &- (P_2 - Q_2 - S_2 - \gamma(\beta)(P_1 - Q_1 - S_1)) + \hat{P}_2. \end{aligned}$$

Again the coefficients are nonnegative (Lemmas 2.8 and 2.4). Using the inequalities (2.13) and the definition

$$(2.27)\qquad \hat{D}_2 = \frac{1}{b_1 K_0}\{|P_2 - Q_2 - S_2| + \gamma(\beta)|P_1 - Q_1 - S_1| + |\hat{P}_2|\}$$

We get

$$\begin{aligned} |\tilde{\epsilon}_3| \le &\frac{1}{b_1}\{(1+\gamma(\beta))b_1 - c_1 - b_2\}|\tilde{\epsilon}_2| \\ &+ \frac{1}{b_1 K_0}\{(1+\gamma(\beta))b_1 - b_2\}|\tilde{\epsilon}_2| L_1(h + |\tilde{\epsilon}_3| + |\tilde{\epsilon}_2| + |\tilde{\epsilon}_1|) \\ &+ (1+\gamma(\beta))b_1 \frac{1}{b_1 K_0}|\tilde{\epsilon}_2| L_1(h + |\tilde{\epsilon}_1| + |\tilde{\epsilon}_2|) + \hat{D}_2 \\ \le &\{\frac{1}{b_1}\{(1+\gamma(\beta))b_1 - c_1 - b_2\} \\ &+ \frac{L_1}{b_1 K_0} h\{2b_1(1+\gamma(\beta)) - b_2\}\}|\tilde{\epsilon}_2| \\ &+ \frac{2L_1}{b_1 K_0}\tilde{\epsilon}_2\{2b_1(1+\gamma(\beta)) - b_2\}|\tilde{\epsilon}_2| \\ &+ \frac{L_1}{b_1 K_0}\{(1+\gamma(\beta))b_1 - b_2\}|\tilde{\epsilon}_3|.|\tilde{\epsilon}_2| + \hat{D}_2. \end{aligned}$$

Consequently,

$\epsilon_3 = \epsilon_2$ or $\tilde{\epsilon}_3(1 - \hat{C}_0\epsilon_2) \le \hat{A}_0(h)\epsilon_2 + \epsilon_2\hat{B}_0\epsilon_2 + \hat{D}_2.$

As result of this section we note the difference inequality,
for $i = 2, 3, .., N-1$,

(2.28)
$$\begin{cases} \epsilon_{i+1} = \epsilon_i \\ \underline{\quad\text{or}\quad} \\ \epsilon_{i+1}(1 - \sum_{j=2}^{i} \hat{C}_{i-j}\epsilon_j) \le \sum_{j=2}^{i} \hat{A}_{i-j}(h)\epsilon_j + \epsilon_i \sum_{j=2}^{i} \hat{B}_{i-j}\epsilon_j + \hat{D}_i. \end{cases}$$

Subpart (Bβ):

According to (2.20) we have $\epsilon_0 \le K_1 h^2$, $\epsilon_1 \le K_1 h^2$, and from (2.8) we get

$$b_1 K_y^{2,1} \tilde{\epsilon}_2 = \tilde{S}_2 + \tilde{Q}_2 - c_2 K_y^{2,0} \tilde{\epsilon}_0 - \{c_1 K_y^{2,1} + b_2 K_y^{2,0}\} \tilde{\epsilon}_1,$$
$$|\tilde{\epsilon}_2| \le \frac{1}{b_1 K_0} \{|\tilde{S}_2| + |\tilde{Q}_2|\} + O(h^2).$$

From (2.6) and (2.16) we get $|\tilde{S}_2| \le (r_2 + r_1) L_2 h^2$, and from (2.3) and Theorem 2.1

$$|\tilde{Q}_2| \le h^{\beta-1} \int_0^{2h} \frac{|K(x_2, t, y(t)) - K(x_2, t, \tilde{y}(t))|}{(2h-t)^\beta}\, dt$$
$$= h^{\beta-1} O(h^2) \int_0^{2h} (2h-t)^{-\beta}\, dt = O(h^2).$$

Thus there exists a constant $\hat{K}_1 \in \mathsf{R}^+$ such that

(2.29) $$\epsilon_j \le \hat{K}_1 h^2 \quad \text{for} \quad j = 0, 1, 2.$$

Now let $i \in \{2, 3, .., N-1\}$. By Lemma 2.4, condition (ii) of our Theorem 2.3, and (2.29), we get for the quantities $\hat{P}_i$, defined in (2.23),

$$\begin{aligned} |\hat{P}_i| &= \{(c_{i-1} + b_i)(1 + \gamma(\beta)) + c_i + b_{i+1} + \gamma(\beta)(c_{i-2} + b_{i-1})\} O(h^2) \\ &= \{d_{i-1}(1 + \gamma(\beta)) + d_i + \gamma(\beta) d_{i-2}\} O(h^2) \\ &\le \{(1 + \gamma(\beta))(i-1)^{-\beta} + i^{-\beta} + \gamma(\beta)(i-2)^{-\beta}\} O(h^2) \end{aligned}$$

where the last " $\le$ " follows by aid of Lemma 2.5. Now, $i - 2 \ge i/3, i - 1 \ge i/2$, $\gamma(\beta) < 1/2$, hence $|\hat{P}_i| = O(h^2) i^{-\beta}$. Furthermore, again by Lemma 2.4,

$$|\hat{P}_2| = \{d_1(1 + \gamma(\beta)) + d_2 + \gamma(\beta) b_1\} O(h^2),$$

hence, because of Lemma 2.5 (i) and $b_1 < 4/3$ for $\beta < 0.5, |\hat{P}_2| = O(h^2) = O(h^2)2^{-\beta}$.

We have shown that that there exists a constant $\hat{L}_6 \in \mathbb{R}^+$ such that

$$|\hat{P}_i| \le \hat{L}_6 h^2 i^{-\beta} \quad \text{for} \quad i = 2, 3, .., n-1. \tag{2.30}$$

Subpart (Bγ):

We have to check the conditions of Lemma 2.1 for the system (2.28) of difference inequalities. We see that we must take $q = 2$ and, for the relevant indices,

$$A(i,j,h) = \hat{A}_{i-j}(h), B(i,j,h) = \hat{B}_{i-j}, C(i,j,h) = \hat{C}_{i-j}, D(i,h) = \hat{D}_i.$$

Put $p = 2$ and conclude from (2.29) that condition (i) is fulfilled. The system (2.28) of difference inequalities is equivalent to condition (ii).

To check the remaining conditions we shall specify constants $\hat{K}_2, \hat{K}_3, \hat{K}_4, \hat{K}_5$, a bound ΔB for h , sets $M_1(i,h), M_2(i,h)$, and quantities $\hat{\delta}(i,h)$. Remember the formulas (2.25) and (2.27) defining the coefficients $\hat{A}_k(h), B_k, C_k, D_k$, and use Lemma 2.4 and 2.5 where appropriate.

Let $i \in \{2, 3, .., N-1\}$ and put

$$\hat{K}_3 = 918 L_1 b_1 B / K_0, \Delta B = \min\{1, \hat{K}_3^{1/(\beta-1)}\} B,$$
$$\hat{K}_2 = 72(B/\Delta B)^\beta + \hat{K}_3, \quad \hat{K}_4 = 36 L_1 b_1 B / K_0,$$
$$\hat{K}_5 = 12\{2(L_4 + L_5 + L_6\}) + \hat{L}_6\}/K_0,$$
$$\hat{\delta}(i,h) = \frac{1}{12 b_1} i^{-\beta}.$$

Then

$$h \sum_{j=2}^{i} \hat{B}_{i-j} < h \sum_{k=0}^{i-1} \hat{B}_k$$
$$\le \frac{3hL_1}{b_1 K_0}\{2b_1(1+\gamma(\beta)) + d_1 - d_2 + d_1 + 2\gamma(\beta) d_1 + b_3 + \gamma(\beta) b_2$$
$$+ \sum_{k=2}^{i-1} (\hat{e}_k k + d_k(1+\gamma(\beta)) + b_{k+2} + b_{k+1}(1+\gamma(\beta)) + \gamma(\beta) b_k)\}$$

$$\leq \frac{3hL_1}{b_1 K_0}\{(2b_1+2d_1)(1+\gamma(\beta))+\gamma(\beta)b_2$$
$$+\sum_{k=2}^{i-1}(\hat{e}_k k + d_k(1+\gamma(\beta)) + b_{k+1}(1+\gamma(\beta)))$$
$$+\sum_{k=0}^{i-1} b_{k+2} + \sum_{k=2}^{i+1}\gamma(\beta)b_k\}.$$

Using Lemmas 2.4 and 2.5, $\frac{1}{b_1}(b_1+d_1) = 2^{2-\beta}-1, \quad \beta < 0.5, \gamma(\beta) < 0.5, \hat{e}_k < e_k$ for $k \geq 2$, we find

$$h\sum_{j=2}^{i}\hat{B}_i \leq \frac{3hL_1}{K_0}\{\frac{15}{2}+\frac{1}{2}+\frac{1}{b_1(1-\beta)}i^{1-\beta}(5+\frac{3}{2}+\frac{3}{2}+\frac{3}{4})\}$$
$$\leq \frac{3hL_1}{K_0}i^{1-\beta}(8+10+3+3+\frac{3}{2})$$
$$\leq \frac{L_1}{K_0}.\frac{153}{2}hNi^{-\beta} = \frac{BL_1}{K_0}.\frac{153}{2}i^{-\beta} = \hat{K}_3\hat{\delta}(i,h).$$

This means that condition (iv) is satisfied.

Furthermore

$$h\sum_{j=2}^{i}\hat{C}_{i-j} = h\sum_{k=0}^{i-2}\hat{C}_k$$

$$\leq \frac{hL_1}{K_0}\{1+\gamma(\beta)-\frac{b_2}{b_1}+\frac{e_1}{b_1}+\frac{\gamma(\beta)}{b_1}(c_1+b_2-b_1)+\frac{1}{b_1}\sum_{k=2}^{i-1}e_k\}$$
$$\leq \frac{hL_1}{K_0}\{1+\frac{\gamma(\beta)}{b_1}c_1+\frac{1}{b_1}(d_1-d_i)\}.$$

Using $c_1 = 1/(2-\beta), \frac{1}{b_1} = (1-\beta)(2-\beta), d_1 = \frac{2^{2-\beta}-2}{(1-\beta)(2-\beta)}, \beta \in (0,1/2), \gamma(\beta) \in (0,1/2)$, we obtain

$$h\sum_{j=2}^{i}\hat{C}_{i-j} \leq \frac{BL_1}{K_0}i^{-\beta}(1+\frac{1}{2}+\frac{3}{2}) = \hat{K}_4\hat{\delta}(i,h),$$

meaning (v).

By aid of (2.17), (2.19), (2.21), (2.30) we get for $i = 2, ..., N-1$ the estimate

$$\hat{D}_i \le \frac{1}{b_1 K_0} h^2\{(L_4 + L_5 + L_6)(i^{-\beta} + \gamma(\beta)(i-1)^{-\beta}) + \hat{L}_6 i^{-\beta}\}$$
$$\le \frac{1}{b_1 K_0} h^2 i^{-\beta}\{2(L_4 + L_5 + L_6) + \hat{L}_6\} = \hat{K}_5 \hat{\delta}(i,h)h^2,$$

that is (vi).

Let now $h \le \Delta B$, and define the sets $M_1(i,h), M_2(i,h)$ as in condition (iii), with $q = 2$. Set again $k^* = \min\{i, [\frac{\Delta B}{h}]\}$. Then $M_1(i,h) \subset \{j | j \in \mathbb{N}, 0 \le i - j \le k^* - 1\}$, and we find

$$\sum_{j \in M_1(i,h)} \hat{A}_{i-j}(h) \le \frac{1}{b_1}\{b_1(1+\gamma(\beta)) - b_2 - c_1 + e_1$$
$$+ \gamma(\beta)(b_2 + c_1 - b_1) + \sum_{k=2}^{i-1} \hat{e}_k\} + h \sum_{j \in M_1(i,h)} \hat{B}_{i-j}.$$

From (2.24) we deduce

$$\sum_{k=2}^{i-1} \hat{e}_k = \sum_{k=2}^{i-1} e_k - \gamma(\beta)\sum_{k=1}^{i-2} e_k = \begin{cases} d_2 - d_i - \gamma(\beta)d_1 + \gamma(\beta)d_{i-1} & \text{for} \quad i > 2 \\ 0 & \text{for} \quad i = 2. \end{cases}$$

Making use of Lemma 2.9 we get

$$\sum_{j \in M_1(i,h)} \hat{A}_{i-j}(h) \le 1 - \frac{d_i}{b_1} + \gamma(\beta)\frac{d_{i-1}}{b_1} + h\sum_{k=0}^{k^*-1} \hat{B}_k$$
$$\le 1 - \frac{1}{6b_1} i^{-\beta} + \frac{153}{2}\frac{hL_1}{K_0}(k^*)^{1-\beta}$$
$$\le 1 - \frac{1}{6b_1} i^{-\beta} + \frac{1}{12b_1} i^{-\beta} = 1 - \hat{\delta}(i,h).$$

The last " $\le$ " follows from

$$(k^*)^{1-\beta} h \le (\frac{\Delta B}{h})^{1-\beta} h \le (\Delta B)^{1-\beta} h^{\beta} < \hat{K}_3^{-1} B^{1-\beta} B^{\beta} i^{-\beta}$$
$$= i^{-\beta} K_0 . (918 L_1 b_1)^{-1}.$$

We have verified condition (iiiα).

If the set $M_2(i,h)$ is not empty (as we assume in the sequel) we have $i + 1 - \frac{\Delta B}{h} > 2$. Notice that also $M_2(i,h) = \{j | j \in \mathbb{N}, [\frac{\Delta B}{h}] \le i - j \le i - 2\}$, abbreviate $\hat{k} = [\frac{\Delta B}{h}]$, and distinguish the two cases $\hat{k} = 1$ and $\hat{k} \geq 2$. In the case $\hat{k} = 1$ we have

$$\sum_{j \in M_2(i,h)} \hat{A}_{i-j}(h) \leq \sum_{k=1}^{i-2} \hat{A}_k(h)$$

$$\leq \frac{1}{b_1}\{e_1 + \gamma(\beta)(b_2 + c_1 - b_1) + \sum_{k=2}^{i-2} \hat{e}_k\} + h \sum_{k=0}^{i-2} \hat{B}_k.$$

Using

$\sum_{k=2}^{i-2} \hat{e}_k \leq \sum_{k=2}^{i-2} e_k < d_2, e_1 + \gamma(\beta)(b_2 + c_1 - b_1) = d_1(1 + \gamma(\beta)) - d_2 - \gamma(\beta) b_1,$
we can proceed with

$$\sum_{j \in M_2(i,h)} \hat{A}_{i-j}(h) \leq \frac{1}{b_1} d_1(1 + \gamma(\beta)) + \hat{K}_3 \hat{\delta}(i,h) \leq \frac{1}{b_1} \frac{4}{3} . \frac{3}{2} . \frac{3}{2} + \hat{K}_3 \hat{\delta}(i,h)$$

$$= \frac{1}{b_1} 3 . [\frac{\Delta B}{h}]^{-\beta} + \hat{K}_3 \hat{\delta}(i,h).$$

In the case $\hat{k} \geq 2$ we find, by aid of Lemma 2.5 (i) and $\hat{e}_j < e_j$ for $j \in \mathbb{N} \backslash \{2\}$,

$$\sum_{j \in M_2(i,h)} \hat{A}_{i-j}(h) \leq \sum_{k=\hat{k}}^{i-2} \hat{A}_k(h)$$

$$\leq \frac{1}{b_1} \sum_{k=\hat{k}}^{i-2} e_k + h \sum_{k=0}^{i-2} \hat{B}_k$$

$$\leq \frac{1}{b_1} d_{\hat{k}} + \hat{K}_3 \hat{\delta}(i,h)$$

$$\leq \frac{3}{2} \frac{1}{b_1} [\frac{\Delta B}{h}]^{-\beta} + \hat{K}_3 \hat{\delta}(i,h).$$

Now, $[x] \geq x/2$ for $x \geq 1$ and $h^\beta = (B/N)^\beta \leq B i^{-\beta}$. Hence, in both cases,

$$\sum_{j \in M_2(i,h)} \hat{A}_{i-j}(h) \leq \frac{3}{b_1} [\frac{\Delta B}{h}]^{-\beta} + \hat{K}_3 \hat{\delta}(i,h)$$

$$\leq \frac{3}{b_1} (\frac{\Delta B}{2h})^{-\beta} + \hat{K}_3 \hat{\delta}(i,h) \leq \frac{6}{b_1} [\frac{\Delta B}{B}]^{-\beta} i^{-\beta} + \hat{K}_3 \hat{\delta}(i,h)$$

$$= (72(\frac{B}{\Delta B})^\beta + \hat{K}_3) \hat{\delta}(i,h) = \hat{K}_2 \hat{\delta}(i,h),$$

and we have, finally, verified (iiiβ).

The proof of Theorem 2.3 is completed.

Chapter III. Numerical realization

III.1. Calculation of coefficients. The application of the discretization scheme (DM) (described at the beginning of III.3) requires calculation of coefficients (defined in Lemma 2.2)

$$a_k = \frac{1}{1-\beta}\{k^{1-\beta} - (k-1)^{1-\beta}\}, \tag{3.1}$$

$$w_k = \frac{1}{(1-\beta)(2-\beta)a_k}\{k^{2-\beta} - (k-1)^{2-\beta} - (2-\beta)(k-1)^{1-\beta}\} \tag{3.2}$$

for indices $k = 1, 2, .., N-1$, computation of $\tilde{y}_0$ and solution of the system (rec) of nonlinear equations.

For large indices k there will happen severe loss of significant digits when we work with a fixed number of decimal digits if a_k and w_k are naively calculated by (3.1) and (3.2). This loss of accuracy can be avoided by aid of Taylor series. For $k \geq 2$ we find

$$a_k = \frac{k^{1-\beta}}{1-\beta}\sum_{n=1}^{\infty}\binom{1-\beta}{n}(-1)^{n+1}k^{-n},$$

$$w_k = \frac{k^{1-\beta}}{(1-\beta)(2-\beta)a_k}\sum_{m=1}^{\infty} n\binom{2-\beta}{n+1}(-1)^{n+1}k^{-n}.$$

Fortunately, in the first series all terms, in the second one all those with index $n \geq 2$, are positive. Thus, calculation of a_k and w_k by these series is stable. One may, e.g., calculate a_k and w_k for $1 \leq k \leq 10$ directly according to (3.1) and (3.2), and for $k \geq 11$ use the series, truncating them at $n = 20$ (summing up to and including $n = 20$.)

One can show (we omit the details) that for $k \geq 2, m \in \mathbb{N}$ the estimates (3.3) and (3.4) are valid.

$$\sum_{n=m+1}^{\infty}\binom{1-\beta}{n}(-1)^{n+1}k^{-n} < \frac{2(1-\beta)}{(m+1)}k^{-(m+1)}, \tag{3.3}$$

$$\sum_{n=m+1}^{\infty} n\binom{2-\beta}{n+1}(-1)^{n+1}k^{-n} \leq \frac{2(2-\beta)(1-\beta)}{m+1}k^{-(m+1)}. \tag{3.4}$$

The relative error of bc th approximations is $< 10^{-20}$ if $m = 20, k \geq 11$.

III.2. Solution of the system of nonlinear equations. The starting value $\tilde{y}_0$ may (if possble) directly taken from the limit relation (see Theorem 1.6) or rom

$$K(0,0,\tilde{y}_0) = (1-\beta)h^{2(\beta-1)}f(h^2). \tag{3.5}$$

If the function satisfies the conditions of Theorem 2.3 there are several standard numerical methods for good approximate solution of the single equation (3.5).

After having computed $\tilde{y}_0$ we must, for $i = 1, 2, .., N$, successively solve for $\tilde{y}_i$ an equation $F_i(\tilde{y}_i) = 0$ with a function $F_i : \mathbb{R} \longrightarrow \mathbb{R}$, given as

(3.6)

$$F(\xi) = K(x_i, (i-1+w_1)h, (1-w_1)\tilde{y}_{i-1} + w_1\xi) - \frac{h^{\beta-1}}{a_1}f(x_i)$$
$$- \frac{1}{a_1}\sum_{j=0}^{i-2} a_{i-j}K(x_i, (j+w_{i-j})h, (1-w_{i-j})\tilde{y}_j + w_{i-j}\tilde{y}_{j+1}).$$

Remember Remark (c) to Theorem 2.3: The solution $\tilde{y}_i$ exists uniquely.

We have used Steffensen's method of iteration (see, e.g.[10]), applied to the fixed point equation $\xi = \phi_i(\xi)$ with $\phi_i(\xi) = \xi + F_i(\xi)$. This method leads, after choice of a starting value ξ_0 (possibly $= \tilde{y}_{i-1}$) with

$$\varphi(\xi) = \xi - \frac{(F_i(\xi))^2}{F_i(\xi + F_i(\xi)) - F_i(\xi)} \quad \text{if} \quad F_i(\xi) \neq 0 \tag{3.7}$$
$$\varphi(\xi) = \xi \quad \text{if} \quad F_i(\xi) = 0,$$

to an iteration $\xi_n = \varphi(\xi_{n-1})$ for $n = 1, 2, 3, ...$

If the conditions of Theorem 2.3, in particular (iii), are fulfilled we have $F_i'(\xi) > 0$ for all $\xi \in \mathbb{R}$, and Steffensen's method is locally quadratically convergent, $\xi_n \longrightarrow \tilde{y}_i$ as $n \longrightarrow \infty$. In practice we hope that $\tilde{y}_{i-1}$ is a good starting value.

III.3. Numerical case studies. With the intention to demonstrate $O(h^2)$ convergence of our discretization
method (DM) we have carried out several numerical case studies. The computations have been done on an Atari MegaSt2 personal computer in programming langguage MODULA-2 with 64-bit LONGREAL arithmetic corresponding to about 15 significant decimal digits. These case studies confirmed $O(h^2)$ convergence, furthermore the approximate values usually were better if $\tilde{y}_0$ was taken as $y(0)$ calculated from the limit relation

$$K(0,0,y(0)) = (1-\beta) \lim_{x\to 0} x^{\beta-1} f(x) \tag{3.8}$$

instead of from the approximating relation

$$K(o,o,\tilde{y}_0) = (1-\beta)h^{2(\beta-1)} f(h^2). \tag{3.9}$$

Although our theory assumes $0.2118 \le \beta < 1$, we made also experiments with the outside value $\beta = 0,1$. Again the numerical results hint to $O(h^2)$ convergence, and it seems to us that the restriction $0.2118 \le \beta < 1$ is simply required by our method of proof. So, somebody should try to extend the proof to $0 < \beta < 0.2118$.

We do not display here all our examples for more of them and for a more detailed discussion we refer to [9]. We restrict ourselves here to show results for

$K(x,t,z) = (1+tx^2)\sqrt{1+z^2}.arctgz, 0 \le t \le x \le B = 6.4, z \in \mathbf{R},$

$f(x) = (1-\beta)^{-1} x^{1-\beta} \cos x, 0 \le x \le B.$

For β the values 0.1, 0.25, 0.75, 0.9 were chosen, and the initial value $\tilde{y}_0$ was taken as $y(0)$ from (3.8).

The exact solution is not known explicitly, and so, to test order of convergence, we assume or hope that there exists an asymptotic expansion of the error in powers of h, beginning with exponent 2. If x is a common grid-point for the three meshes with step-lengths $2h = B/(N/), h = B/N, h/2 = B/(2N)$, with a not too small even natural number N , we may expect that the ratio

$$\rho(x;N) = \frac{y_{2h}(x) - y_h(x)}{y_h(x) - y_{h/2}(x)} = \frac{\tilde{y}(x;N/2) - \tilde{y}(x;N)}{\tilde{y}(x;N) - \tilde{y}(x;2N)}$$

generically is not far off from $2^2 = 4$. Here $y_h(x) = \tilde{y}(x; N)$ is the approximate solution obtained via the step-length $h = B/N$, and $y_{2h}(x) = \tilde{y}(x; N/2)$, $y_{h/2}(x) = \tilde{y}(x; 2N)$ are obtained via the step-lengths $B/(N/2), B/(2N)$, respectively. In the following tables we display rounded numerical results. If you observe that for $x = 0.1$ the ratio $\rho(x; N)$ is not very near to 4 please consider the fact that 0.1 is near 0 and that at $x = 0$ the ratio ρ cannot be formed because we have always taken precisely $\tilde{y}(0) = y(0)$. Thus, convergence of $\rho(0.1; N) \to 4$ as $N \to \infty$ may be slower than for larger values x. Notice that in the tables $E-1$ means $10^{-1}, E-3$ means 10^{-3}.

N	$\tilde{y}(0.1;N)$	$\rho(0.1;N)$	$\tilde{y}(1;N)$	$\rho(1;N)$	$\tilde{y}(B;N)$	$\rho(B;N)$
64	9.02930922E-1		-2.06180875E-1		-1.73186474E-3	
128	8.99083455E-1	18.056	-2.03170633E-1	4.4037	-1.71896628E-3	4.0079
256	8.98870368E-1	3.0265	-2.02487059E-1	4.0866	-1.71574805E-3	3.9989
512	8.98799959E-1		-2.02319785E-1		-1.71494327E-3	

Table 1. *Numerical resuts for* $\beta = 0.1$

N	$\tilde{y}(0.1;N)$	$\rho(0.1;N)$	$\tilde{y}(1;N)$	$\rho(1;N)$	$\tilde{y}(B;N)$	$\rho(B;N)$
64	9.03537079E-1		-1.35725467E-1		3.36179535E-3	
128	9.00715215E-1	10.099	-1.33217622E-1	4.0744	3.37140750E-3	3.9887
256	9.00435795E-1	3.0572	-1.32602112E-1	4.0124	3.37381735E-3	3.9913
512	9.00344396E-1		-1.324487709E-1		3.37442113E-3	

Table 2. *Numerical resuts for* $\beta = 0.25$

N	$\tilde{y}(0.1;N)$	$\rho(0.1;N)$	$\tilde{y}(1;N)$	$\rho(1;N)$	$\tilde{y}(B;N)$	$\rho(B;N)$
64	9.06163865E-1		2.11573809E-1		4.56024125E-3	
128	9.05961186E-1	3.2672	2.11917632E-1	3.4659	4.56142214E-3	4.2273
256	9.05899151E-1	3.3825	2.12016834E-1	3.5495	4.56170149E-3	4.2500
512	9.05880811E-1		2.12044728E-1		4.56176722E-3	

Table 3. *Numerical resuts for* $\beta = 0.9$

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